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On the scaling rules for the anomaly-induced effective action of metric and electromagnetic field

Ana M. Pelinson [a], Ilya L. Shapiro [b,*,1]

[a] Departamento de Física, CFM, Universidade Federal de Santa Catarina, 88040-900, Florianópolis, SC, Brazil
[b] Departamento de Física, ICE, Universidade Federal de Juiz de Fora, 36036-330, MG, Brazil

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ABSTRACT

The anomaly-induced effective action is a useful tool for deriving the contributions coming from quantum effects of massless conformal fields. It is well known that such corrections in the higher derivative vacuum sector of the gravitational action provide the same exponential inflation (Starobinsky model) as the cosmological constant term. At the same time, the presence of a classical electromagnetic field breaks down the exponential solution. In this Letter we explore the role of the anomaly-induced term in the radiation sector and, furthermore, derive the "equation of state" and the scaling laws for all terms in the Einstein equations. As one could expect, the scaling law for the vacuum anomaly-induced effective action is the same as for the cosmological constant.

1. Introduction

It is well known that the conformal anomaly is useful for various applications Quantum Field Theory. In particular, the anomaly-induced effective action has been explored in the cosmological setting about three decades ago [1]. Soon it was discovered that, in the absence of matter fields or radiation, the quantum anomaly-induced contributions lead to the Starobinsky model of inflation [2] (see also [3–8] for an alternative work and further developments). The traditional version of this inflationary model is based on the unstable exponential solution [2], that implies some special choice of the number of the quantum fields with different spins [7]. For instance, the present-day universe with (presumably) only photon being an active quantum field, or the early universe where the active quantum content is described by the Minimal Standard Model of particle physics, satisfy the condition of unstable inflation.

An alternative possibility is to consider the supersymmetric matter content of active fields, that leads to the stable inflationary solution at the initial stage of inflation. The transition from stable to unstable inflationary regimes can be associated to the decoupling of the massive *s*-particles [9,10] in the universe where the inflation is slowing down because of the quantum effects of massive fields [10]. The same effect holds in the presence of the cosmological constant, which actually plays only a small role in this story [11–13]. In both cases of stable and unstable inflationary solutions one usually assume that the universe is empty, that means there are no matter fields and/or radiation.

All the mentioned massive or massless fields of different spins are virtual ones, they manifest themselves only through their contributions to the vacuum action. If the real radiation is present, there is no exponential solution for the conformal factor $a(t)$ and the last tends to the corresponding FRW solution if the particle content corresponds to the unstable case and if the initial data are chosen in an appropriate way [1]. After a while, the dynamical system describing the universe enters the regime where the effect of higher derivative terms becomes negligible [6,7,11], and the behavior of the conformal factor is essentially the same as in the purely classical universe dominated by classical radiation content.

The above conclusion is based on the analysis of the theory with the action which includes Einstein–Hilbert term, classical radiation and higher-derivative anomaly-induced gravitational contributions. However, there may be one missed component in this consideration. In the case when the background radiation is present, one may need to take into account also the anomaly-induced contribution to the electromagnetic part. Indeed, the clas-

* Corresponding author.
E-mail addresses: ana.pelinson@gmail.com (A.M. Pelinson), shapiro@fisica.ufjf.br (I.L. Shapiro).
1 On leave from Tomsk State Pedagogical University, Tomsk, Russia.

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sical radiation does decouple from the equation for the conformal factor, and the radiation density manifests itself only via the first integral of this equation, that is the first Friedmann equation. So, it looks interesting to check what is the effect of the anomaly-induced term, e.g., whether it is capable to produce some significant change in the acceleration of the universe. The same problem can be explored, also, for the radiation-dominated epoch after the inflation ends.

One can consider a bit more general formulation of the problem. We can derive the "equation of state" for all components of the gravitational action, namely the Einstein–Hilbert term, cosmological constant term, free radiation, quantum anomaly-induced contribution to the radiation part and quantum anomaly-induced contribution to the vacuum part. Then we can check how the corresponding "energy densities" depend on the scale factor. For instance, the comparison of these dependencies for the anomaly-induced vacuum terms and the cosmological constant can better explain why the two kind of vacuum actions produce similar exponential behavior. This issue may be also interesting in view of the recent attempts to deal with the cosmological constant problem by taking the anomaly-induced contributions into account [14,15].

The Letter is organized as follows. In Section 2 we write down the anomaly-induced terms in both gravitational and electromagnetic sectors and consider the relations between timelike and spacelike components of the diagonalized equations for the metric (which are Energy–Momentum Tensors in the electromagnetic field case). These relations can be seen as equation of state for all the terms in the modified Einstein equations in the cosmological setting. In Section 3 we explore what is the effect of the anomaly-induced electromagnetic term for the rate of expansion of the universe in the two different situations, namely when the higher derivative metric dependent terms are present or not. Finally, in Section 4 we draw our conclusions and discuss some possible applications of the results.

2. Classical and anomalous terms in the effective action

The conformal anomaly is the typical theoretical phenomenon for massless conformal invariant quantum fields on some nontrivial external background. In case of massless conformal fields the action of vacuum (gravitational one) has to include, at least, the conformal invariant higher derivative part (see, e.g., [16] for the introduction and [17] for a recent review of Quantum Field Theory in curved space)

$$S_{HD} = \int d^4x\sqrt{-g}\big(a_1 C^2 + a_2 E + a_3\Box R\big). \tag{1}$$

Here $C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + (1/3)R^2$ is the square of the Weyl tensor and $E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2$ is the integrand of the Gauss–Bonnet topological term. The terms in the Lagrangian of (1) satisfy the conformal Noether identity and, furthermore, do not affect the dynamical equation for the conformal factor of the metric. At the quantum level, however, the conformal symmetry is violated and this also affects the cosmological solution.

In the cosmological setting, massless conformal invariant quantum fields corresponds to the early epoch when the energy of the photons is much greater than masses of at least some of the charged spinor fields. This condition can be easily satisfied in the inflationary period, especially in the framework of the Starobinsky model, which has, usually, very high values of the typical energies at the end of inflationary period. Furthermore, this condition can be fulfilled in the radiation-dominated period after inflation, where many massive fields approximately can be approximately treated as massless.

2.1. Anomaly-induced terms

Consider the approximation of massless fields. In case of both gravitational and electromagnetic background fields, the conformal anomaly has the form

$$\big\langle T^\mu_\mu\big\rangle = -\big(wC^2 + bE + c\Box R + \beta F^2\big), \tag{2}$$

where $F^2 = F^2_{\mu\nu}$ is square of the strength tensor of the electromagnetic fields, w, b, c are the β-functions for the parameters of the vacuum action and β is proportional to the electromagnetic charge β-function. At one loop order, using the Minimal Subtraction scheme of renormalization, we get

$$\beta = -\frac{2e^2}{3(4\pi)^2}\sum_f N_f - \frac{e^2}{6(4\pi)^2}\sum_s N_s \tag{3}$$

as a sum over charged fermions and scalars with the multiplicities N_f and N_s correspondingly.[2] The one-loop values of w, b, and c, can be found, e.g. in [18,16,17].

It is well known that taking into account the conformal anomaly in the cosmological case leads to the Starobinsky exponential solution [2] for the conformal factor, if there are no matter fields. At the same time, if the radiation is present, there is no such solution. One can naturally ask whether the anomalous electromagnetic term in (2) can change this situation. And more general, whether this term can affect the expansion of the universe at the early stage of its history.

In order to use field quantities in the cosmological setting, one has to perform some space averaging. Obviously, $\langle F^2\rangle \sim \langle E^2\rangle - \langle H^2\rangle$ equals zero for a free radiation. But this does not apply, e.g., to the radiation-dominated early universe, because in this case there is also a hot plasma of other particles and the content of the universe does not reduce to a free electromagnetic radiation. As a qualitative simplest estimate we shall suppose that $F^2 \neq 0$ and set its scale-factor dependence in accordance to its conformal property. One can assume, for instance, that at some fixed scale the magnitude of this term is proportional to the ρ_r^0, that is the classical radiation energy density. This radiation density is supposed to describe not only electromagnetic fields, but also a hot plasma which fills the Universe. It is important to note that such nontrivial material content of the universe is indeed possible at the last stage of the stable inflation, where we observe oscillations of the conformal factor [10] and, consequently, production of photons and charged particles.

The anomaly-induced effective action can be easily derived as a functional of the new variables $\bar g_{\mu\nu}$ and σ, where $g_{\mu\nu} = \bar g_{\mu\nu}\cdot e^{2\sigma}$ and the metric $\bar g_{\mu\nu}$ has fixed determinant. Disregarding the conformal invariant term in the effective action we arrive at the following expression [22]:

$$\bar\Gamma = \int d^4x\sqrt{-\bar g}\left\{w\sigma\bar C^2 + b\sigma\left(\bar E - \frac{2}{3}\bar\Box\bar R\right) + 2b\sigma\,\bar\Delta_4\sigma + \beta\sigma\bar F^2\right\} - \frac{3c+2b}{36}\int d^4x\sqrt{-g}R^2, \tag{4}$$

where $\bar F^2 = \bar g^{\mu\alpha}\bar g^{\nu\beta}F_{\alpha\beta}F_{\alpha\beta} = e^{-4\sigma}F^2$ and Δ_4 is a fourth derivative conformally covariant operator acting on dimensionless scalar

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}R_{;\mu}\nabla^\mu. \tag{5}$$

[2] The detailed discussion of the anomaly-induced action of electromagnetic field and its relation to the more general result coming from the physical renormalization scheme can be found in [19], see also [20,21].

The expression (4) is the quantum correction to the classical action of vacuum. Let us note that the covariant (nonlocal and local) forms of the anomaly-induced action are well known [22,19,20], but Eq. (4) is sufficient for our present purposes. The total action has the form

$$S_t = S_{EH} + \bar{\Gamma} + \text{conf. invariant terms}, \tag{6}$$

where the conformal invariant terms include the classical actions of radiation and of hot charged particles and S_{EH} is the Einstein–Hilbert term

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}(R+2\Lambda). \tag{7}$$

In the case of the cosmological, FRW metric, the classical massless fields decouple from gravity. Taking into account the conformal properties, it proves useful to rewrite the expression (6) in a more detailed form

$$S_t = S_{EH} + S_{HD} + S_r^0 + \bar{\Gamma}_{HD} + \bar{\Gamma}_\beta. \tag{8}$$

Here S_{HD} and S_r^0 are classical higher derivative metric and radiation (including massless charged fields) conformal invariant actions. $\bar{\Gamma}_{HD}$ and $\bar{\Gamma}_\beta$ are parts of anomalous action (4). In what follows we will use the same indications for all quantities, including the trace of the stress tensor, $T_i = (T_{EH}, T_{HD}, \bar{T}_{HD}, T_r^0, T_\beta)$, energy density ρ_i and pressure p_i, with $T_i = \rho_i - 3p_i$ in the corresponding reference frame. On the top of that, we will sometimes use notation for the total expression in the radiation sector, like $\rho_r = \rho_r^o + \rho_\beta$.

2.2. Energy density and pressure

Consider the stress–energy tensor, whose components are given by the variational derivative of the total effective action (8),

$$T^{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S_t}{\delta g_{\alpha\beta}}. \tag{9}$$

In order to calculate separately the contributions of the Einstein–Hilbert term from the terms with high derivatives (HD) which come from the quantum contributions and the electromagnetic one, we present the trace of the stress–energy tensor in the form

$$T = \frac{1}{a^3}\frac{\delta S_t}{\delta a} = T_{EH} + T_r^0 + T_{HD} + \bar{T}_{HD} + T_\beta = \rho_t - 3p_t, \tag{10}$$

where

$$\rho_t = \rho_{EH} + \rho_r^0 + \rho_{HD} + \bar{\rho}_{HD} + \rho_\beta = -T_0^0 \tag{11}$$

and (in Cartesian coordinates)

$$p_t = p_{EH} + p_r^0 + p_{HD} + \bar{p}_{HD} + p_\beta = T_1^1 = T_2^2 = T_3^3 \tag{12}$$

are density-like and pressure-like components, correspondingly. The indices and bars of all quantities are in accordance with the ones of the actions in the r.h.s. of (8). We find useful to introduce such notations even for the Einstein–Hilbert term, despite the physical sense of the quantities is different in this case (as it is, of course, for the higher derivative vacuum terms, too).

Our purpose is to see how all ρ's and p's depend on the scale factor $a(t)$ and also how they behave during and after the inflationary period. One can find the densities for the components by assuming that the conservation law is satisfied separately for each of the stress–energy tensors $T_i = (T_{EH}, T_r^0, T_{HD}, \bar{T}_{HD}, T_\beta)$ in (10). In terms of the cosmic scale factor the conservation law can be expressed as

$$d(\rho_i a^3) = -p_i d(a^3), \quad \text{where } p_i = \frac{\rho_i - T_i}{3}. \tag{13}$$

Following this standard procedure, we can immediately see that since the trace of the classical radiation stress tensor is zero, the equation of state is as it is supposed to be,

$$T_r^0 = 3p_r^0 - \rho_r^0 = 0, \quad \text{hence } p_r^0 = \frac{\rho_r^0}{3}. \tag{14}$$

Correspondingly, this term does not contribute to the equation of motion for $a(t)$, that is to

$$\frac{\delta S_r}{\delta a(t)} = 0.$$

Of course, this term does influence the expansion through the first integral of the equation of motion, that is through Friedmann equation $H^2 = (8\pi G/3)\rho$.

Let us consider another terms. The first observation is that, since the trace is zero for the S_{HD} term, the corresponding equation of state is exactly the same as for the free radiation, $p_{HD} = \rho_{HD}/3$. For other three terms we obtain, in terms of conformal time η (as usual, $dt = a(\eta)\,d\eta$),

$$T_{EH} = \frac{3}{4\pi G}\left[\frac{a''}{a^3} - \frac{2\Lambda}{3}\right], \tag{15}$$

$$\bar{T}_{HD} = 6c\left[-\frac{a''''}{a^5} + 4\frac{a'''a'}{a^6} + 3\left(\frac{a''}{a^3}\right)^2 - 6\frac{a''a'^2}{a^7}\right] - 24b\left[\left(\frac{a'}{a^2}\right)^4 - \frac{a''a'^2}{a^7}\right], \tag{16}$$

and

$$T_\beta = \frac{\beta\bar{F}^2}{a^4}. \tag{17}$$

Using Eqs. (13), the solution for all the densities ρ_i come from the differential equations of the form

$$\frac{d\rho_i}{da^3} + \frac{4}{3}\frac{\rho_i}{a^3} = \frac{T_i}{3a^3}. \tag{18}$$

A general solution for this nonhomogeneous equation (18) is

$$\rho_i(a) = C(a)a^{-4}, \tag{19}$$

where the coefficient $C(a)$ is obtained by the integration of

$$\frac{dC}{d\eta} = T_i a^3 a'. \tag{20}$$

Integrating (20) for each of remaining stress–energy tensor components above and substituting them into Eq. (19), we arrive at the following results:

$$\rho_{EH} = \frac{3}{8\pi G}\left(\frac{a'^2}{a^4} - \frac{\Lambda}{3}\right),$$
$$p_{EH} = -\frac{1}{8\pi G}\left(2\frac{a''}{a^3} - \frac{a'^2}{a^4} - \Lambda\right), \tag{21}$$

$$\bar{\rho}_{HD} = -6c\left[\frac{a'''a'}{a^6} - \frac{1}{2}\left(\frac{a''}{a^3}\right)^2 - 2\frac{a''a'^2}{a^7}\right] + 6b\left(\frac{a'}{a^2}\right)^4,$$
$$\bar{p}_{HD} = -2c\left[5\frac{a'''a'}{a^6} - \frac{a''''}{a^5} + \frac{5}{2}\left(\frac{a''}{a^3}\right)^2 - 8\frac{a''a'^2}{a^7}\right] + 8b\left[3\left(\frac{a'}{a^2}\right)^4 - \frac{a''a'^2}{a^7}\right], \tag{22}$$

$$\rho_\beta = \frac{\beta\bar{F}^2}{a^4}\ln a, \qquad p_\beta = \frac{\beta\bar{F}^2}{3a^4}(\ln a - 1). \tag{23}$$

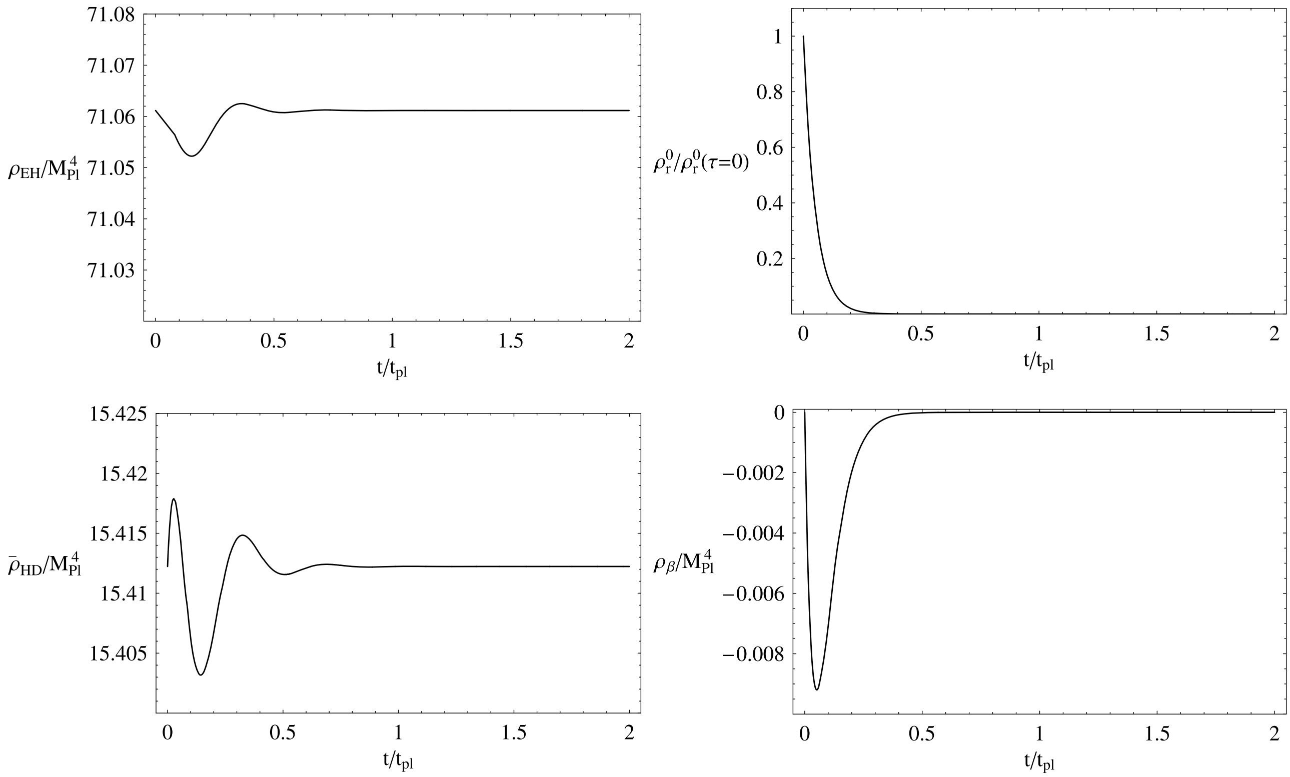

Fig. 1. We have assumed here the MSSM particle content with $N_{1,1/2,0} = (12, 48, 104)$ and took the numerical value $\beta \bar{F}^2 = -0.1$. The plots for $\bar{\rho}_{HD}$ and ρ_{EH} rapidly tend to constants, because the inflation is stable and in the exponential regime the curvature components behave like constants. In this regime, it shows exactly the same scaling law as the density of the cosmological constant.

Indeed, Eqs. (22) derived here are well known, they are exactly the same as the ones obtained in [1], and also recalculated in [8]. It is easy to see that these formulas are quite different from the ones for the cosmological constant in (21). One can expect that for the general form of $a(\eta)$ these two different equations of state will definitely produce different contributions. However, in short we will see the effect of the two terms is equal for the exponential inflation case.

3. Cosmological solutions with anomalous terms

Here we consider the effect of radiation anomalous term on the behavior of the conformal factor of the metric, in the framework of an FLRW cosmology, with and without higher derivative anomalous terms.

3.1. Stable anomaly-induced inflation with radiation term

As a first step, consider the equation of motion including the higher derivative terms. The most useful choice of variable is the conformal factor as a function of cosmic time, $\sigma(t)$. The last quantity is defined as $\sigma = \ln a$. The equation of motion can be obtained from the 00-component [1,2] or directly from the trace $T = \rho_t - 3p_t = 0$ [7]. In terms of $\tau = t/t_{Pl} = M_{Pl}t$, where t_{Pl} is the Planck time unit $t_{Pl} \simeq 5.3 \times 10^{-44}$ s, the equation has the form

$$\ddddot{\sigma} + 7\dot{\sigma}\dddot{\sigma} + 4\left(3 - \frac{b}{c}\right)\dot{\sigma}^2\ddot{\sigma} + 4\ddot{\sigma}^2 - \frac{4b}{c}\dot{\sigma}^4 - \frac{1}{c}\left(\ddot{\sigma} + 2\dot{\sigma}^2 - \frac{2}{3}\frac{\rho_\Lambda}{M_{Pl}^4}\right) - \frac{1}{6c}\left(\frac{\beta\bar{F}^2}{M_{Pl}^4}\right)e^{-4\sigma} = 0. \quad (24)$$

In this equation the contribution of the cosmological constant term is written in terms of vacuum energy density $\rho_\Lambda = \Lambda/(8\pi G) = \Lambda M_{Pl}^2$, where $M_{Pl} = 1/\sqrt{8\pi G} = 2.44 \times 10^{18}$ GeV is the reduced Planck mass.

The direct inspection shows that, in the presence of the $\beta\bar{F}^2$-term, there is no exponential solution. This demonstrates that the importance of vacuum for such solution hold also when we take the anomaly in the radiation sector into account.

The next question is what is the role of the $\beta\bar{F}^2$-term for the case of a stable inflation. In principle, one can expect two different situations: (i) the anomalous term slows down the exponential inflation, as it happens with the terms generated by the quantum effects of massive light fields [10,11]; (ii) the anomalous $\beta\bar{F}^2$-term decreases very fast and soon becomes negligible. The numerical analysis show that this last behavior actually takes place. For the illustration we present the corresponding plots for the case of Minimal Supersymmetric Standard Model (MSSM) in Fig. 1. As we have already mentioned in Section 1, the supersymmetric particle content is the most interesting here, because it provides stable inflation, making the possible effect of the radiation term (or the absence of such effect) the most explicit and clear.

In order to find the cosmological evolution of all the densities, we solve Eq. (24) numerically and then replace the solution into the expressions which directly follow, in particular, from (21), (22) and (23). We present these results, in Fig. 1, as functions of $\tau = t/t_{Pl}$, where

$$\frac{\rho_{EH}(\tau)}{M_{Pl}^4} = 3\dot{\sigma}^2 - \frac{\rho_\Lambda}{M_{Pl}^4}, \quad (25)$$

$$\rho_r^0(\tau) = e^{-4\sigma}\rho_r^0(\tau = 0), \quad (26)$$

$$\frac{\bar{\rho}_{HD}(\tau)}{M_{Pl}^4} = -6c\left(\dot{\sigma}\dddot{\sigma} + 3\dot{\sigma}^2\ddot{\sigma} - \frac{1}{2}\ddot{\sigma}^2 - \frac{b}{c}\dot{\sigma}^4\right), \quad (27)$$

$$\frac{\rho_\beta(\tau)}{M_{Pl}^4} = \frac{\beta \bar{F}^2}{M_{Pl}^4}\sigma e^{-4\sigma}, \tag{28}$$

for the MSSM particle content with $N_{1,1/2,0} = (12, 48, 104)$.

3.2. Radiation-dominated evolution after inflation

In this section we shall investigate the effect of quantum corrections at the period after the anomaly-induced inflation ends and the higher derivative terms in (4) become negligible. Then the relevant part of the total action has the form

$$S_t = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_{class.matter} + \int d^4x\sqrt{-\bar{g}}\beta\sigma\bar{F}^2, \tag{29}$$

where $S_{class.matter}$ is the classical action of the matter fields. We are interested in the period when matter and radiation are very hot and can be treated as conformal. Then, the classical massless fields in $S_{class.matter}$ decouple from the conformal factor of the metric and the effective equation of motion, in terms of conformal time, has the form

$$\frac{3}{4\pi G}a'' + \frac{\beta\bar{F}^2}{a} = 0. \tag{30}$$

Let us, as before, denote the derivative with respect to the cosmic time t by a point. Then Eq. (30) becomes

$$\ddot{a} + \frac{\dot{a}^2}{a} - \frac{\mu^2}{2a^3} = 0. \tag{31}$$

In the last equation we have introduced a useful notation[3]

$$\mu^2 = -\frac{8\pi G\beta\bar{F}^2}{3}. \tag{32}$$

As a first step in solving Eq. (31) we find the relation between $a(t)$ and $H = \dot{a}/a$

$$H^2 = \frac{C + \mu^2\ln a}{a^4}, \tag{33}$$

where C is an integration constant. In order to clarify the sense of this constant, let us consider the standard classical model with $\mu = 0$. In this case, from Eq. (33) follows $a(t) = C(t - t_0)^{1/2}$, where t_0 is some fixed instant of time. Different choices of t_0 can be compensated by the renormalization of C, so we set $t_0 = 0$. On the other hand, by solving the Friedmann equation we obtain

$$a(t) = \left[\frac{32\pi G\rho_r^0(t=0)}{3}\right]^{1/4}\cdot a_0\cdot\sqrt{t}, \tag{34}$$

where $\rho_r^0(t=0)$ and a_0 are the energy density of the electromagnetic field and the scale factor of the metric at the instant $t = 0$. The comparison of the two expressions for $a(t)$ lead to the relation $C = \frac{8\pi G\rho_r^0(t=0)}{3}$. It is natural to fix that the a_0 corresponds to the fiducial metric $\bar{g}_{\mu\nu}$. In what follows we put $a_0 = 1$. Then the elements of the solution (33) satisfy the relation $\mu^2 \ll C$ if the anomalous contribution and energy density satisfy the relation $|\beta\bar{F}^2| \ll 4\rho_r^{(0)}$. As we have indicated above, this relation is quite natural, because for the free radiation $\bar{F}^2 = 0$ and the presence of the anomalous term is due to the interaction with other fields which have energy density much smaller than the one of radiation.

Finally, the general analytic solution of (31) can be presented in the form

$$t = \frac{2e^{-2C/\mu^2}}{\mu}\int_{\sqrt{C}/\mu}^{\sqrt{\sigma+C/\mu^2}} e^{2z^2}\,dz$$
$$= \frac{e^{-2C/\mu^2}}{\mu}\sqrt{\frac{\pi}{2}}\left[\mathrm{Erfi}\left(\sqrt{2\sigma + 2C/\mu^2}\right) - \mathrm{Erfi}(\sqrt{2C}/\mu)\right]. \tag{35}$$

The disadvantage of this formula is that it becomes singular in the classical limit $\mu \to 0$. In order to solve this difficulty, one can derive an approximate solution by treating the term with μ as a small perturbation,

$$\sqrt{C}t \cong \frac{a^2}{2}\left(1 - \frac{\mu^2}{2C}\cdot\ln a\right). \tag{36}$$

As one can see from the last relation, the expansion of the universe performs slightly faster as a result of the quantum effects related to the electromagnetic anomalous term βF^2.

When the universe expands, the radiation temperature is decreasing. It is instructive to find the temperature relations when the quantum term in the solution (36) is relevant. The lower bound for the relevant temperature is defined by the energy corresponding to the moment when the lightest charged fermion decouples and the upper bound is the energy scale when the inflation ends and the higher derivative terms in Eq. (4) become negligible. In the framework of the modified Starobinsky model [9–11], the scale of the graceful exit from the anomaly-induced inflation depends on the scale of the supersymmetry breaking, and may vary from $H = 300$ GeV to $H = 10^{14}$ GeV for different gauge theories. Let us notice that, contrary to the case of a purely gravitational background, the mixed electromagnetic-gravitational background corresponds to the loop diagrams with external lines of both electromagnetic potential and metric perturbations. Indeed, the decoupling scale is defined by the energy of the photons which is much greater than the energy of the gravitons.

In order to find the temperature of the radiation corresponding to (36), we can use the Friedmann equations

$$\frac{\dot{a}^2}{a^2} = H^2 = \frac{8\pi G}{3}\rho_r, \qquad \frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -8\pi G p_r, \tag{37}$$

where p_r is the radiation pressure and the thermodynamical relation is close to $\rho_r \approx \rho_r^0 = \frac{\pi^2}{15}T^4$. Then we arrive at the formula

$$T^4 = \frac{45}{\pi^2}H^2\cdot M_{Pl}^2. \tag{38}$$

The next requires the equation of state for the anomalous term, which was obtained in the previous section. Using this result directly leads us to

$$\rho_r = \frac{3}{8\pi G}\cdot\frac{C + \mu^2\ln a}{a^4} \tag{39}$$

and

$$p_r = \frac{C + \mu^2(\ln a - 1)}{8\pi G a^4} = \frac{1}{3}\rho_r + \frac{1}{3}\frac{|\beta|\bar{F}^2}{a^4}. \tag{40}$$

As one can see here, the quantum effect on the background electromagnetic fields decreases the radiation pressure. The dependence between the temperature and the scale factor is given by

[3] Let us note that $\bar{F}^2 = 2(\langle H^2\rangle - \langle E^2\rangle)$ is typically positive and that, according to our definition (3), $\beta < 0$.

$$T=\frac{1}{a}\left[\frac{45(C+\mu^2\ln a)}{8\pi^3 G}\right]^{1/4}=\frac{1}{a}\left[\frac{15}{\pi^2}\left(\rho_r^0-|\beta|\bar{F}^2\ln a\right)\right]^{1/4}. \tag{41}$$

Naturally, the quantum effects produce some deviation from the usual classical formulas, namely, the ρ_r is a bit larger than ρ_r^0 for a given temperature.

4. Conclusions and discussions

We have considered a cosmological applications of the vacuum quantum effects in the radiation-dominated universe and found that the βF^2 term in the conformal anomaly leads to a slight modification of the evolution law and the thermal history of the universe. In the transitional period between inflation and radiation dominated universe the βF^2 gives a nonzero contribution to the acceleration of the universe, that is different from the classical radiation. It would be interesting to explore further physical consequences of this effect.

The relation for the anomaly-induced effective action in the radiation sector can be useful for investigating the general features of the gravity with anomaly-induced quantum corrections. In particular, it would be very interesting to explore the stability of the corresponding semi-classical solution, for a realistic particle content, that means nonstable Starobinsky inflation. This problem is well known as a problem of stability of Minkowski and de Sitter spaces (see, e.g., [23–25] and further reference therein). Our previous analysis also shows that the stability conditions for the conformal factor of the metric may be different for the flat space from one side and for the dS space from another one [11]. It would be very interesting to check out what are the conditions of stability of the classical solution in a general case, for different stages of the universe expansion. The effect of radiation in the anomaly-induced action (refgeneral solution) is potentially relevant on this respect. We hope to report on this issue in a close future.

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