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Generalisation of a Waiting-Time Relation

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A generalisation of a waiting-time relation is developed by the use of Laplace transform theory. The generalisation produces an infinite series and it is demonstrated how it may be summed by representation in closed form. Extensions and examples of the waiting-time relation are given.

1. INTRODUCTION

It seems that the sum, after a rearrangement to suit the following work,

$$\sum_{n=0}^{\infty} (-1)^n (abe^{ab})^n \frac{(t+n)^n}{n!} = \frac{e^{-abt}}{1+ab} \tag{1.1}$$

first appeared in the work of Jensen [11]. Jensen's work was based on an extension of the binomial theorem due to Abel and an application of Lagrange's formula.

In the analysis of the delay in the answering of telephone calls, Erlang [7] obtains an integro-differential-difference equation from which a similar result to (1.1) is quoted. Likewise a series similar to (1.1) later appeared in the works of Bruwier [3, 4] in his analysis of differential-difference equations. In fact, the result (1.1) arises in a number of areas including the

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works of Feller [8] on ruin problems, Hall [10] on coverage processes, Smith [13] on renewal theory, and Tijms [14] on stochastic modelling, just to name a few. To date no extension to (1.1) appears to be available. It is therefore the aim of this paper to give a novel technique for the development and generalisation of (1.1). Recurrence relations will be developed and further extensions to the results indicated.

2. THE DIFFERENTIAL-DIFFERENCE EQUATION

Consider the differential-difference equation

$$\sum_{n=0}^{R} \binom{R}{R-n} c^{R-n} \sum_{r=0}^{n} \binom{n}{r} b^{n-r} f^{(r)}(t-(R-n)a) = 0, \qquad t > Ra,$$

$$\sum_{r=0}^{R} \binom{R}{r} b^{R-r} f^{(r)}(t) = 0, \qquad 0 < t \le Ra, \tag{2.1}$$

with $f^{(R-1)}(0) = 1$ and all other initial conditions at rest, where a, b, and c are real constants. Erlang [2] considered (2.1) in his work on the delay in the answering of telephone calls for the case of $R = 1$ only. For the case of R servers, Erland derived a differential-difference equation different from (2.1) and this will be the subject of a forthcoming paper.

It has become commonplace to analyse differential-difference equations by the use of Laplace transform theory. In this paper Laplace transform techniques will be used to bring out the essential features that are required for the results.

Taking the Laplace transform of (2.1) results in

$$F(p) = L\{f(t)\} = \frac{1}{(p+b+c\exp(-ap))^{R}}$$

$$= \sum_{n=0}^{\infty} \binom{n+R-1}{n} (-1)^{n} \frac{c^{n}\exp[-an(p+b)]\exp(anb)}{(p+b)^{n+R}}. \tag{2.2}$$

The inverse Laplace transform is

$$f(t) = \sum_{n=0}^{\infty} \binom{n+R-1}{n} (-1)^{n} c^{n} \exp[-b(t-an)]$$

$$\times \frac{(t-an)^{n+R-1}}{(n+R-1)!} H(t-an), \tag{2.3}$$

where $H(x)$ is the unit Heaviside step function.

The solution to (2.1) by Laplace transform theory may be written as

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}[g(p)]^{-1}\,dp$$

for an appropriate choice of γ such that all the zeros of the characteristic equation

$$g(p) = (p + b + ce^{-ap})^R \tag{2.4}$$

are contained to the left of the line in the Bromwich contour.

Now, using the residue theorem,

$$f(t) = \sum \text{Residues of } \left\{e^{pt}[g(p)]^{-1}\right\},$$

which suggests the solution of $f(t)$ may be written in the form $f(t) = \sum_r Q_r \exp(p_r t)$ where the sum is over all the characteristic roots p_r of $g(p) = 0$ and Q_r is the contribution of the residues in $F(p)$ at $p = p_r$.

The poles of (2.2) depend on the zeros of the characteristic equation (2.4), the roots of $g(p) = 0$. The dominant root p_0 of $g(p) = 0$ has the greatest real part and therefore asymptotically

$$f(t) \sim \sum_{k=0}^{R-1} k!Q_{-(R-k)} \frac{t^{R-k-1}}{(R-k-1)!}\exp(p_0 t)$$

and so from (2.3)

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} (-1)^n \frac{c^n \exp[-b(t-an)](t-an)^{n+R-1}}{n!(R-1)!} H(t-an) \\ &\sim \sum_{k=0}^{R-1} k!Q_{-(R-k)} \frac{t^{R-k-1}}{(R-k-1)!}\exp(p_0 t), \end{aligned} \tag{2.5}$$

where the contribution $Q_{-(R-k)}$ to the residue is

$$k!Q_{-(R-k)} = \lim_{p \to p_0} \left\{ \frac{d^{(k)}}{dp^{(k)}} \left(\frac{(p-p_0)^R}{g(p)} \right) \right\}, \qquad k = 0, 1, 2, \ldots, R-1, \tag{2.6}$$

since the right-hand side of (2.2) has a pole of order R for $1 - ac \neq 0$ at the dominant root $p = p_0$.

It seems reasonable to suggest that if in (2.5) t is large, more and more terms in the expression on the left-hand side will be included. Therefore it is *conjectured* that, for all positive integer values of R,

$$\sum_{n=0}^{\infty}(-1)^n \frac{c^n \exp[-b(t-an)](t-an)^{n+R-1}}{n!(R-1)!} = \sum_{k=0}^{R-1} k! Q_{-(R-k)} \frac{t^{R-k-1}}{(R-k-1)!} \exp(p_0 t) \tag{2.7}$$

for all real values of t, in the region where the infinite series converges.

By the use of the ratio test it can be seen that the series on the left-hand side of (2.7) converges in the region

$$|ac \exp(1+ab)| < 1.$$

A proof of (2.7) will now be given for the case of $R = 2$. An application of Burmann's theorem will be utilised.

3. THE CONJECTURE PROVED

Burmann's theorem [15] essentially allows for the expansion of a function in positive powers of another function, and can be stated as follows:

BURMANN'S THEOREM. *Let ϕ be a simple function in a domain D, zero at a point β of D, and let*

$$\theta(z) = \frac{z-\beta}{\phi(z)}, \qquad \theta(\beta) = \frac{1}{\phi'(\beta)}.$$

If $f(z)$ is analytic in D, then, $\forall z \in D$,

$$f(z) = f(\beta) + \sum_{r=1}^{n} \frac{\{\phi(z)\}^r}{r!} \frac{d^{r-1}}{dt^{r-1}} \left[f'(t)\{\theta(t)\}^r\right]_{t=\beta} + R_{n+1},$$

where

$$R_{n+1} = \frac{1}{2\pi i} \int_{\Gamma} d\nu \int_{c} \left[\frac{\phi(\nu)}{\phi(t)}\right]^n \frac{f'(t)\phi'(\nu)}{\phi(t)-\phi(\nu)}\, dt.$$

The ν-integral is taken along a contour Γ in D from β to z, and the t-integral along a closed contour C in D encircling Γ once positively.

APPLICATION OF BURMANN'S THEOREM. Without loss of generality, choosing $b + c = 0$ and $1 + ab > 0$ allows the dominant root of the characteristic equation (2.4) to occur at $p_0 = 0$.

Let $t = -a\tau$ and so from (2.7)

$$\sum_{n=0}^{\infty} (-1)^n \frac{(abe^{ab})^n (\tau+n)^{n+R-1}}{n!} = e^{-ab\tau} \sum_{k=0}^{R-1} Q_{-(R-k)} (-1)^k \binom{R-1}{k} \frac{\tau^{R-k-1}}{a^k}. \qquad (3.1)$$

Putting $ab = -\rho$ in (3.1) gives

$$\sum_{n=0}^{\infty} (\rho e^{-\rho})^n \frac{(\tau+n)^{n+R-1}}{n!} = e^{\rho\tau} \sum_{k=0}^{R-1} Q_{-(R-k)} (-1)^k \binom{R-1}{k} \frac{\tau^{R-k-1}}{a^k}. \qquad (3.2)$$

In the case of $R = 2$ and evaluating $Q_{-(2-k)}$ for $k = 0, 1$ from (2.6) allows (3.2) to be written as

$$\tau + \sum_{n=1}^{\infty} (\rho e^{-\rho})^n \frac{(\tau+n)^{n+1}}{n!} = e^{\rho\tau} \left[\frac{\tau}{(1-\rho)^2} + \frac{\rho}{(1-\rho)^3} \right]. \qquad (3.3)$$

Equation (3.3) is now shown to be true by applying Burmann's theorem. Let

$$f(z) = e^{xz} \left[\frac{x}{(1-z)^2} + \frac{z}{(1-z)^3} \right],$$

$$\theta(z) = \frac{z}{\phi(z)} = e^z, \qquad \phi(z) = ze^{-z}, \qquad f(\beta)_{\beta=0} = x$$

and it may be shown that $R_{n+1} \to 0$ as $n \to \infty$. From

$$f(t) = e^{xt} \left[\frac{x}{(1-t)^2} + \frac{t}{(1-t)^3} \right],$$

$$f'(t) = e^{xt} \left[\frac{x^2}{(1-t)^2} + \frac{xt + 2x + 1}{(1-t)^3} + \frac{3t}{(1-t)^4} \right]$$

and so

$$f'(t)\{\theta(t)\}^r = \exp[t(r+x)]\, \chi(t),$$

where

$$\chi(t) = \frac{x^2}{(1-t)^2} + \frac{xt + 2x + 1}{(1-t)^3} + \frac{3t}{(1-t)^4}$$
$$= \sum_{j=0}^{\infty} \frac{j+1}{2}\left[2x^2 + (2x+1)(j+2) + j(x+j+2)\right]t^j$$

and the coefficients in this last expression are the same as those in a Taylor series expansion

$$\chi^{(j)}(0) = \frac{(j+1)!}{2}\left[2x^2 + x(3j+4) + (j+1)(j+2)\right],$$
$$j = 0, 1, 2 \ldots .$$

Now let

$$B_r(t) = \frac{d^{r-1}}{dt^{r-1}}\left[f'(t)\{\theta(t)\}^r\right]$$
$$= \frac{d^{r-1}}{dt^{r-1}}\left[\exp[t(r+x)]\,\chi(t)\right] = \exp[t(r+x)]$$
$$\times\left[\binom{r-1}{0}(r+x)^{r-1}\chi^{(0)}(t) + \binom{r-1}{1}(r+x)^{r-2}\chi'(t)\right.$$
$$+\binom{r-1}{2}(r+x)^{r-3}\chi''(t) + \cdots + \binom{r-1}{r-2}(r+x)\chi^{(r-2)}(t)$$
$$\left.+\binom{r-1}{r-1}(r+x)^0\chi^{(r-1)}(t)\right].$$

Hence

$$B_r(0) = (r+x)^{r-1}(x+1)^2 + (r-1)(r+x)^{r-2}(2x+3)(x+2)$$
$$+ 3(r-1)(r-2)(r+x)^{r-3}(x+2)(x+3) + \cdots$$
$$+ \frac{(r-1)(r+x)(r-1)!}{2}\left[2x^2 + x(3r-2) + r(r-1)\right]$$
$$+ \frac{r!}{2}\left[2x^2 + x(3r+1) + r(r+1)\right].$$

Put $y = r + x$, giving

$$\begin{aligned} B_r(0) &= y^{r+1} + y^r[-2(r-1) + 2(r-1)] \\ &\quad + y^{r-1}\left[(r-1)^2 - (r-1)(4r-7) + 3(r-1)(r-2)\right] + \cdots \\ &\quad + y\left[\frac{(r-1)(r-1)!}{2}r + \frac{r!}{2}(-r+1)\right] \\ &\quad + y^0\left[\frac{r!}{2}(2r^2 - 3r^2 - r + r^2 + r)\right] \\ &= y^{r+1} = (r+x)^{r+1}. \end{aligned}$$

Hence it follows that

$$e^{xz}\left[\frac{x}{(1-z)^2} + \frac{z}{(1-z)^3}\right] = x + \sum_{r=1}^{\infty} \frac{(ze^{-z})^r}{r!}(r+x)^{r+1}.$$

Using the same technique, (3.1) can be proved for $R = 3, 4, 5 \ldots$. A proof of (2.7) for the case $R = 1$ can be found in the work of Cerone and Sofo [5].

4. A RECURRENCE RELATION FOR Q_r

The following lemma regarding moments of the convolution of the generator function $\phi(x)$ will be proved and required in the evaluation of a recurrence relation for the contribution $Q_{-(R-m)}$ to the residues.

LEMMA. *The nth moment of the Rth convolution of* $\phi(x) = -bH(a-x)$ *is* $(-ab)^R(-1)^n a^n n! C_n^R$.

Proof. Consider the rectangular wave $\phi(x) = -bH(a-x) = b(-1 + H(x-a))$, which has a Laplace transform of

$$\Phi(p) = \frac{b(-1 + e^{-ap})}{p}.$$

The Rth convolution of $\Phi(p)$ can be expressed as

$$\begin{aligned}\Phi^R(p) &= b^R\left(\frac{-1+e^{-ap}}{p}\right)^R \\ &= b^R\left(\frac{-1+\sum_{r=0}^{\infty}(-1)^r\frac{(ap)^r}{r!}}{p}\right)^R, \qquad R=1,2,3\ldots. \\ &= b^R\left(-\sum_{r=0}^{\infty}a(-1)^r\frac{(ap)^r}{(r+1)!}\right)^R \\ &= (-ab)^R\sum_{r=0}^{\infty}(-1)^r C_r^R a^r p^r. \end{aligned} \tag{4.1}$$

The convolution constants C_r^R in (4.1) can be evaluated recursively as follows:

$$\begin{aligned} C_r^1 &= \beta_r = \frac{1}{(r+1)!}, \qquad R=1, \\ C_r^R &= \sum_{j=0}^{r}\beta_{r-j}C_j^{R-1}, \qquad R=2,3,4\ldots. \end{aligned} \tag{4.2}$$

Moreover, the convolution constants are polynomials in R of degree r, so that

$$C_0^R = 1, \qquad C_1^R = \frac{R}{2}, \qquad C_2^R = R(3R+1)/24, \qquad \text{and so on.}$$

These convolution constants are related to Stirling polynomials and details may be found in the work of Cerone and Sofo [6].

The nth moment of the Rth convolution can be obtained by differentiating (4.1) n times with respect to p, so that

$$\frac{d^n}{dp^n}\left[\Phi^R(p)\right] = (-ab)^R\sum_{r=n}^{\infty}(-1)^r \dot{C}_r^R a^r r(r-1)\cdots(r-n+1)p^{r-n}.$$

Therefore the nth moment of $\Phi^R(p)$ is

$$\lim_{p\to 0}\frac{d^n}{dp^n}\left[\Phi^R(p)\right] = (-ab)^R(-1)^n a^n n! C_n^R.$$

The proof of the lemma is complete.

This result will now be used in the determination of a recurrence relation for Q_r. The contribution $Q_{-(R-m)}$ to the residue can be obtained from (2.6). However, a recurrence relation for $Q_{-(R-m)}$ is developed which is argued to be more computationally efficient than using (2.6) directly and better allows for an induction-type proof of (3.1).

From (2.6)

$$\begin{aligned} m!Q_{-(R-m)} &= \lim_{p\to 0} \frac{d^m}{dp^m}\left[\left(\frac{1}{1-\Phi(p)}\right)^R\right], \qquad m = 0,1,2,\ldots,(R-1) \\ &= \lim_{p\to 0} \frac{d^m}{dp^m}\left[\left(1+\frac{\Phi(p)}{1-\Phi(p)}\right)^R\right] \\ &= \lim_{p\to 0} \sum_{k=0}^{R} \frac{d^m}{dp^m}\left[\binom{R}{k}\Phi^k(p)\frac{1}{(1-\Phi(p))^k}\right] \\ &= \lim_{p\to 0} \sum_{k=0}^{R}\binom{R}{k}\sum_{r=0}^{m}\binom{m}{r}\frac{d^{m-r}}{dp^{m-r}}\left[\Phi^k(p)\right]\frac{d^r}{dp^r}\left\{\frac{1}{(1-\Phi(p))^k}\right\}. \end{aligned}$$

Now utilising the lemma for the $(m-r)$th moment of $\Phi^k(p)$ implies that

$$\begin{aligned} m!Q_{-(R-m)} &= \sum_{k=0}^{R}\binom{R}{k}\sum_{r=0}^{m}\binom{m}{r}(-ab)^k(-a)^{m-r}(m-r)!C^k_{m-r}r!Q_{-(k-r)}, \\ Q_{-(R-m)} &= \sum_{k=0}^{R}\binom{R}{k}\sum_{r=0}^{m}(-a)^{m-r}(-ab)^k C^k_{m-r}Q_{-(k-r)}. \end{aligned}$$

Using the fact that $C^R_0 = 1$ and taking the term at $k = R$, $r = m$ on the left-hand side results in

$$Q_{-(R-m)} = \frac{1}{\left(1-(-ab)^R\right)}\left[\sum_{k=0}^{R}\binom{R}{k}\sum_{r=0}^{m}(-a)^{m-r}(-ab)^k \times C^k_{m-r}Q_{-(k-r)} - (-ab)^R Q_{-(R-m)}\right]. \quad (4.3)$$

Equation (4.3) allows for the recursive evaluation of the contribution to the residues, $Q_{-(R-m)}$, with the initial values $C^0_0 = 1$, $Q_{-(0-0)} = 1$.

It is instructive to follow an example through so that the flavour of the calculations for $Q_{-(R-m)}$ can be gleaned.

Consider (4.3) and let $m = 1$. Then

$$Q_{-(R-1)} = \frac{1}{\left(1-(-ab)^R\right)}\left[\sum_{k=0}^{R}\binom{R}{k}(-ab)^k(-a)C_1^kQ_{-(k-0)} + \sum_{k=0}^{R-1}\binom{R}{k}(-ab)^kC_0^kQ_{-(k-1)}\right].$$

Since $C_1^0 = 0$, $Q_{-(0-1)} = 0$ and from previous recursive calculations $C_0^k = 1$, $C_1^k = k/2$,

$$Q_{-(k-0)} = \frac{1}{(1+ab)^k}, \qquad Q_{-(k-1)} = \frac{-ka(-ab)}{2(1-(-ab))^{k+1}}.$$

Then

$$\begin{aligned} Q_{-(R-1)} &= \frac{-a}{\left(1-(-ab)^R\right)}\left[\sum_{k=1}^{R}\binom{R}{k}\frac{(-ab)^k k}{2(1-(-ab))^k} + \sum_{k=1}^{R-1}\binom{R}{k}(-ab)^{k+1}\frac{k}{2(1-(-ab))^{k+1}}\right] \\ &= \frac{-a(-ab)R}{2\left(1-(-ab)^R\right)(1-(-ab))^R}\sum_{k=0}^{R-1}\binom{R}{k}(-ab)^k \\ &\quad \times (1-(-ab))^{R-k-1}. \end{aligned}$$

Using the definition of the Bernstein polynomial [1],

$$B^n(x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}.$$

Then $Q_{-(R-1)}$ may be expressed as

$$Q_{-(R-1)} = \frac{a^2bR\left[B^{R-1}(-ab)\right]}{2(1-(-ab))^{R+1}\sum_{k=0}^{R-1}(-ab)^k}$$

and so

$$Q_{-(R-1)} = \frac{a^2bR}{2(1+ab)^{R+1}}.$$

If the value of m is specified at the outset, (4.3) may be simplified to produce the following recurrence relations on R only, so that for $m = 0$,

$$Q_{-((R+1)-0)} = \left[Q_{-(1-0)}\right]^{R+1},$$

for $m = 1$,

$$Q_{-((R+1)-1)} = (R+1)Q_{-(1-1)}\left[Q_{-(1-0)}\right]^{R},$$

for $m = 2$,

$$Q_{-((R+1)-2)} = (R+1)\left[Q_{-(1-0)}Q_{-(1-2)} + R\{Q_{-(1-1)}\}^2\right]\left[Q_{-(1-0)}\right]^{R-1},$$

and, for $m = 3$,

$$Q_{-((R+1)-3)} = (R+1)\Big[Q_{-(2-0)}Q_{-(1-3)} + 3RQ_{-(1-0)}Q_{-(1-1)}Q_{-(1-2)} + R(R-1)\{Q_{-(1-1)}\}^3\Big]\left[Q_{-(1-0)}\right]^{R-2}.$$

Table 1 gives for some of the $Q_{-(R-m)}$ the contribution to the residues from using the recurrence relation (4.3) or those following it.

From (3.1) and using the residue calculations at (4.3), or from Table 1, the results for the right-hand side of (3.1) are listed in Table 2.

These elegant results, expressing the infinite series in closed form, can be generated from (3.1) for any positive integer value of R.

The results at (2.7) or (3.1) can be used as a basis for the generation of other infinite series which may be expressed in closed form. This will be investigated in the next section.

5. GENERATING FUNCTION

The basic equations at (2.7) or (3.1) can be differentiated and integrated to produce more identities in closed form.

Integrating the result at (2.7) will yield the same result as when considering the differential-difference equation (2.1) with an exponential- or polynomial-type forcing term, respectively. The analysis can also be achieved with a polynomial−exponential-type forcing term.

From (2.7), with $b + c = 0$,

$$\frac{d^j}{dt^j}\left[\sum_{n=0}^{\infty} b^n \exp[-b(t-an)]\,\frac{(t-an)^{n+R-1}}{n!}\right]$$
$$= \frac{d^j}{dt^j}\left[(R-1)!\sum_{k=0}^{R-1} Q_{-(R-k)}\frac{t^{R-k-1}}{(R-k-1)!}\right], \qquad 0 < j \le R-1.$$

TABLE 1
Values of $Q_{-(R-m)}$ for $m = 0, 1, 2, 3, 4, 5$

m	$Q_{-(R-m)}$
0	$\dfrac{1}{(1+ab)^R}$
1	$\dfrac{Ra^2b}{2(1+ab)^{R+1}}$
2	$\dfrac{-Ra^3b(4-ab(3R-1))}{12(1+ab)^{R+2}}$
3	$\dfrac{Ra^4b\left[2-4abR+a^2b^2R(R-1)\right]}{8(1+ab)^{R+3}}$
4	$\dfrac{-Ra^5b\left[48-ab(56+200R)+a^2b^2(-16+40R+120R^2)-a^3b^3(2+5R-30R^2+15R^3)\right]}{240(1+ab)^{R+4}}$
5	$\dfrac{Ra^6b\left[16-ab(64+128R)+a^2b^2(-8+36R+140R^2)+a^3b^3(16R+40R^2-40R^3)+a^4b^4(2R+5R^2-10R^3+3R^4)\right]}{96(1+ab)^{R+5}}$

TABLE 2
Closed-Form Expression of (3.1) for $R = 1, 2, 3, 4$

R	Right-Hand Side of (3.1)
1	$e^{-ab\tau}\left[\frac{1}{1+ab}\right]$
2	$e^{-ab\tau}\left[\frac{\tau}{(1+ab)^2} - \frac{ab}{(1+ab)^3}\right]$
3	$e^{-ab\tau}\left[\frac{\tau^2}{(1+ab)^3} - \frac{3ab\tau}{(1+ab)^4} - \frac{ab(1-2ab)}{(1+ab)^5}\right]$
4	$e^{-ab\tau}\left[\frac{\tau^3}{(1+ab)^4} - \frac{6ab\tau^2}{(1+ab)^5} - \frac{ab(4-11ab)\tau}{(1+ab)^6} - \frac{ab(1-8ab+6a^2b^2)}{(1+ab)^7}\right]$

So that

$$\sum_{n=0}^{\infty} b^n (n+R-1)(n+R-2)\cdots(n+R-j)$$
$$\times \exp[-b(t-an)] \frac{(t-an)^{n+R-(j+1)}}{n!}$$
$$= \sum_{r=0}^{j} \binom{j}{r} b^{j-r} \frac{d^r}{dt^r}\left[(R-1)! \sum_{k=0}^{R-1} Q_{-(R-k)} \frac{t^{R-k-1}}{(R-k-1)!}\right]. \quad (5.1)$$

Differentiation is permissible within the radius of convergence of the infinite series, which for (5.1) is

$$|-ab\exp(1+ab)| < 1. \quad (5.2)$$

For $R = 2$ and $j = 1$,

$$\sum_{n=0}^{\infty} (n+1) b^n \exp[-b(t-an)] \frac{(t-an)^n}{n!}$$
$$= \frac{b}{(1+ab)^2}\left[t + \frac{a^2 b}{1+ab} + \frac{1}{b}\right].$$

Integrating (2.7) ν times with $b + c = 0$ results in

$$\sum_{n=0}^{\infty} \frac{b^n e^{abn}(t - an)^{n+R-1+\nu}}{n!(n + R - 1 + 1)(n + R - 1 + 2)\cdots(n + R - 1 + \nu)}$$

$$= \underbrace{\int \cdots \int}_{\nu \text{ times}} e^{bt}(R - 1)! \sum_{k=0}^{R-1} Q_{-(R-k)} \frac{t^{R-k-1}}{(R - k - 1)!}\, dt, \qquad \nu = 1, 2, 3, \dots .$$

This integration is permissible within the radius of convergence (5.2).

For $R = 2$ and $\nu = 1$,

$$\sum_{n=0}^{\infty} (n + 1) b^n \exp[-b(t - an)] \frac{(t - an)^{n+2}}{(n + 2)!}$$

$$= \frac{1}{(1 + ab)^2}\left[\frac{t}{b} + \frac{a^2b^2 - 1 - ba}{b^2(1 + ab)}\right] + \frac{\exp(-bt)}{(b \exp(ab))^2}.$$

In the case when (2.1) has an impulsive-type forcing term of the form $w(t) = \delta(t - \mu)$ and all initial conditions at rest, then, by a change of variable $t - \mu = T$, $\mu \in \mathbb{R}^+$, the relation (2.7) holds with t replaced by T.

Further results may be obtained by considering a forcing term of the form

$$w(t) = e^{-bt} \frac{t^{m-1}}{(m - 1)!}$$

on the right-hand side of the system (2.1).

The following section develops specific functional relationships for (3.1).

6. FUNCTIONAL RELATIONS

For the case of $R = 1$, Pyke and Weinstock [12] gave a functional relationship of (3.1). The following lemma states a functional relationship for (3.1) in the general case with $R - 1 = \nu$.

LEMMA. *Given that*

$$f_\nu(\tau) = \sum_{n=0}^{\infty} (-1)^n \gamma^n \frac{(\tau + n)^{n+\nu}}{n!}, \qquad \nu = 0, 1, 2, \dots,$$

then

$$f_\nu(\tau) + \gamma f_\nu(\tau + 1) = \tau f_{\nu-1}(\tau)$$

and

$$f_\nu(\tau) = q_\nu(\tau)\exp\left[-\frac{\tau\gamma^a f_\nu(\alpha)}{(ab)^{\alpha-1}q_\nu(\alpha)}\right], \qquad \alpha = 1, 2, 3, \ldots .$$

Proof. From (3.1) let $\gamma = abe^{ab}$. Then

$$\begin{aligned} f_\nu(\tau) &= \sum_{n=0}^{\infty}(-1)^n\gamma^n\frac{(\tau+n)^{n+\nu}}{n!} \\ &= e^{-ab\tau}\sum_{k=0}^{\nu}Q_{-((\nu+1)-k)}(-1)^k\binom{\nu}{k}\frac{1}{a^k}\tau^{\nu-k}, \\ f_\nu(\tau)+\gamma f_\nu(\tau+1) &= \sum_{n=0}^{\infty}(-1)^n\gamma^n\frac{(\tau+n)^{n+\nu}}{n!} \\ &\quad + \gamma\sum_{n=0}^{\infty}(-1)^n\gamma^n\frac{(\tau+1+n)^{n+\nu}}{n!} \\ &= \tau\sum_{n=0}^{\infty}(-1)^n\gamma^n\frac{(\tau+n)^{n+\nu-1}}{n!} \\ &= \tau f_{\nu-1}(\tau). \end{aligned}$$

From the right-hand side of (3.1)

$$\begin{aligned} f_\nu(\tau) &= e^{-ab\tau}\left[\sum_{k=0}^{\nu}Q_{-((\nu+1)-k)}(-1)^k\binom{\nu}{k}\frac{1}{a^k}\tau^{\nu-k}\right. \\ &\qquad \left. +ab\sum_{k=0}^{\nu}Q_{-((\nu+1)-k)}(-1)^k\binom{\nu}{k}\frac{1}{a^k}(\tau+1)^{\nu-k}\right] \\ &= e^{-ab\tau}\left[\sum_{k=0}^{\nu}Q_{-((\nu+1)-k)}(-1)^k\binom{\nu}{k}\frac{1}{a^k}\left\{\tau^{\nu-k}+ab(\tau+1)^{\nu-k}\right\}\right] \\ &= e^{-ab\tau}q_\nu(\tau) \end{aligned}$$

and it follows, after some algebraic manipulation, that

$$f_\nu(\tau) = q_\nu(\tau)\exp\left[-\tau\gamma^\alpha f_\nu(\alpha)/(ab)^{\alpha-1}q_\nu(\alpha)\right] \qquad \text{for } \alpha = 1, 2, 3, \ldots .$$

7. CONCLUSION

A novel technique has been developed and utilised in the summing of infinite series. A host of infinite series can be expressed in closed form by the use of this procedure. This generalisation of a waiting-time relation apart from the case of $R = 1$ does not seem to appear in the literature, such as the work of Gradshteyn and Ryzhik [9].

In a subsequent paper the authors will extend the techniques developed here to consider noninteger values of R and other cases in which more than one dominant root of the characteristic equation will affect the closed-form solution of the infinite series.

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