A non-hamiltonian cyclically 4-edge connected bicubic graph with 50 vertices is constructed. This is the smallest non-hamiltonian 3-connected bicubic graph known. © 1989 Academic Press, Inc.

A graph is bicubic if it is bipartite and 3-regular. A graph is hamiltonian if it contains a circuit that includes every vertex. A graph is cyclically 4-edge connected if any set of edges whose removal separates the graph into two parts both containing circuits, contains at least 4 edges.

In 1971 Tutte [5] published a conjecture that no non-hamiltonian 3-connected bicubic graph existed. Horton found the first such graph on 96 vertices (see [1, p. 240]). Horton [4] later published another one on 92 vertices. Ellingham [2] constructed an infinite family of these graphs, the smallest on 78 vertices. He noted that up to that point no cyclically 4-edge connected bicubic graph had been found. Ellingham and Horton [3] constructed an infinite family of cyclically 4-edge connected such graphs with the smallest on 54 vertices. In this note we construct another infinite family of cyclically 4-edge connected bicubic graphs whose smallest member has 50 vertices.

Let $uv$ be an edge in a graph. Let $w$ be a vertex not in the graph. We say that $w$ is inserted along $uv$ if $w$ is added to the vertex set and edge $uv$ is replaced by edges $uw$ and $wv$. If vertices $a_1$ and $a_2$ are inserted along $uv$, then $a_1$ and $a_2$ are added to the vertex set and $uv$ is replaced by edges $ua_1$, $a_1a_2$, and $a_2v$.

Let $T$ be the trivalent tree on 6 vertices, as shown in Fig. 1.

Let $A$ be a cyclically 4-edge connected bicubic graph with the property that no hamiltonian circuit passes through both of two particular non-adjacent edges. Let $ab$ and $cd$ be the specified edges. Such a graph on 18 vertices is shown in Fig. 2. This graph was used by Ellingham and Horton in their constructions.
Let $B$ be the graph obtained by inserting two vertices along each of the specified edges in $A$. Let $e, f, g, h$ be inserted along $ab$ and $cd$, respectively. We note that $B$ is bipartite and non-hamiltonian with four vertices of degree 2, namely $e, f, g, h$.

Let $B_1$ and $B_2$ be copies of $B$. For $i = 1$ and 2, let $e_i, f_i, g_i, h_i$ be the vertices of degree 2 in $B_i$. We join with edges these vertices of degree 2 and the vertices of degree 1 in a copy of $T$, as indicated in Fig. 3.

It can be easily shown that the resulting graph $G$ is bicubic and cyclically 4-edge connected. To show that $G$ is non-hamiltonian, we assume to the contrary that $G$ does have a hamiltonian circuit $H$. Consequently, for each $i$, either 2 or 4 of the edges joining $B_i$ and $T$ are included in $H$. We will consider each of the four cases separately.

Case 1. $H$ includes 2 edges joining $B_1$ and $T$ and 2 edges joining $B_2$ and $T$. For each $i$, the restriction of $H$ to $B_i$ is a hamiltonian path. Since $B_i$
is non-hamiltonian, bipartite, and has an even number of vertices $e_i$ and $h_i$ or $f_i$ and $g_i$ are the endpoints of this path. If $e_1$ and $h_1$ are the endpoints for the restriction of $H$ to $B_1$, and $e_2$ and $h_2$ are the endpoints for the restriction of $H$ to $B_2$, the edges $e_1s, h_1q, e_2s$, and $h_2p$ are included in $H$. These edges, however, make it impossible for $H$ to include vertices $y$ and $r$. Similar results occur when $e_1$ and $h_1$ and $f_2$ and $g_2$, $f_1$ and $g_1$ and $e_2$ and $h_2$, $f_1$ and $g_1$ and $f_2$ and $g_2$, are the endpoints for the restriction of $H$ to $B_1$ and $B_2$, respectively. Since $H$ must include every vertex in $G$, we conclude that $H$ cannot include 2 edges joining $B_1$ and $T$ and 2 edges joining $B_2$ and $T$.

**Case 2.** $H$ includes 2 edges joining $B_1$ and $T$ and 4 edges joining $B_2$ and $T$. $H$ includes edges $e_2s, f_2q, g_2r$, and $h_2p$. The restriction of $H$ to $B_2$ forms two disjoint paths which include every vertex in $B_2$ and have endpoints $e_2, f_2, g_2$, and $h_2$. One path has endpoints $e_2$ and $f_2$, and the other path has endpoints $g_2$ and $h_2$, for, otherwise, these paths and edges $e_2f_2$ and $g_2h_2$ would form a hamiltonian circuit in $B_2$, which is impossible. Let $P_1$ and $P_2$ denote these paths, respectively.

The restriction of $H$ to $B_1$ has endpoints $e_1$ and $h_1$ or $f_1$ and $g_1$. If $e_1$ and $h_1$ are its endpoints, then $e_1s$ and $h_1q$ are in $H$. The restriction of $H$ to $B_1$, path $P_1$, and edges $e_1s, h_1q, e_2s$, and $f_2q$ form a circuit that misses vertices $x$ and $y$. Similarly, if $f_1$ and $g_1$ are the endpoints for the restriction of $H$ to $B_1$, $H$ contains the circuit formed by this restriction, path $P_2$, and edges $f_1p, g_1r$, and $h_2p$, which misses vertices $x$ and $y$. Thus, $H$ cannot include 2 edges joining $B_1$ and $T$ and 4 edges joining $B_2$ and $T$.

**Case 3.** $H$ includes 4 edges joining $B_1$ and $T$ and 2 edges joining $B_2$ and $T$. By using arguments similar to those found in Case 2, we determine that $H$ contains a circuit which does not include vertices $x$ and $y$, and so $H$ cannot occur in this case.

**Case 4.** $H$ includes 4 edges joining $B_1$ and $T$ and 4 edges joining $B_2$ and $T$. Since $e_1s$ and $e_2s$ are in $H$, $ys$ cannot be in $H$. Similar arguments
show that edges $px$, $qx$, and $ry$ cannot be in $H$. As a result, $H$ cannot include vertices $x$ and $y$, a contradiction.

The smallest non-hamiltonian cyclically 4-edge connected bicubic graph obtained by using this technique has 50 vertices and appears in Fig. 4. Replacing the subgraph induced by vertices $x$ and $y$ with a suitable graph in the general construction also produces such a graph. As a result, we can obtain a non-hamiltonian cyclically 4-edge connected bicubic graph on $2n$ vertices, $2n > 50$.

REFERENCES