Oscillation Criteria for Second-Order Nonlinear Differential Equations with Impulses

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Abstract—The present paper is devoted to the investigation of the oscillatory behavior of a kind of extensively studied second-order nonlinear delay differential equations with impulses. Some interesting results are obtained, which illustrate that impulses play a very important role in giving rise to the oscillations of equations. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Consider the impulsive delay differential equation

\[
\left\{ \begin{array}{l}
(a(t) (x'(t))^{\sigma})' + f(t, x(t), x(t - \tau)) = 0, \quad t \neq t_k, \\
x(t_k^+) = I_k(x(t_k)), \quad x'(t_k^+) = \dot{I}_k(x'(t_k)),
\end{array} \right.
\]

where \( \tau > 0, \, 0 < \sigma = p/q \) with \( p \) and \( q \) odd integers (odd/odd), \( 0 < t_1 < t_2 < \cdots < t_k < \cdots \) and \( \lim_{t \to \infty} t_k = \infty, \, t_{k+1} - t_k > \tau, \) and

\[
x'(t_k) = \lim_{h \to 0^+} \frac{x(t_k + h) - x(t_k)}{h}, \\
x'(t_k^+) = \lim_{h \to 0^+} \frac{x(t_k + h) - x(t_k^+)}{h}.
\]

Throughout this paper, assume that the following conditions hold:

(i) \( f(t, u, v) \) is continuous in \([t_0 - \tau, +\infty) \times (-\infty, +\infty) \times (-\infty, +\infty), \) where \( t_0 \geq 0, \) \( u f(t, u, v) > 0 \) (\( uv > 0 \)), and \( \frac{f(t, u, v)}{v} \geq p(t) \) (\( v \neq 0 \)), where \( p(t) \) is continuous in \([t_0 - \tau, +\infty), \) \( p(t) \geq 0, \) and \( x \varphi(x) > 0 \) (\( x \neq 0 \)), \( \varphi'(x) \geq 0; \)

(ii) \( I_k(x), \dot{I}_k(x) \) are continuous in \((-\infty, +\infty), \) and there exist positive numbers \( c_k, \bar{c}_k, \underline{d}_k, \bar{d}_k \) such that

\[
c_k \leq I_k(x) : x \leq \bar{c}_k, \quad \underline{d}_k \leq \dot{I}_k(x) : x \leq \bar{d}_k;
\]

(iii) \( a(t) \) is a positive continuous function in \([t_0 - \tau, +\infty) \) and \( A(t) = \int_{t_0}^t \frac{ds}{a^{1/\sigma}(s)} \).

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Recently, there is increasing interest on the oscillatory/nonoscillatory behavior of first-order linear delay differential equations with impulses (see [1-6]), and good results are being obtained. But there is little in the way of results for second-order nonlinear delay differential equations with impulses. The present paper is devoted to the study of the oscillatory behavior of a type of extensively studied second-order nonlinear delay differential equations with impulses. Some interesting results are gained here. In addition, some examples show that, even though some delay differential equations without impulses are nonoscillatory, they may become oscillatory if some impulses are added to them. That is, in some cases, impulses play a dominating part in causing the oscillations of equations. For some related results on the oscillatory behavior of some second-order nonlinear ODE with impulses, please see [7]. For the theory of delay differential equations and impulsive differential equations, we refer to the recent books by Győri and Ladas [8] and Lakshmikantham, Bainov and Simeonov [9], respectively.

We introduce the following notation:

\[ \text{PC}_\beta = \{ x : [\beta - \tau, \beta] \to \mathbb{R} \mid x(t) \text{ is twice continuously differentiable for } t \in [\beta - \tau, \beta] \} \]

\[ \text{PK}_\beta = \{ x : (\tau, +\infty) \to \mathbb{R} \mid x(t) \text{ is continuous for } t > \tau, x(t) \text{ and } x(t +) \text{ exist and } x(t) = x(t), x'(t) = x'(t) \text{ for } t \in [\beta - \tau, \beta] \} \]

\[ x(t) = \phi(t), \quad t \in [\beta - \tau, \beta] \]

For any \( \beta > 0, \phi \in \text{PC}_\beta \), a function \( x : [\beta - \tau, +\infty) \to \mathbb{R} \) is called a solution of equation (1) satisfying the initial value condition

\[ x(t) = \phi(t), \quad t \in [\beta - \tau, \beta] \]

if \( x \in \Omega_\beta \) and satisfies (1) and (2).

Using the method of steps, one can show that for any \( \tau > 0 \) and \( \phi \in \text{PC}_\beta \), there exists a unique solution \( x \in \Omega_\beta \) of the initial value problems (1) and (2).

A solution of (1) is said to be nonoscillatory if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

### 2. SOME LEMMAS

**Lemma 1.** Let \( x(t) \) be a solution of equation (1). Suppose that there exists some \( T \geq t_0 \) such that \( x(t) > 0 \) for \( t \geq T \). If

\[ A(t_{j+1}) - A(t_j) + \sum_{k=1}^{j+1} \frac{d_{j+k} A(t_{j+k+1}) - A(t_{j+k})}{c_j c_{j+1}} + \ldots + \frac{d_{j+n} A(t_{j+n+1}) - A(t_{j+n})}{c_j c_{j+1} \ldots c_{j+n}} + \ldots = +\infty \]

for some \( t_j(\geq t_1), \) where \( A(t) = \int_{t_0}^{t} \frac{ds}{a \sqrt{S(s)}}, \) then

\[ x'(t) \geq 0 \quad \text{and} \quad x'(t) \geq 0 \]

for \( t \in (t_k, t_{k+1}], \) where \( t_k \geq T. \)

**Proof.** At first, we prove that \( x'(t) \geq 0 \) for any \( t \geq T. \) If not, then there exists some \( j \) such that \( t_j \geq T, \) \( x'(t_j) < 0, \) and \( x'(t_j^+) = \int_{t_j}^{t_j^+} \frac{ds}{a(t)} < 0. \) Let

\[ a(t) \left(x'(t)^+\right)^{\sigma} = -\beta^\sigma \quad (\beta > 0) \]

and

\[ S(t) = a(t) \left(x'(t)^{\sigma}\right). \]

By (1), for \( t \in (t_{j-i-1}, t_j], i = 1, 2, \ldots \) we have

\[ (a(t) \left(x'(t)^{\sigma}\right))' = -f(t, x(t), x(t - \tau)) \leq -p(t) \varphi(x(t - \tau)) \leq 0. \]

This completes the proof. \( \Box \)
Hence, \( S(t) \) is monotonically decreasing in \((t_{j+1-1}, t_{j+1}]\). So
\[
a(t_{j+1}) \left( x' \left( t_{j+1} \right) \right)^\sigma \leq a \left( t_{j+1}^+ \right) \left( x' \left( t_{j+1}^+ \right) \right)^\sigma = -\beta^\sigma < 0
\]
and
\[
a(t_{j+2}) \left( x' \left( t_{j+2} \right) \right)^\sigma \leq a \left( t_{j+1}^+ \right) \left( x' \left( t_{j+1}^+ \right) \right)^\sigma = a \left( t_{j+1} \left( \int x' \left( t_{j+1} \right) \right)^\sigma \right)
\leq \left( d_{j+1} \right)^\sigma a \left( t_{j+1} \left( x' \left( t_{j+1} \right) \right)^\sigma \right)
\leq -\left( d_{j+1} \right)^\sigma \beta^\sigma < 0.
\]
By induction, we obtain
\[
a \left( t_{j+n} \right) \left( x' \left( t_{j+n} \right) \right)^\sigma \leq - \left( d_{j+1} d_{j+2} \cdots d_{j+n-1} \right)^\sigma \beta^\sigma < 0
\]for any natural number \( n \geq 2 \). Now, we claim that
\[
x \left( t_{j+n} \right) \leq c_{j+1} c_{j+2} \cdots c_{j+n-1} \times \left[ x \left( t_{j+1}^+ \right) - \beta \left( A \left( t_{j+1} \right) - A \left( t_{j} \right) \right) - \frac{d_{j+1}}{c_{j+1}} \beta \left( A \left( t_{j+2} \right) - A \left( t_{j+1} \right) \right) - \cdots - \frac{d_{j+1} d_{j+2} \cdots d_{j+n-1}}{c_{j+1} c_{j+2} \cdots c_{j+n-1}} \beta \left( A \left( t_{j+n} \right) - A \left( t_{j+n-1} \right) \right) \right]\]
for any natural number \( n \geq 2 \). Since \( S(t) \) is monotonically decreasing in \((t_{j+i-1}, t_{j+i}]\), we get
\[
a(t) \left( x' \left( t \right) \right)^\sigma \leq a \left( t_{j+1}^+ \right) \left( x' \left( t_{j+1}^+ \right) \right)^\sigma , \quad t \in \left( t_{j+i-1}, t_{j+i} \right]
\]
and
\[
x' \left( t \right) \leq \frac{a \left( t_{j+1}^+ \right) \left( x' \left( t_{j+1}^+ \right) \right)^\sigma}{a^{1/\sigma} \left( t \right)} .
\]
Integrating the above inequality, we have
\[
x \left( t \right) \leq x \left( s \right) + \left[ a \left( t_{j+1}^+ \right) \left( x' \left( t_{j+1}^+ \right) \right)^\sigma \right]^{1/\sigma} \int_s^t \frac{du}{a^{1/\sigma} \left( u \right)}
\]
for \( t_j < s < t < t_{j+1} \). Let \( t \rightarrow t_{j+1}, \ s \rightarrow t_{j+1}^+ \), we get
\[
x \left( t_{j+1} \right) \leq x \left( t_{j+1}^+ \right) + \left[ a \left( t_{j+1}^+ \right) \left( x' \left( t_{j+1}^+ \right) \right)^\sigma \right]^{1/\sigma} \left( A \left( t_{j+1} \right) - A \left( t_{j} \right) \right)
\leq x \left( t_{j+1}^+ \right) - \beta \left( A \left( t_{j+1} \right) - A \left( t_{j} \right) \right) .
\]
Similar to (7), we have
\[
x \left( t_{j+2} \right) \leq x \left( t_{j+1}^+ \right) + \left[ a \left( t_{j+1}^+ \right) \left( x' \left( t_{j+1}^+ \right) \right)^\sigma \right]^{1/\sigma} \left( A \left( t_{j+2} \right) - A \left( t_{j+1} \right) \right) .
\]
By Condition (ii) and (4),(7),(8), we obtain
\[
x \left( t_{j+2} \right) \leq I_{j+1} \left( x \left( t_{j+1} \right) \right) + \left( t_{j+1} + 1 \right)^{1/\sigma} \frac{d_j}{c_j} \left[ x \left( t_{j+1} \right) - \beta \left( A \left( t_{j+1} \right) - A \left( t_{j} \right) \right) \right] + \left( A \left( t_{j+2} \right) - A \left( t_{j+1} \right) \right)
\leq c_{j+1} x \left( t_{j+1} \right) + d_{j+1} a^{1/\sigma} \left( t_{j+1} \right) x' \left( t_{j+1} \right) \left( A \left( t_{j+2} \right) - A \left( t_{j+1} \right) \right)
\leq c_{j+1} \left[ x \left( t_{j+1}^+ \right) - \beta \left( A \left( t_{j+1} \right) - A \left( t_{j} \right) \right) \right] + d_{j+1} \left[ a \left( t_{j+1}^+ \right) \left( x' \left( t_{j+1}^+ \right) \right)^\sigma \right]^{1/\sigma} \left( A \left( t_{j+2} \right) - A \left( t_{j+1} \right) \right)
\leq c_{j+1} \left[ x \left( t_{j+1}^+ \right) - \beta \left( A \left( t_{j+1} \right) - A \left( t_{j} \right) \right) - \frac{d_{j+1}}{c_{j+1}} \beta \left( A \left( t_{j+2} \right) - A \left( t_{j+1} \right) \right) \right] .
\]
Then (6) holds for \( n = 2 \). Now we suppose that (6) holds for \( n = N \), i.e.,

\[
\begin{align*}
    x(t_{j+N}) &\leq c_{j+1}c_{j+2}\cdots c_{j+N-1} \times \left\{ x(t_{j+N}) - \beta(A(t_{j+N}) - A(t_j)) \\
    - \frac{d_{j+1}}{c_{j+1}} &\beta(A(t_{j+N}) - A(t_{j+N-1})) - \cdots - \frac{d_{j+1}d_{j+N-1}}{c_{j+1}c_{j+2}\cdots c_{j+N-1}} &\beta(A(t_{j+N}) - A(t_{j+N-1})) \right\}.
\end{align*}
\]

(9)

We shall prove that (6) holds for \( n = N+1 \). Since \( S(t) = a(t)(x'(t))^\sigma \) is monotonically decreasing in \((t_{j+N-1}, t_{j+N}]\), we have

\[
a(t)(x'(t))^\sigma \leq a \left( t_{j+N}^+ \right) \left( x' \left( t_{j+N}^+ \right) \right)^\sigma, \quad t \in (t_{j+N}, t_{j+N+1}]
\]

and

\[
x'(t) \leq \frac{a \left( t_{j+N}^+ \right) \left( x' \left( t_{j+N}^+ \right) \right)^\sigma}{a^{1/\sigma}(t)}.
\]

Integrating the above formula, in view of Condition (ii) and (4),(5),(9), we obtain

\[
x(t_{j+N+1}) \leq x(t_{j+N}^+) + \left[ a \left( t_{j+N}^+ \right) \left( x' \left( t_{j+N}^+ \right) \right)^\sigma \right]^{1/\sigma} \left( A(t_{j+N+1}) - A(t_{j+N}) \right)
\]

\[
\leq c_{j+N} x(t_{j+N}) + d_{j+N} \left[ a \left( t_{j+N} \right) \left( x' \left( t_{j+N} \right) \right)^\sigma \right]^{1/\sigma} \left( A(t_{j+N+1}) - A(t_{j+N}) \right)
\]

\[
\leq c_{j+N}c_{j+N-1} \cdots c_{j+1} \left[ x(t_{j+1}^+) - \beta(A(t_{j+1}) - A(t_j)) \right]
\]

\[
- \frac{d_{j+1}}{c_{j+1}} &\beta(A(t_{j+N}) - A(t_{j+N-1})) \right]
\]

\[
- d_{j+1}d_{j+2} \cdots d_{j+N} &\beta(A(t_{j+N+1}) - A(t_{j+N}))
\]

\[
\leq c_{j+N}c_{j+N-1} \cdots c_{j+1} \left[ x(t_{j+1}^+) - \beta(A(t_{j+1}) - A(t_j)) \right]
\]

\[
- \frac{d_{j+1}d_{j+2} \cdots d_{j+N-1}}{c_{j+1}c_{j+2} \cdots c_{j+N-1}} &\beta(A(t_{j+N}) - A(t_{j+N-1}))
\]

\[
- \frac{d_{j+1}d_{j+2} \cdots d_{j+N}}{c_{j+1}c_{j+2} \cdots c_{j+N}} &\beta(A(t_{j+N+1}) - A(t_{j+N}))
\].

Hence, (6) holds for \( n = N + 1 \). By induction, (6) holds for any natural number \( n \geq 2 \). Since \( x(t_k) \geq 0 \) \((t_k \geq T)\), one finds that (6) contradicts (3). Therefore,

\[
x' \left( t_k \right) \geq 0, \quad (t_k \geq T).
\]

By Condition (ii), we have, for any \( t_k \geq T \), \( x'(t_k^+) \geq d_k \), \( x'(t_k) \geq 0 \). Because \( S(t) \) is decreasing in \((t_{j+N-1}, t_{j+N}]\), we get, for \( t \in (t_{j+N-1}, t_{j+N}] \), \( S(t) \geq 0 \), which implies \( x'(t) \geq 0 \). The proof of this lemma is complete.

REMARK 1. In the case that \( x(t) \) is eventually negative, if (3) holds true, then \( x'(t_k^+) \leq 0 \) and \( x'(t) \leq 0 \), for \( t \in (t_{j+N-1}, t_{j+N}] \), where \( t_k \geq T \).

3. MAIN RESULTS

THEOREM 1. Assume that (3) holds and there exists a positive integer \( k_0 \) such that \( c_k^0 \geq 1 \) for \( k \geq k_0 \). If

\[
\int_{t_0}^{t_1} p(s) \, ds + \frac{1}{(d_1^1)^{\sigma}} \int_{t_1}^{t_2} p(s) \, ds + \frac{1}{(d_1^1d_2^2)^{\sigma}} \int_{t_2}^{t_3} p(s) \, ds \\
+ \cdots + \frac{1}{(d_1^1d_2^2\cdots d_n^n)^{\sigma}} \int_{t_n}^{t_{n+1}} p(s) \, ds \right) = +\infty.
\]

(10)

Then every solution of (1) is oscillatory.
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PROOF. Without loss of generality, we can assume \( k_0 = 1 \). If (1) has a nonoscillatory solution \( x(t) \), we might as well assume that \( x(t) > 0 \) \( (t \geq t_0) \). It follows from Lemma 1 that \( x'(t) \geq 0 \) for \( t \in (t_k, t_{k+1}] \), where \( k = 1, 2, \ldots \). Let

\[
w(t) = \frac{a(t) \left( x'(t) \right)^\sigma}{\varphi \left( x(t - \tau) \right)}.
\]

Then \( w(t^+_{k}) \geq 0 \) \( (k = 1, 2, \ldots) \), \( w(t) \geq 0 \) \( (t \geq t_0) \). Using Condition (i) and equation (1), we get

\[
w'(t) = -\frac{\int f(t, x(t), x(t - \tau))}{\varphi \left( x(t - \tau) \right)} - \frac{a(t) \left( x'(t) \right)^\sigma \varphi'(x(t - \tau)) x'(t - \tau)}{\varphi^2 \left( x(t - \tau) \right)} \leq -p(t),
\]

\( t \neq t_k, t_k + \tau \).

It follows from the continuity of \( a(t) \), Condition (ii), \( c_k^* \geq 1 \), and \( \varphi'(x) \geq 0 \) that

\[
w(t^+_{k}) = \frac{a \left( t^+_{k} \right) \left( x' \left( t^+_{k} \right) \right)^\sigma}{\varphi \left( x \left( t^+_{k} - \tau \right) \right)} \leq \frac{(d^+_{k})^\sigma a \left( t_k \right) \left( x' \left( t_k \right) \right)^\sigma}{\varphi \left( x \left( t_k - \tau \right) \right)} = (d^+_{k})^\sigma w(t_k)
\]

and

\[
w(t^+_{k} + \tau) = \frac{a \left( t^+_{k} + \tau \right) \left( x' \left( t^+_{k} + \tau \right) \right)^\sigma}{\varphi \left( x \left( t^+_{k} \right) \right)} \leq \frac{a \left( t_k + \tau \right) \left( x' \left( t_k + \tau \right) \right)^\sigma}{\varphi \left( c_k^* x \left( t_k \right) \right)} \leq \frac{a \left( t_k + \tau \right) \left( x' \left( t_k + \tau \right) \right)^\sigma}{\varphi \left( x \left( t_k \right) \right)} = w(t_k + \tau).
\]

Integrating (12) from \( s \) to \( t \), we have

\[
w(t) \leq w(s) - \int_s^t p(u) \, du,
\]

where \( t_0 < s < t < t_1 \). Let \( s \to t_0^+ \) and \( t \to t_1 \). In view of (13) and (14), we get

\[
w \left( t_1^+ \right) \leq \left( d^+_1 \right)^\sigma w \left( t_1 \right) \leq \left( d^+_1 \right)^\sigma \left[ w \left( t_0^+ \right) - \int_{t_0^+}^{t_1} p \left( s \right) \, ds \right] \leq \left( d^+_1 \right)^\sigma w \left( t_0^+ \right) - \left( d^+_1 \right)^\sigma \int_{t_0^+}^{t_1} p \left( s \right) \, ds.
\]

Similarly, the following inequality holds:

\[
w \left( t_2^+ \right) \leq \left( d^+_2 \right)^\sigma w \left( t_2 \right) \leq \left( d^+_2 \right)^\sigma \left[ w \left( t_1^+ + \tau \right) - \int_{t_1^+ + \tau}^{t_2} p \left( s \right) \, ds \right]
\]

\[
\leq \left( d^+_2 \right)^\sigma \left[ w \left( t_1 + \tau \right) - \int_{t_1 + \tau}^{t_2} p \left( s \right) \, ds \right] \leq \left( d^+_2 \right)^\sigma \left[ w \left( t_1^+ \right) - \int_{t_1^+}^{t_2} p \left( s \right) \, ds \right]
\]

\[
\leq \left( d^+_1 d^+_2 \right)^\sigma \left[ w \left( t_1^+ \right) - \int_{t_1}^{t_2} p \left( s \right) \, ds \right] \leq \left( d^+_1 d^+_2 \right)^\sigma \int_{t_1}^{t_2} p \left( s \right) \, ds.
\]

By induction, we have

\[
w \left( t_n^+ \right) \leq \left( d^+_1 d^+_2 \cdots d^+_n \right)^\sigma \left[ w \left( t_{n-1}^+ \right) - \int_{t_{n-1}}^{t_n} p \left( s \right) \, ds \right] - \cdots - \left( d^+_n \right)^\sigma \int_{t_{n-2}}^{t_{n-1}} p \left( s \right) \, ds - \left( d^+_n \right)^\sigma \int_{t_{n-3}}^{t_{n-2}} p \left( s \right) \, ds
\]

\[
\leq \left( d^+_1 d^+_2 \cdots d^+_n \right)^\sigma \left\{ w \left( t_1^+ \right) - \int_{s_0}^{t_1} p \left( s \right) \, ds - \frac{1}{\left( d^+_1 \right)^\sigma} \int_{t_1}^{t_2} p \left( s \right) \, ds
\]

\[
- \cdots - \frac{1}{\left( d^+_1 d^+_2 \cdots d^+_{n-2} \right)^\sigma} \int_{t_{n-2}}^{t_{n-1}} p \left( s \right) \, ds \right\}.
\]

In view of (10), (17), and \( w(t) \geq 0 \), we get a contradiction. Hence, every solution of (1) is oscillatory. The proof of Theorem 1 is complete.
THEOREM 2. Assume that (3) holds and \( \varphi(ab) \geq \varphi(a)\varphi(b) \) for any \( ab > 0 \). If

\[
\int_{t_0}^{t_1} p(s) \, ds + \frac{\varphi(c_1^*)}{(d_1^*)^\sigma} \int_{t_1}^{t_2} p(s) \, ds + \frac{\varphi(c_2^*)}{(d_1^*d_2^*)^\sigma} \int_{t_2}^{t_3} p(s) \, ds + \cdots + \frac{\varphi(c_n^*)}{(d_1^*d_2^*\cdots d_n^*)^\sigma} \int_{t_n}^{t_{n+1}} p(s) \, ds + \cdots = +\infty,
\]

then every solution of (1) is oscillatory.

PROOF. If (1) has a nonoscillatory solution \( x(t) \), without loss of generality, we can assume \( x(t) > 0 \) \( (t > t_0) \). Let \( w(t) \) be defined by (11). Then

\[
w(t^+ _k) \geq 0 \quad (k = 1, 2, \ldots), \quad w(t) \geq 0 \quad (t \geq t_0).
\]

Relation (1) and Condition (i) yield

\[
w'(t) \leq -p(t), \quad t \neq t_k, \; t_k + \tau.
\]

It is easy to see that

\[
w(t^+_k) = \frac{a(t^+_k)(x'(t^+_k))^\sigma}{\varphi(x(t^+_k - \tau))} - \frac{a(t_k)(x'(t_k))^\sigma}{\varphi(x(t_k - \tau))} = \left(\frac{d_k^*}{d_1^*d_2^*\cdots d_n^*}\right)^\sigma w(t_k)
\]

and

\[
w(t^+_k + \tau) = \frac{a(t^+_k + \tau)(x'(t^+_k + \tau))^\sigma}{\varphi(x(t^+_k))} - \frac{a(t_k + \tau)(x'(t_k + \tau))^\sigma}{\varphi(c_k^*x(t_k))} \leq \frac{1}{\varphi(c_k^*)}w(t_k + \tau).
\]

Similarly to the proof of (17) of Theorem 1, by induction, we get, for any natural number \( n \)

\[
w(t^+_k) \leq \frac{\varphi(c_1^*) \varphi(c_2^*) \cdots \varphi(c_{n-2}^*)}{(d_1^*d_2^*\cdots d_{n-2}^*)^\sigma} \left[ w(t^+_0) - \int_{t_0}^{t_1} p(s) \, ds - \frac{\varphi(c_1^*)}{(d_1^*)^\sigma} \int_{t_1}^{t_2} p(s) \, ds - \cdots - \frac{\varphi(c_{n-1}^*)}{(d_1^*d_2^*\cdots d_{n-1}^*)^\sigma} \int_{t_{n-1}}^{t_n} p(s) \, ds \right] \int_{t_{n-1}}^{t_n} p(s) \, ds.
\]

Relations (18), (21), and \( w(t^+_k) \geq 0 \) \( (k = 1, 2, \ldots) \) lead to a contradiction. Hence, every solution of (1) is oscillatory. The proof is complete.

Using Theorems 1 and 2, we can obtain some corollaries as follow.

COROLLARY 1. Assume that (3) holds and there exists a positive integer \( k_0 \) such that \( c_k^* \geq 1 \), \( d_k^* \leq 1 \) for \( k \geq k_0 \). If

\[
\int_{-\infty}^{+\infty} p(s) \, ds = +\infty,
\]

then every solution of (1) is oscillatory.

PROOF. Without loss of generality, let \( k_0 = 1 \). With \( d_1^* \leq 1 \), we know

\[
\int_{t_0}^{t_1} p(s) \, ds + \frac{1}{(d_1^*)^\sigma} \int_{t_1}^{t_2} p(s) \, ds + \frac{1}{(d_1^*d_2^*)^\sigma} \int_{t_2}^{t_3} p(s) \, ds + \cdots + \frac{1}{(d_1^*d_2^*\cdots d_n^*)^\sigma} \int_{t_n}^{t_{n+1}} p(s) \, ds \geq \int_{t_0}^{t_1} p(s) \, ds + \int_{t_1}^{t_2} p(s) \, ds + \cdots + \int_{t_{n-1}}^{t_n} p(s) \, ds = \int_{t_n}^{t_{n+1}} p(s) \, ds.
\]

Let \( n \to \infty \). It follows from (22) and (23) that

\[
\int_{t_0}^{t_1} p(s) \, ds + \frac{1}{(d_1^*)^\sigma} \int_{t_1}^{t_2} p(s) \, ds + \frac{1}{(d_1^*d_2^*)^\sigma} \int_{t_2}^{t_3} p(s) \, ds + \cdots + \frac{1}{(d_1^*d_2^*\cdots d_n^*)^\sigma} \int_{t_n}^{t_{n+1}} p(s) \, ds = +\infty.
\]

By Theorem 1, we get that all solutions of (1) are oscillatory.
COROLLARY 2. Assume that (3) holds and there exists a positive integer \( k_0 \) and a constant \( \alpha > 0 \) such that
\[
\left( \frac{c_k}{d_k^\sigma} \right)^\alpha \geq \left( \frac{t_{k+1}}{t_k} \right)^\alpha, \quad \text{for } k \geq k_0
\] (24)
and
\[
\int_{t}^{+\infty} t^\alpha p(t) \, dt = +\infty. \tag{25}
\]
Then every solution of (1) is oscillatory.

PROOF. Without loss of generality, let \( k_0 = 1 \). Relation (24) yields
\[
\begin{align*}
\int_{t_0}^{t_1} p(s) \, ds &+ \frac{1}{(d_1^1)^\sigma} \int_{t_1}^{t_2} p(s) \, ds + \frac{1}{(d_1^1 d_2^1)^\sigma} \int_{t_2}^{t_3} p(s) \, ds \\
&+ \cdots + \frac{1}{(d_1^1 d_2^1 \cdots d_n^1)^\sigma} \int_{t_n}^{t_{n+1}} p(s) \, ds \\
&\geq \frac{1}{(d_1^1)^\sigma} \int_{t_1}^{t_2} p(s) \, ds + \frac{1}{(d_1^1 d_2^1)^\sigma} \int_{t_2}^{t_3} p(s) \, ds + \cdots + \frac{1}{(d_1^1 d_2^1 \cdots d_n^1)^\sigma} \int_{t_n}^{t_{n+1}} p(s) \, ds
\end{align*}
\] (26)
Let \( n \to \infty \), it follows from (25) and (26) that
\[
\int_{t_0}^{t_1} p(s) \, ds + \frac{1}{(d_1^1)^\sigma} \int_{t_1}^{t_2} p(s) \, ds + \frac{1}{(d_1^1 d_2^1)^\sigma} \int_{t_2}^{t_3} p(s) \, ds \\
+ \cdots + \frac{1}{(d_1^1 d_2^1 \cdots d_n^1)^\sigma} \int_{t_n}^{t_{n+1}} p(s) \, ds + \cdots = +\infty.
\]
According to Theorem 1, we can conclude that equation (1) is oscillatory.

COROLLARY 3. Assume that (3) holds and \( \varphi(ab) \geq \varphi(a)\varphi(b) \) for any \( ab > 0 \). Furthermore, suppose that there exist a positive integer \( k_0 \) and a constant \( \alpha > 0 \) such that
\[
\frac{\varphi(c_k^\alpha)}{(d_k^\sigma)^\alpha} \geq \left( \frac{t_{k+1}}{t_k} \right)^\alpha, \quad \text{for } k \geq k_0
\] (27)
and
\[
\int_{t}^{+\infty} t^\alpha p(t) \, dt = +\infty. \tag{28}
\]
Then every solution of (1) is oscillatory.

Corollary 3 can be deduced from Theorem 2. Its proof is similar to that of Corollary 2 and is omitted.

4. EXAMPLES

EXAMPLE 1. Consider the impulsive delay differential equation
\[
\begin{align*}
x'' + \frac{1}{4t^2} x \left( t - \frac{1}{5} \right) &= 0, \quad t \neq k, \quad k = 1, 2, 3, \ldots, \\
x(k^+) &= \left( \frac{k+1}{k} \right) x(k), \quad x'(k^+) = x'(k), \quad k = 1, 2, 3, \ldots
\end{align*}
\] (29)
where \( c_k = c_k^* = (k + 1)/k \), \( d_k = d_k^* = 1 \), \( t_0 = 2/5 \), \( A(t) = \int_{t/5}^t \frac{ds}{a(s)} = t - 2/5 \), \( p(t) = 1/4t^2 \), \( t_k = k \), and \( \varphi(x) = x \). Obviously, Conditions (i) and (ii) are satisfied and

\[
A(t_1) - A(t_0) + \frac{d_1}{c_1} (A(t_2) - A(t_1)) + \frac{d_1 d_2}{c_1 c_2} (A(t_3) - A(t_2)) + \cdots + \frac{d_1 d_2 \cdots d_n}{c_1 c_2 \cdots c_n} (A(t_{n+1}) - A(t_n)) + \cdots = 3/5 + 1/2 + 1/3 + \cdots + 1/n + \cdots = +\infty.
\]

Let \( k_0 = 1, \alpha = 1 \). Then

\[
\varphi(c_k^*) = c_k^* = \frac{k + 1}{k} = \frac{t_{k+1}}{t_k}
\]

and

\[
\int_{t_{k+1}}^{+\infty} t^\alpha p(t) dt = \int_{t_{k+1}}^{+\infty} tp(t) dt = \int_{t_{k+1}}^{+\infty} t \times \frac{1}{2t^2} dt = +\infty.
\]

By Corollary 2, we know that every solution of equation (29) is oscillatory. But, by [12], the delay differential equation \( x'' + (1/2t^2)x(t - 1/5) = 0 \), when no impulse is added to it, is nonoscillatory.

**EXAMPLE 2.** Consider the superlinear impulsive equation

\[
x'' + \frac{1}{t^3} x^{2n-1} \left( t - \frac{1}{3} \right) = 0, \quad t \neq k, \quad k = 1, 2, \ldots,
\]

where \( n \geq 2 \) is a natural number, and \( c_k = c_k^* = (k + 1)/k, d_k = d_k^* = 1, t_0 = 2/3, A(t) = \int_{t/3}^t 1/(a^*(s)) ds = t - 2/3, p(t) = 1/t^3, t_k = k \), and \( \varphi(x) = x^{2n-1} \). Let \( k_0 = 1, \alpha = 3 \). Obviously, Conditions (i), (ii), and (3) are satisfied and

\[
\varphi(c_k^*) = c_k^* = \left( \frac{k + 1}{k} \right)^{2n-1} = \left( \frac{t_{k+1}}{t_k} \right)^\alpha
\]

and

\[
\int_{t_{k+1}}^{+\infty} t^\alpha p(t) dt = \int_{t_{k+1}}^{+\infty} t^3 p(t) dt = \int_{t_{k+1}}^{+\infty} t^3 \times \frac{1}{t^3} dt = +\infty.
\]

By Corollary 2, we know that every solution of equation (30) is oscillatory. But the delay differential equation \( x'' + (1/t^3)x^{2n-1}(t - 1/3) = 0 \), by [11], is nonoscillatory when no impulse is added to it.

**EXAMPLE 3.** Consider the sublinear impulsive equation

\[
x'' + \frac{1}{t^2} x^{1/3} \left( t - \frac{1}{12} \right) = 0, \quad t \neq k, \quad k = 1, 2, \ldots,
\]

where \( c_k = c_k^* = 1, d_k = d_k^* = k/(k + 1), A(t) = \int_{t/12}^t \frac{ds}{a^*(s)} = t - 1/2, t_0 = 1/2, p(t) = 1/t^2, t_k = k \), and \( \varphi(x) = x^{1/3} \). Let \( k_0 = 1, \alpha = 1 \). Obviously, Conditions (i), (ii), and (3) are satisfied and \( \varphi(ab) = \varphi(a)\varphi(b) \) for any \( ab > 0 \) and

\[
\varphi(c_k^*) = c_k^* = \frac{k + 1}{k} = \frac{t_{k+1}}{t_k}
\]

and

\[
\int_{t_{k+1}}^{+\infty} t^\alpha p(t) dt = \int_{t_{k+1}}^{+\infty} (t^3 p(t) dt = \int_{t_{k+1}}^{+\infty} t \times \frac{1}{t^2} dt = +\infty.
\]

By Corollary 3, we know that every solution of equation (31) is oscillatory. But the delay differential equation \( x'' + (1/t^2)x^{1/3}(t - 1/12) = 0 \), by [11], is nonoscillatory when no impulse is added to it.
REFERENCES