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Additive results for the generalized Drazin inverse in a Banach algebra ☆

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Abstract

In this paper we investigate additive properties of the generalized Drazin inverse in a Banach algebra. We find some new conditions under which the generalized Drazin inverse of the sum $a + b$ could be explicitly expressed in terms of a, a^{d}, b, b^{d}. Also, some recent results of Castro and Koliha [New additive results for the g-Drazin inverse, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 1085–1097] are extended.

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1. Introduction

Let $\mathscr{A}$ be a complex Banach algebra with the unit 1. By $\mathscr{A}^{-1}$, $\mathscr{A}^{\mathsf{nil}}$, $\mathscr{A}^{\mathsf{qnil}}$ we denote the sets of all invertible, nilpotent and quasinilpotent elements in $\mathscr{A}$, respectively. Let us recall that the Drazin inverse of $a \in \mathscr{A}$ [1] is the element $x \in \mathscr{A}$ (denoted by a^{D}) which satisfies

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$$xax = x, \quad ax = xa, \quad a^{k+1}x = a^k \tag{1}$$

for some nonnegative integer k. The least such k is the index of a, denoted by $ind(a)$. When $ind(a) = 1$ then the Drazin inverse a^{D} is called the group inverse and it is denoted by $a^{\#}$. The conditions (1) are equivalent to

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathscr{A}^{\mathsf{nil}}. \tag{2}$$

The concept of the generalized Drazin inverse in a Banach algebra was introduced by Koliha [2]. The condition $a - a^2x \in \mathscr{A}^{\mathsf{nil}}$ was replaced by $a - a^2x \in \mathscr{A}^{\mathsf{qnil}}$. Hence, the generalized Drazin inverse of a is the element $x \in \mathscr{A}$ (written a^{d}) which satisfies

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathscr{A}^{\mathsf{qnil}}. \tag{3}$$

We mention that an alternative definition of the generalized Drazin inverse in a ring is also given in [3–5]. These two concepts of generalized Drazin inverse are equivalent in the case when the ring is actually a complex Banach algebra with a unit. It is well-known that a^{d} is unique whenever it exists [2]. The set $\mathscr{A}^{\mathsf{d}}$ consists of all $a \in \mathscr{A}$ such that a^{d} exists. For interesting properties of Drazin inverse see [6–8].

Let $a \in \mathscr{A}$ and let $p \in \mathscr{A}$ be a idempotent ($p = p^2$). Then we write

$$a = pap + pa(1-p) + (1-p)ap + (1-p)a(1-p)$$

and use the notations

$$a_{11} = pap, \quad a_{12} = pa(1-p), \quad a_{21} = (1-p)ap, \quad a_{22} = (1-p)a(1-p).$$

Every idempotent $p \in \mathscr{A}$ induces a representation of an arbitrary element $a \in \mathscr{A}$ given by the following matrix:

$$a = \begin{bmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p. \tag{4}$$

Let a^{π} be the spectral idempotent of a corresponding to $\{0\}$. It is well-known that $a \in \mathscr{A}^{\mathsf{d}}$ can be represented in the following matrix form:

$$a = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_p,$$

relative to $p = aa^{\mathsf{d}} = 1 - a^{\pi}$, where a_{11} is invertible in the algebra $p\mathscr{A}p$ and a_{22} is quasinilpotent in the algebra $(1-p)\mathscr{A}(1-p)$. Then the generalized Drazin inverse is given by

$$a^{\mathsf{d}} = \begin{bmatrix} a_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p.$$

The motivation for this paper was the paper of Djordjević and Wei [9] and the paper of Castro and Koliha [10]. In both of these papers the conditions under which the generalized Drazin inverse $(a+b)^{\mathsf{d}}$ could be expressed in terms of a, a^{d}, b, b^{d} were considered. In [9] this problem is investigated for a bounded linear operator on an arbitrary complex Banach space under assumption that $AB = 0$ and these results are the generalizations of the results from [11] where the same problem was considered for matrices. Castro and Koliha [10] considered the same problem for the elements of the Banach algebra with unit under some weaker conditions. They generalized the results from [9].

In the present paper we investigate additive properties of the generalized Drazin inverse in a Banach algebra and find an explicit expression for the generalized Drazin inverse of the sum $a + b$ under various conditions.

In the first part of the paper we find some new conditions, which are nonequivalent to the conditions from [10], allowing for the generalized Drazin inverse of $a+b$ to be expressed in terms of $a, a^{\mathsf{d}}, b, b^{\mathsf{d}}$. It is interesting to note that in some cases we obtain the same expression for $(a+b)^{\mathsf{d}}$ as in [10]. In the rest of the paper we generalize recent results from [10].

2. Results

First we state the following result which is proved in [12] for matrices, extended in [13] for a bounded linear operator and in [10] for arbitrary elements in a Banach algebra.

Theorem 2.1. *Let $x \in \mathscr{A}$ and let*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p,$$

relative to the idempotent $p \in \mathscr{A}$.

(1) *If $a \in (p\mathscr{A}p)^{\mathsf{d}}$ and $b \in ((1-p)\mathscr{A}(1-p))^{\mathsf{d}}$, then x is generalized Drazin invertible and*

$$x^{\mathsf{d}} = \begin{bmatrix} a^{\mathsf{d}} & u \\ 0 & b^{\mathsf{d}} \end{bmatrix}_p, \tag{5}$$

where $u = \sum_{n=0}^{\infty}(a^{\mathsf{d}})^{n+2}cb^n b^{\pi} + \sum_{n=0}^{\infty} a^{\pi}a^n c(b^{\mathsf{d}})^{n+2} - a^{\mathsf{d}}cb^{\mathsf{d}}$.

(2) *If $x \in \mathscr{A}^{\mathsf{d}}$ and $a \in (p\mathscr{A}p)^{\mathsf{d}}$, then $b \in ((1-p)\mathscr{A}(1-p))^{\mathsf{d}}$ and x^{d} is given by (5).*

Now, we state an auxiliary result.

Lemma 2.1. *Let $a, b \in \mathscr{A}^{\mathsf{qnil}}$ and let $ab = ba$ or $ab = 0$, then $a + b \in \mathscr{A}^{\mathsf{qnil}}$.*

Proof. If $ab = ba$ we have that

$$r(a+b) \leqslant r(a) + r(b),$$

which gives that $a + b \in \mathscr{A}^{\mathsf{qnil}}$. The case when $ab = 0$ follows from the equation

$$(\lambda - a)(\lambda - b) = \lambda(\lambda - (a+b)). \qquad \square$$

Considering the previous lemma, the first idea was to replace the basic condition $ab = 0$ which was used in the papers [11,9] by the condition $ab = ba$. As we expected, this condition was not enough to derive a formula for $(a+b)^{\mathsf{d}}$. Hence, to this aim we assume the following three conditions for $a, b \in \mathscr{A}^{\mathsf{d}}$:

$$a = ab^{\pi}, \quad b^{\pi}ba^{\pi} = b^{\pi}b \quad \text{and} \quad b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab. \tag{6}$$

Instead of the condition $ab = ba$ we assume the weaker condition $b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab$. Notice that

$$a = ab^{\pi} \Leftrightarrow ab^{\mathsf{d}} = 0 \Leftrightarrow \mathscr{A}a \subseteq Ab^{\pi}, \tag{7}$$

$$b^{\pi}ba^{\pi} = b^{\pi}b \Leftrightarrow b^{\pi}ba^{\mathsf{d}} = 0 \Leftrightarrow \mathscr{A}b^{\pi}b \subseteq \mathscr{A}a^{\pi}, \tag{8}$$

$$b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab \Leftrightarrow (ba - ab)\mathscr{A} \subseteq (b^{\pi}a^{\pi})^{\circ}, \tag{9}$$

where for $u \in \mathscr{A}$, $u^{\circ} = \{x \in \mathscr{A} : ux = 0\}$.

For matrices and bounded linear operators on a Banach space the conditions (7)–(9) are equivalent to

$$\mathcal{N}(b^{\pi}) \subseteq \mathcal{N}(a), \quad \mathcal{N}(a^{\pi}) \subseteq \mathcal{N}(b^{\pi}b), \quad \mathcal{R}(ba-ab) \subseteq \mathcal{N}(b^{\pi}a^{\pi}).$$

Remark that conditions (6) are not symmetric in a, b like the conditions (3.1) from [10], so our expression for $(a+b)^{\mathsf{d}}$ is not symmetric in a, b at all.

In the next theorem under the assumption that for $a, b \in \mathscr{A}^{\mathsf{d}}$ the conditions (6) hold, we offer the following expression for $(a+b)^{\mathsf{d}}$.

Theorem 2.2. *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ be such that (6) is satisfied. Then $a+b \in \mathscr{A}^{\mathsf{d}}$ and*

$$\begin{aligned}(a+b)^{\mathsf{d}} = {} & \left(b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n\right)a^{\pi} \\ & - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n(a^{\mathsf{d}})^{k+2}b(a+b)^{k+1} \\ & + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n a^{\mathsf{d}}b - \sum_{n=0}^{\infty} b^{\mathsf{d}}a(a^{\mathsf{d}})^{n+2}b(a+b)^n.\end{aligned} \tag{10}$$

Before proving Theorem 2.2 we have to prove the following result which is a special case of this theorem:

Theorem 2.3. *Let $a \in \mathscr{A}^{\mathsf{qnil}}$, $b \in \mathscr{A}^{\mathsf{d}}$ are such that $b^{\pi}ab = b^{\pi}ba$ and $a = ab^{\pi}$. Then (6) is satisfied, $a+b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n. \tag{11}$$

Proof. First, suppose that $b \in \mathscr{A}^{\mathsf{qnil}}$. Then $b^{\pi} = 1$ and from $b^{\pi}ab = b^{\pi}ba$ we obtain that $ab = ba$. Using Lemma 2.1, $a+b \in \mathscr{A}^{\mathsf{qnil}}$ and (11) holds. Now, we assume that b is not quasinilpotent and we consider the matrix representation of a and b relative to the $p = 1 - b^{\pi}$. We have

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \qquad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $b_1 \in (p\mathscr{A}p)^{-1}$ and $b_2 \in ((1-p)\mathscr{A}(1-p))^{\mathsf{qnil}} \subset \mathscr{A}^{\mathsf{qnil}}$. From $a = ab^{\pi}$, it follows that $a_{11} = 0$ and $a_{21} = 0$. We denote $a_1 = a_{12}$ and $a_2 = a_{22}$. Hence,

$$a+b = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}_p.$$

The condition $b^{\pi}ab = b^{\pi}ba$ implies that $a_2b_2 = b_2a_2$. Hence, using Lemma 2.1, we get $a_2 + b_2 \in ((1-p)\mathscr{A}(1-p))^{\mathsf{qnil}}$. Now, by Theorem 2.1, we obtain that $a+b \in \mathscr{A}^{\mathsf{d}}$ and

$$(a+b)^{\mathrm{d}} = \begin{bmatrix} b_1^{-1} & \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1 (a_2+b_2)^n \\ 0 & 0 \end{bmatrix}_p$$

$$= b^{\mathrm{d}} + \sum_{n=0}^{\infty} (b^{\mathrm{d}})^{n+2} a (a+b)^n. \qquad \square$$

Let us observe that the expressions for $(a+b)^{\mathrm{d}}$ in (11) and in (3.6), Theorem 3.3 [10] are exactly the same. If we assume that $ab = ba$ instead of $b^{\pi}ab = b^{\pi}ba$, we will get a much simpler expression for $(a+b)^{\mathrm{d}}$.

Corollary 2.1. *Let $a \in \mathscr{A}^{\mathrm{qnil}}$, $b \in \mathscr{A}^{\mathrm{d}}$ are such that $ab = ba$ and $a = ab^{\pi}$, then $a+b \in \mathscr{A}^{\mathrm{d}}$ and*

$$(a+b)^{\mathrm{d}} = b^{\mathrm{d}}.$$

Proof. From the condition $a = ab^{\pi}$, as we mentioned before, it follows that $ab^{\mathrm{d}} = 0$. Now, because the Drazin inverse b^{d} is double commutant of a, we have that

$$(b^{\mathrm{d}})^{n+2} a (a+b)^n = a (b^{\mathrm{d}})^{n+2} (a+b)^n = 0. \quad \square$$

Proof of the Theorem 2.2. If b is quasinilpotent we can apply Theorem 2.3. Hence, we assume that b is neither invertible nor quasinilpotent and consider the following matrix representation of a and b relative to the $p = 1 - b^{\pi}$:

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \qquad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $b_1 \in (p\mathscr{A}p)^{-1}$ and $b_2 \in ((1-p)\mathscr{A}(1-p))^{\mathrm{qnil}}$. As in the proof of Theorem 2.3, from $a = ab^{\pi}$ it follows that $a = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_p$ and

$$a + b = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}_p.$$

From the conditions $b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab$ and $b^{\pi}ba^{\pi} = b^{\pi}b$, we obtain that $a_2^{\pi}b_2a_2 = a_2^{\pi}a_2b_2$ and $b_2 = b_2a_2^{\pi}$. Now, by Theorem 2.3 it follows that $(a_2+b_2) \in ((1-p)\mathscr{A}(1-p))^{\mathrm{d}}$ and

$$(a_2+b_2)^{\mathrm{d}} = a_2^{\mathrm{d}} + \sum_{n=0}^{\infty} (a_2^{\mathrm{d}})^{n+2} b_2 (a_2+b_2)^n. \tag{12}$$

By Theorem 2.1 we get

$$(a+b)^{\mathrm{d}} = \begin{bmatrix} b_1^{-1} & u \\ 0 & (a_2+b_2)^{\mathrm{d}} \end{bmatrix}_p,$$

where $u = \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1 (a_2+b_2)^n (a_2+b_2)^{\pi} - b_1^{-1} a_1 (a_2+b_2)^{\mathrm{d}}$ and by b_1^{-1} we denote the inverse of b_1 in the algebra $p\mathscr{A}p$. Using (12), we have that

$$u = \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1 (a_2+b_2)^n = a_2^{\pi} - \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1 (a_2+b_2)^n a_2^{\mathrm{d}} b_2$$

$$\times \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b_1)^{-(n+2)}a_1(a_2+b_2)^n(a_2^{\mathsf{d}})^{k+2}b_2(a_2+b_2)^{k+1} - b_1^{-1}a_1a_2^{\mathsf{d}}$$
$$-\sum_{n=0}^{\infty} b_1^{-1}a_1(a_2^{\mathsf{d}})^{n+2}b_2(a_2+b_2)^n$$

By a straightforward computation we obtain that (10) holds. □

Corollary 2.2. *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ are such that $ab = ba$, $a = ab^{\pi}$ and $b^{\pi} = ba^{\pi} = b^{\pi}b$, then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}}.$$

Let us also observe that if a, b are such that a is invertible and b is group invertible than the conditions (8) and (9) are satisfied, so we have to assume just that $a = ab^{\pi}$. In the opposite case when b is invertible we get $a = 0$.

As we mentioned before, Hartwig et al. in [11] for matrices and Djordjević and Wei [9] for operators used the condition $AB = 0$ to derive the formula $(a+b)^{\mathsf{d}}$. Castro and Koliha [10] relaxed this hypothesis by assuming the following three conditions symmetric in $a, b \in \mathscr{A}^{\mathsf{d}}$,

$$a^{\pi}b = b, \quad ab^{\pi} = a, \quad b^{\pi}aba^{\pi} = 0. \tag{13}$$

It is easy to see that $ab = 0$ implies (13), but the converse is not true (see Example 3.1, [10]).

It is interesting to remark that the conditions (13) and (6) are independent, neither of them implies the other one, but in some cases we obtain the same expressions for $(a+b)^{\mathsf{d}}$.

If we consider the algebra $\mathscr{A}$ of all complex 3×3 matrices and $a, b \in \mathscr{A}$ which are given in the Example 3.1 [10], we can see that the conditions (13) are satisfied, but the conditions (6) are not satisfied. In the following example we have the opposite case. We construct matrices a, b in the algebra $\mathscr{A}$ of all complex 3×3 matrices such that (6) is satisfied but (13) is not satisfied. If we assume that $ab = ba$ in Theorem 2.2 the expression for $(a+b)^{\mathsf{d}}$ will be exactly the same as in Theorem 3.5 [10] (in this paper Corollary 2.4).

Example. Let

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$a^{\pi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $b^{\pi} = 1$. Now, we can see that $a = ab^{\pi}$, $a^{\pi}ab = a^{\pi} = ba$ and $ba^{\pi} = b$ i.e., (6) is satisfied. Also, $a^{\pi}b = 0 \neq b$, so (13) is not satisfied.

In the rest of the paper we will present a generalization of the results from [10]. We will use some weaker conditions than in [10]. For example in the next theorem which is the generalization of Theorem 3.3 [10] we will assume that $e = (1 - b^\pi)(a + b)(1 - b^\pi) \in \mathscr{A}^{\mathsf{d}}$ instead of $ab^\pi = a$. If $ab^\pi = a$ then $e = (1 - b^\pi)b = \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}_p$ for $p = 1 - b^\pi$ and $e^{\mathsf{d}} = b^{\mathsf{d}}$.

Theorem 2.4. *Let $b \in \mathscr{A}^{\mathsf{d}}$, $a \in \mathscr{A}^{\mathsf{qnil}}$ be such that*

$$e = (1 - b^\pi)(a + b)(1 - b^\pi) \in \mathscr{A}^{\mathsf{d}} \quad \text{and} \quad b^\pi ab = 0,$$

then $a + b \in \mathscr{A}^{\mathsf{d}}$ and

$$(a + b)^{\mathsf{d}} = e^{\mathsf{d}} + \sum_{n=0}^{\infty} (e^{\mathsf{d}})^{n+2} ab^\pi (a + b)^n.$$

Proof. The case when $b \in \mathscr{A}^{\mathsf{qnil}}$ follows from Lemma 2.1. Hence, we assume that b is not quasinilpotent,

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p \quad \text{and} \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $p = 1 - b^\pi$. From $b^\pi ab = 0$ we have that $b^\pi a(1 - b^\pi) = 0$, i.e., $a_{21} = 0$. Denote $a_1 = a_{11}$, $a_{22} = a_2$ and $a_{12} = a_3$. Then,

$$a + b = \begin{bmatrix} a_1 + b_1 & a_3 \\ 0 & a_2 + b_2 \end{bmatrix}_p.$$

Also, $b^\pi ab = 0$ implies that $a_2 b_2 = 0$, so $a_2 + b_2 \in ((1 - p)\mathscr{A}(1 - p))^{\mathsf{qnil}}$, according to Lemma 2.1. Now, applying Theorem 2.1, we obtain that

$$(a + b)^{\mathsf{d}} = \begin{bmatrix} (a_1 + b_1)^{\mathsf{d}} & u \\ 0 & 0 \end{bmatrix}_p,$$

where $u = \sum_{n=0}^{\infty} ((a_1 + b_1)^{\mathsf{d}})^{(n+2)} a_3 (a_2 + b_2)^n$. By a direct computation we verify that

$$(a + b)^{\mathsf{d}} = e^{\mathsf{d}} + \sum_{n=0}^{\infty} (e^{\mathsf{d}})^{n+2} ab^\pi (a + b)^n. \qquad \square$$

Now, as a corollary we obtain Theorem 3.3 from [10].

Corollary 2.3. *Let $b \in \mathscr{A}^{\mathsf{d}}$, $a \in \mathscr{A}^{\mathsf{qnil}}$ and let $ab^\pi = a$, $b^\pi ab = 0$. Then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a + b)^{\mathsf{d}} = b^{\mathsf{d}} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a(a + b)^n.$$

The next result is a generalization of Theorem 3.5 in [10]. For simplicity we use the following notation:

$$\begin{aligned} e &= (1 - b^\pi)(a + b)(1 - b^\pi) \in \mathscr{A}^{\mathsf{d}}, \\ f &= (1 - a^\pi)(a + b)(1 - a^\pi), \\ \mathscr{A}_1 &= (1 - a^\pi)\mathscr{A}(1 - a^\pi), \\ \mathscr{A}_2 &= (1 - b^\pi)\mathscr{A}(1 - b^\pi), \end{aligned}$$

where $a, b \in A^{\mathsf{d}}$ are given.

We also prove the next result which is the generalization of Theorem 3.5 [10].

Theorem 2.5. *Let* $a, b \in \mathscr{A}^{\mathsf{d}}$ *be such that* $(1-a^\pi)b(1-a^\pi) \in \mathscr{A}^{\mathsf{d}},\ f \in \mathscr{A}_1^{-1}$ *and* $e \in \mathscr{A}_2^{\mathsf{d}}$. *If*

$$(1-a^\pi)ba^\pi = 0, \quad b^\pi aba^\pi = 0, \quad a^\pi = a(1-b^\pi)a^\pi = 0$$

then $a+b \in \mathscr{A}^{\mathsf{d}}$ *and*

$$\begin{aligned}(a+b)^{\mathsf{d}} = {} & \left(b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n\right)a^\pi + \sum_{n=0}^{\infty} b^\pi (a+b)^n a^\pi b (f)_{\mathscr{A}_1}^{-(n+2)} \\ & - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathsf{d}})^{k+1}a(a+b)^{n+k}a^\pi b(f)_{\mathscr{A}_1}^{-(n+2)} - b^{\mathsf{d}}a^\pi b(f)_{\mathscr{A}_1}^{-1} \\ & - \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n a^\pi b (f)_{\mathscr{A}_1}^{-1} + (f)_{\mathscr{A}_1}^{-1},\end{aligned}$$

where by $(f)_{\mathscr{A}_1}^{-1}$ *we denote the inverse of* f *in* $\mathscr{A}_1$.

Proof. Obviously, if a is invertible, then the statement of the theorem holds. If a is quasinilpotent, then the result follows from Theorem 2.4. Hence, we assume that a is neither invertible nor quasinilpotent. As in the proof of Theorem 2.2, we have that

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \qquad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_p,$$

where $p = 1 - a^\pi$, $a_1 \in (p\mathscr{A}p)^{-1}$ and $a_2 \in ((1-p)\mathscr{A}(1-p))^{\mathsf{qnil}}$. From $(1-a^\pi)ba^\pi = 0$, we have that $b_{12} = 0$. Denote $b_1 = b_{11}$, $b_{22} = b_2$ and $b_{21} = b_3$. Then,

$$a+b = \begin{bmatrix} a_1+b_1 & 0 \\ b_3 & a_2+b_2 \end{bmatrix}_p.$$

The condition $a^\pi b^\pi aba^\pi = 0$ expressed in the matrix form yields

$$a^\pi b^\pi aba^\pi = \begin{bmatrix} 0 & 0 \\ 0 & b_2^\pi \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & b_2^\pi a_2 b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly, $a^\pi a(1-b^\pi) = 0$ implies that $a_2 b_2^\pi = a_2$. From Corollary 2.3 we get that $a_2 + b_2 \in \mathscr{A}^{\mathsf{d}}$ and

$$(a_2+b_2)^{\mathsf{d}} = b_2^{\mathsf{d}} + \sum_{n=0}^{\infty}(b_2^{\mathsf{d}})^{n+2}a_2(a_2+b_2)^n.$$

Now, using Theorem 2.1 we obtain that $a+b \in \mathscr{A}^{\mathsf{d}}$ and

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} (a_1+b_1)^{\mathsf{d}} & 0 \\ u & (a_2+b_2)^{\mathsf{d}} \end{bmatrix}_p,$$

where

$$u = \sum_{n=0}^{\infty} b_2^\pi (a_2+b_2)^n b_3 (f)_{\mathscr{A}_1}^{-(n+2)}$$

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b_2^{\mathsf{d}})^{k+1}a_2(a_2+b_2)^{n+k}b_3(f)_{\mathscr{A}_1}^{-(n+2)} - b_2^{\mathsf{d}}b_3(f)_{\mathscr{A}_1}^{-1}$$
$$-\sum_{n=0}^{\infty}(b_2^{\mathsf{d}})^{n+2}a_2(a_2+b_2)^n b_3(f)_{\mathscr{A}_1}^{-1}.$$

By a straightforward computation we obtain that the result holds. □

Corollary 2.4. *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ satisfy the conditions (13). Then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = \left(b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n\right)a^{\pi} + \sum_{n=0}^{\infty}b^{\pi}(a+b)^n b(a^{\mathsf{d}})^{(n+2)}$$
$$-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathsf{d}})^{k+1}a(a+b)^{n+k}b(a^{\mathsf{d}})^{(n+2)} + b^{\pi}a^{\mathsf{d}}$$
$$-\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n ba^{\mathsf{d}}.$$

Proof. We have that $f = (1 - a^{\pi})a$, so $(f)_{\mathscr{A}_1}^{-1} = a^{\mathsf{d}}$. □

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