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The three-dimensional flow past a stretching sheet and the homotopy perturbation method

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Abstract

An approximate analytical solution is obtained of the steady, laminar three-dimensional flow for an incompressible, viscous fluid past a stretching sheet using the homotopy perturbation method (HPM) proposed by He. The flow is governed by a boundary value problem (BVP) consisting of a pair of non-linear differential equations. The solution is simple yet highly accurate and compares favorably with the exact solutions obtained early in the literature. The methodology presented in the paper is useful for solving the BVPs consisting of more than one differential equation.

Keywords: Stretching sheet; Three-dimensional flow; Homotopy perturbation method; Boundary value problem

1. Introduction

The homotopy perturbation method introduced by He has come to be accepted as an elegant tool in the hands of researchers looking for simple yet highly effective solutions to complicated problems in many diverse arenas of science and technology. In a series of papers He [1–13] has outlined and refined the HPM, showing its usefulness by solving algebraic, non-linear ordinary differential equations, bifurcation, partial differential equations and problems involving discontinuities. As a rule, HPM tends to produce much more elegant solutions as compared to the other competing techniques such as homotopy analysis method, regular perturbation methods etc., yet it is not at the cost of accuracy. In general the solutions produced by the HPM are as accurate as the solutions given by the other approximate methods. Recently, He has provided a highly readable account of the finesses and intricacies of the HPM in a monograph [14].

The steady two-dimensional laminar flow of an incompressible, viscous fluid past a stretching sheet has become a classical problem in fluid dynamics as it admits an unusually simple closed form solution, first discovered by Crane [15]. Since then the problem has been extensively studied by taking into account many different physical features either separately or in various combinations. In most of the cases the solution turns out to be again in a closed form closely mimicking the solution of the original problem. Thus the effect of suction was analyzed by Gupta and Gupta [16], that of a uniform transverse magnetic field, when the fluid is electrically conducting, by Anderson [17]. The closed form solution for a viscoelastic fluid was pointed out by Troy et al. [18] and Chang [19] who also

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discovered an extraneous solution for a specific value of the viscoelastic fluid parameter — for general values of the parameter the solution was derived by Ariel [20], though these solutions have been shown to be unstable. The joint effect of the viscoelasticiy and magnetic field on Crane's problem has been investigated by Ariel [21]. Another feature, the interest in which has been revived recently, is that of the partial slip of the fluid past the sheet. Wang [22] presented the solution of the problem. The heat transfer aspects of the Crane's problem have been studied by many researchers starting with Gupta and Gupta [16]. In all these investigations the existence of the closed form solution considerably facilitated the analysis.

The flow past a stretching sheet need not be necessarily two-dimensional because the stretching of the sheet can take place in a variety of ways. The flow becomes axisymmetric when the stretching is radial, and for this problem there is no closed form exact analytical solution. Its numerical solution was first provided by Wang [23] who considered the problem as a special case of the general problem of the three-dimensional flow in which the stretching could be sheared i.e., need not be the same in all the directions. Wang also gave a perturbation solution for the general case taking his primary solution as Crane's solution. It turned out that his solution, though quite complicated, was acceptable only for small deviations from the two-dimensional case. The effects of viscoelasticity on the axisymmetric flow past a stretching sheet have been analyzed by Ariel for an elastico-viscous fluid [24] and a second grade fluid [25]. A non-iterative solution for the MHD flow has been given by Ariel [26] using the technique of Samuel and Hall [27]. Recently Ariel et al. [28] applied the HPM to derive the analytical solution to the axisymmetric flow past a stretching sheet. They also included the effects of suction and magnetic field, separately, and jointly and reported that there is hardly any degradation in the performance of the HPM solution even when thin boundary layers develop on account of strong suction and/or magnetic field — the usual perturbation techniques or even the shooting methods perform rather poorly in such situations.

Besides Wang [23], the solution of the generalized three-dimensional flow past a stretching sheet has been given by Ariel [29] using the Ackroyd method [30] of an infinite series of negative exponentials and recently [31] by the technique of Samuel and Hall [27] which generates the solution non-iteratively. Unlike the case of the two-dimensional or the axisymmetric flow, the generalized three-dimensional flow is governed by a BVP consisting of a pair of non-linear differential equations. Ariel et al. [28] were able to apply the HPM for the axisymmetric flow as the BVP had a single differential equation. Our endeavor in the present paper is look into the suitability of the HPM when the problem is characterized by BVPs embodying more than one differential equation. Therefore in the present work we re-examine the steady, laminar three-dimensional flow of a viscous, incompressible fluid past a stretching sheet, and attempt to obtain its solution using the HPM.

2. Equations of motion

We take the stretching sheet in the XOY plane and assume that the fluid occupies the domain $z > 0$. The entire motion is caused by the stretching of the sheet by a velocity $(u, v, 0)$ given by

$$u = Ax \quad \text{and} \quad v = By \tag{1}$$

where A and B are the constants of proportionality.

Wang [23] has demonstrated that by using the similarity variables

$$u = Axf'(\eta), \quad v = Ayg'(\eta), \quad w = -\sqrt{A\nu}(f+g), \tag{2}$$

where

$$\eta = \sqrt{\frac{A}{\nu}}z \tag{3}$$

(ν being the kinematic viscosity) the Navier–Stokes equations can be reduced to the following pair of ordinary differential equations:

$$f''' + (f+g)f'' - f'^2 = 0 \tag{4}$$

$$g''' + (f+g)g'' - g'^2 = 0. \tag{5}$$

In Eqs. (2), (4) and (5), a prime denotes the derivative with respect to η.

The boundary conditions for the problem are

$$f(0) + g(0) = 0, \quad f'(0) = 1, \quad g'(0) = \beta \tag{6}$$

$$f'(\infty) = 0, \quad g'(\infty) = 0 \tag{7}$$

where

$$\beta = B/A. \tag{8}$$

The pressure p at any point is given by

$$p = p_0 + \rho\left(\nu\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{1}{2}w^2\right) \tag{9}$$

where ρ is the density of the fluid, and p_0 is the pressure at some reference point (the origin in the present problem).

3. Homotopy perturbation solution

The essence of the HPM is the introduction of the homotopy perturbation parameter p (not to be confused with the pressure) which takes the values from 0 to 1. When $p = 0$, the system of equations takes a simplified form whose solution can be readily obtained analytically. As p is increased and it eventually takes the value 1, the system of equations evolves to the required form, and it is expected that solution would approach the desired value. The variables of interest are expressed as a power series in p.

The elegance of the method is achieved by introducing yet other parameter(s) which can be chosen suitably to fulfill certain desirable criteria, such as the absence of the secular terms in the solution. We follow the procedure successfully implemented in [28] and apply it to both the equations governing the motion. We rewrite Eqs. (4) and (5) as

$$f''' - b^2 f' + p[(f+g)f'' - f'^2 + b^2 f'] = 0, \tag{10}$$

$$g''' - b^2 g' + p[(f+g)g'' - g'^2 + b^2 g'] = 0. \tag{11}$$

In accordance with the standard practice of HPM, we seek the solutions for f and g as

$$f = f_0 + pf_1 + p^2 f_2 + \cdots, \tag{12}$$

$$g = g_0 + pg_1 + p^2 g_2 + \cdots. \tag{13}$$

If we substitute for f and g from Eqs. (12) and (13) in Eqs. (10) and (11), and boundary conditions (6) and (7), we obtain the following system of equations, upon equating the like powers of p:
Zeroth order system:

$$f_0''' - b^2 f_0' = 0, \tag{14}$$

$$g_0''' - b^2 g_0' = 0, \tag{15}$$

$$f_0(0) + g_0(0) = 0, \quad f_0'(0) = 1, \quad g_0'(0) = \beta, \tag{16}$$

$$f_0'(\infty) = 0, \quad g_0'(\infty) = 0. \tag{17}$$

First order system:

$$f_1''' - b^2 f_1' + (f_0 + g_0) f_0'' - f_0'^2 + b^2 f_0' = 0, \tag{18}$$

$$g_1''' - b^2 g_1' + (f_0 + g_0) g_0'' - g_0'^2 + b^2 g_0' = 0, \tag{19}$$

$$f_1(0) + g_1(0) = 0, \quad f_1'(0) = 0, \quad g_1'(0) = 0, \tag{20}$$

$$f_1'(\infty) = 0, \quad g_1'(\infty) = 0. \tag{21}$$

Second order system:

$$f_2''' - b^2 f_2' + (f_0 + g_0) f_1'' - 2 f_0' f_1' + (f_1 + g_1) f_0'' + b^2 f_1' = 0, \tag{22}$$

$$g_2''' - b^2 g_2' + (f_0 + g_0) g_1'' - 2 g_0' g_1' + (f_1 + g_1) g_0'' + b^2 g_1' = 0, \tag{23}$$

$$f_2(0) + g_2(0) = 0, \quad f_2'(0) = 0, \quad g_2'(0) = 0, \tag{24}$$

$$f_2'(\infty) = 0, \quad g_2'(\infty) = 0. \tag{25}$$

If necessary the higher order systems can be written down readily. However, one of the hallmarks of the HPM is that one rarely needs the solution beyond the second order system. Sufficiently accurate solutions are obtained by restricting to the second degree or even the first degree terms in (12) and (13).

Solving the BVP (14)–(17), we obtain

$$f_0' = \mathrm{e}^{-b\eta}, \quad g_0' = \beta \mathrm{e}^{-b\eta}, \quad f_0 + g_0 = \frac{1}{b}(1 + \beta)(1 - \mathrm{e}^{-b\eta}). \tag{26}$$

Substituting from (26) into (18) and (19), the following solution is obtained for the first order system:

$$f_1' = \frac{\beta}{3b^2}(\mathrm{e}^{-b\eta} - \mathrm{e}^{-2b\eta}) + \frac{b^2 - 1 - \beta}{2b} \eta \mathrm{e}^{-b\eta}, \tag{27}$$

$$g_1' = \frac{\beta}{3b^2}(\mathrm{e}^{-b\eta} - \mathrm{e}^{-2b\eta}) + \frac{b^2 - 1 - \beta}{2b} \beta \eta \mathrm{e}^{-b\eta}, \tag{28}$$

$$f_1 + g_1 = \frac{\beta}{3b^3}(1 - \mathrm{e}^{-b\eta})^2 + \frac{b^2 - 1 - \beta}{2b^3}(1 + \beta)(1 - \mathrm{e}^{-b\eta} - b\eta \mathrm{e}^{-b\eta}). \tag{29}$$

The auxiliary parameter b can be chosen in several ways. However, as He [8] has demonstrated convincingly, it should be chosen so that the solutions are free of the secular terms such as $\eta \mathrm{e}^{-b\eta}$. Indeed in [28] and several other investigations the optimum results were obtained by invoking precisely this criterion for b. We therefore seek the solutions that are free of secular terms. This is achieved if we set

$$b^2 = 1 + \beta. \tag{30}$$

The first order system solution, in which case, simplifies to

$$f_1' = g_1' = \frac{\beta}{3b^2}(\mathrm{e}^{-b\eta} - \mathrm{e}^{-2b\eta}), \quad f_1 + g_1 = \frac{\beta}{3b^3}(1 - \mathrm{e}^{-b\eta})^2. \tag{31}$$

Once the value of b is determined, we no longer have control over the presence of the secular terms in the solutions of the higher order systems. Thus, we cannot avoid the secular terms in the solution for the second order system. Therefore, if we persist with the present methodology, we must truncate the solution after the first degree terms in Eq. (12) and (13) and be satisfied with the solution

$$f = f_0 + f_1, \quad g = g_0 + g_1 \tag{32}$$

obtained by terminating the series after the first degree terms and setting $p = 1$. The velocity components are thus given by

$$f' = \mathrm{e}^{-b\eta} + \frac{\beta}{3b^2}(\mathrm{e}^{-b\eta} - \mathrm{e}^{-2b\eta}), \tag{33a}$$

$$g' = \beta \mathrm{e}^{-b\eta} + \frac{\beta}{3b^2}(\mathrm{e}^{-b\eta} - \mathrm{e}^{-2b\eta}) \tag{33b}$$

$$f + g = (1 - \mathrm{e}^{-b\eta}) \left[\frac{1}{b}(1 + \beta) + \frac{\beta}{3b^3}(1 - \mathrm{e}^{-b\eta}) \right]. \tag{33c}$$

We need not be unduly concerned about the accuracy of the solution since it is only limited to the first order, because as seen in [28] (the case $\beta = 1$), the error in the solution is minimal (less than 0.4%), and, of course, for the case $\beta = 0$, we get the exact solution.

Table 1
Illustrating the variation of $-f''(0)$ and $-g''(0)$ with β, using the homotopy perturbation method, exact solution (Ariel [29]) and the first order perturbation solution in β (Wang [23])

β	$-f''(0)$			$-g''(0)$		
	HPM	Exact	Wang	HPM	Exact	Wang
0	1	1	1	0	0	0
0.1	1.017027	1.020260	1.020902	0.073099	0.066847	0.058198
0.2	1.034587	1.039495	1.041804	0.158231	0.148737	0.116395
0.3	1.052470	1.057955	1.062705	0.254347	0.243360	0.174593
0.4	1.070529	1.075788	1.083607	0.360599	0.349209	0.232791
0.5	1.088662	1.093095	1.104509	0.476290	0.465205	0.290988
0.6	1.106797	1.109947	1.125411	0.600833	0.590529	0.349186
0.7	1.124882	1.126398	1.146312	0.733730	0.724532	0.407384
0.8	1.142879	1.142489	1.167214	0.874551	0.866683	0.465581
0.9	1.160762	1.158254	1.188116	1.022922	1.016539	0.523779
1.0	1.178511	1.173721	1.209018	1.178511	1.173721	0.581977

In the next section we compare the solution obtained by the HPM with the exact solution obtained by Ariel [29] and the regular first order perturbation solution obtained by Wang [23].

4. Comparison of the results

There can be several criteria for comparison of the results obtained by the HPM and those obtained by Ariel [29] and Wang [23]. If some representative numbers are to be chosen, a natural choice is the values of $-f''(0)$ and $-g''(0)$, which respectively are the measures of the stress at the wall in the x- and y-directions. For the HPM we have from Eqs. (26) and (30)–(32)

$$-f''(0) = \frac{3+2\beta}{3\sqrt{1+\beta}}, \quad -g''(0) = \frac{\beta(2+3\beta)}{3\sqrt{1+\beta}}. \tag{34}$$

In Table 1, the values of these quantities are presented for different values of β.

It can be seen from the table that the homotopy perturbation method gives much more accurate results than those obtained by the perturbation solution, except for the values of $f''(0)$ near $\beta = 0$, which is to be expected as the Wang's solution is developed around $\beta = 0$. However, the perturbation solution soon degrades as the value of β is increased, whereas the HPM continues to give fully acceptable results for all values of β. This is one great advantage of using the HPM in that it gives uniformly accurate solution for all values of the physical parameters.

5. Conclusion

In the present work we have applied the homotopy perturbation method (HPM) to compute the steady, laminar, three-dimensional flow of an incompressible, viscous fluid near a stretching sheet. The solution, though restricted to the first order expansion, is quite elegant and fully acceptable in accuracy. It has been demonstrated that the HPM can be applied advantageously even when the flow is governed by a BVP consisting of more than one differential equation. In the present problem we were assisted by the "similarity" of the pair of equations which permitted the use of one undetermined parameter. In other flows, it is more likely that more than one parameter might be required to obtain the solution free of the secular terms. It is proposed to apply the HPM to such problems in future. The results of the investigations will be reported in due course of time.

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