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Reducing belief revision to circumscription (and vice versa) ★

Paolo Liberatore [1], Marco Schaerf *

Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", via Salaria 113, I-00198 Roma, Italy

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Abstract

Nonmonotonic formalisms and belief revision operators have been introduced as useful tools to describe and reason about evolving scenarios. Both approaches have been proven effective in a number of different situations. However, little is known about their relationship. Previous work by Winslett has shown some correlations between a specific operator and circumscription. In this paper we greatly extend Winslett's work by establishing new relations between circumscription and a large number of belief revision operators. This highlights similarities and differences between these formalisms. Furthermore, these connections provide us with the possibility of importing results in one field into the other one. © 1997 Elsevier Science B.V.

Keywords: Knowledge representation; Circumscription; Belief revision; Computational complexity

1. Introduction

During the last years, many formalisms have been proposed in the AI literature to model commonsense reasoning. Particular emphasis has been put in the formal modeling of a distinct feature of commonsense reasoning, that is, its nonmonotonic nature. The AI goal of providing a logic model of human agents' capability of reasoning in the presence of incomplete or contradictory information has proven to be a very hard one. Nevertheless, many important formalisms have been put forward in the literature.

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* Corresponding author. E-mail: schaerf@dis.uniroma1.it. Work partially supported by ASI (Italian Space Agency).

[1] E-mail: liberatore@dis.uniroma1.it.

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Two main approaches have been proposed to handle the nonmonotonic aspects of commonsense reasoning. The first one deals with this problem, by defining a new logic equipped with a nonmonotonic consequence operator. Important examples of this approach are *default logic* proposed in [32] and *circumscription* introduced in [29]. The second one relies on preserving a classical (monotonic) inference operator, but introduces a *revision* operator that accommodates a new piece of information into an existing body of knowledge. Specific revision operators have been introduced, among the others, in [10, 18]. A general framework for revision has been proposed by Alchourron, Gärdenfors and Makinson in [1, 17]. A close variant of revision is *update*. The general framework for update has been studied in [23,24] and specific operators have been proposed in [15,35].

As pointed out by Winslett in [35], the large variety of candidate semantics for belief revision and update varies widely in motivations, goals and area of application. It is generally believed that there is no "best" method and that each one is suited for a particular domain of application.

In this paper we investigate the relationship between circumscription and many operators for belief revision and update. A first study of these relations has been done in [34], where she relates her operator to circumscription. We expand her results showing similar connections between several other belief revision operators and circumscription. To this end, we also introduce a variant of circumscription based on cardinality, rather than set-containment.

The established correlations highlight the relations between the two fields. Moreover, as side benefits, they provide us with the opportunity to import results in one field into the other one. A practical result is the possibility of directly using algorithms developed for circumscription also for belief revision. In the last years, many algorithms for circumscription have appeared in the literature (see, for example, [19,30,31]), while, to the best of our knowledge, only Winslett in [35] has proposed an algorithm for belief revision. Using our reductions, it is possible to reduce a reasoning problem of belief revision into one in circumscription, thus taking advantage of the large number of algorithms and reasoning systems already developed.

In this paper we focus our attention on propositional languages, since some of the belief revision operators have only been defined in this setting. However, in Section 9 we briefly explain how and when our results also apply to full first-order circumscription and belief revision.

The paper is organized as follows: In Section 2 we recall some key definitions and results for belief revision and circumscription, introduce a variant of circumscription (NCIRC), define the kind of relations we want to establish and explain the notation used throughout the following sections. In Sections 3, 4, 5 and 6 we show the main relations between the various kinds of revision operators and circumscription, while in Section 7 we show relations and reductions between the various operators. In Section 8 we focus on syntactically-restricted knowledge bases. Section 9 discusses possible applications of our results with particular attention to the computational complexity analysis. While in Section 10 we draw some conclusions. Finally, in the appendices we have the proofs of most theorems and a brief description of some complexity classes used for some of the results.

2. Preliminaries

In this section we (very briefly) present the background and terminology needed to understand the results presented later in the paper. For the sake of simplicity, throughout this paper we restrict our attention to a (finite) propositional language. In Section 9 we briefly discuss how these results also apply to full first-order languages.

The *alphabet* of a propositional formula α is the set of all propositional atoms occurring in it and is denoted by $V(\alpha)$. Formulae are built over a finite alphabet of propositional letters using the usual connectives $\neg$ (not), $\vee$ (or) and $\wedge$ (and). Additional connectives are used as shorthands, $\alpha \rightarrow \beta$ denotes $\neg\alpha \vee \beta$, $\alpha \equiv \beta$ is a shorthand for $(\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$.

An *interpretation* of a formula is a truth assignment to the atoms of its alphabet. A *model* M of a formula F is an interpretation that satisfies F (written $M \models F$). Interpretations and models of propositional formulae will be denoted as sets of atoms (those which are mapped into 1). A theory T is a set of formulae. An interpretation is a model of a theory if it is a model of every formula of the theory. Given a theory T and a formula F we say that T entails F, written $T \models F$, if F is true in every model of T. Given a propositional formula or a theory T, we denote with $\mathcal{M}(T)$ the set of its models. We say that T is consistent, written $T \not\models \bot$, if $\mathcal{M}(T)$ is non-empty.

2.1. Belief revision and update

Belief revision is concerned with the modeling of accommodating a new piece of information (the revising formula) into an existing body of knowledge (the knowledge base), where the two might contradict each other. A slightly different perspective is taken by knowledge update. An analysis of the differences between belief revision and update is out of the scope of this paper, for an interesting discussion we refer the reader to the work [24]. We assume that both the revising formula and the knowledge base can be either a single formula or a theory.

In the literature, the first formal studies on the principles of belief revision have been presented by Alchourron, Gärdenfors and Makinson in [1,17]. In these papers they present a set of postulates that all revision operators should satisfy. These postulates, known as the AGM postulates, assume that the revision operator applies to a deductively-closed set of formulae. In order to make the presentation more homogeneous, we present the reformulation of these postulates where the revision operator applies to propositional formulae. More precisely, we denote with K the knowledge base (that is the existing logical theory), with A the revising formula (that is the new information) and with $*$ the revision operator. This formulation has been presented by Katsuno and Mendelzon in [25], where they prove this set of postulates equivalent to the original one.

Thus, the AGM postulates for (finite) propositional knowledge bases are:

AGM1. $K * A$ implies A.

AGM2. If $K \wedge A$ is satisfiable then $K * A \equiv K \wedge A$.

AGM3. If A is satisfiable then $K * A$ is also satisfiable.

AGM4. If $K_1 \equiv K_2$ and $A_1 \equiv A_2$ then $K_1 * A_1 \equiv K_2 * A_2$.

AGM5. $(K * A) \wedge B$ implies $K * (A \wedge B)$.

AGM6. If $(K * A) \wedge B$ is satisfiable then $K * (A \wedge B)$ implies $(K * A) \wedge B$.

The intuitive meaning of the postulates is simple to understand. AGM1 states that the new information A is always retained in the revision. AGM2 postulates that, if no inconsistency arises, A is simply added to K. AGM3 states that inconsistency cannot be introduced unless A is inconsistent. Furthermore, because of AGM4 the revision operator obeys the Principle of Irrelevance of Syntax and postulates AGM5 and AGM6 impose constraints on the behavior of revision in the presence of conjunctions.

Katsuno and Mendelzon in [25] have shown that, to any revision operator satisfying AGM1–AGM6, corresponds a family of reflexive, transitive and total orderings over the set of interpretations, one for each formula K. Given a revision $*$ and its corresponding family of orderings, the following relation holds.

$$\mathcal{M}(K * A) = \min(\mathcal{M}(A), \leqslant_K) \tag{1}$$

Any given ordering $\leqslant_K$ has the so-called property of *faithfulness*, that can be summarized as follows:

(i) If $I \in \mathcal{M}(K)$ then $I \leqslant_K J$ for any interpretation J.

(ii) If $I \in \mathcal{M}(K)$ and $J \notin \mathcal{M}(K)$ then $J \leqslant_K I$ does not hold.

Roughly speaking, the models of $\mathcal{M}(K)$ are exactly the minimal elements of $\leqslant_K$, and the other interpretations are ordered according to their distance from models of K: given two models I and J, it holds $I \leqslant_K J$ if and only if I is considered more plausible to an agent believing K. In this sense, achieving the principle of minimal change, Eq. (1) means that the result of a revision is constituted by the models of A that are closer to K.

We now recall the different approaches to revision and update, classifying them into formula-based and model-based ones. A more thorough exposition can be found in [13]. We use the following conventions: the expression $\mathrm{card}(S)$ denotes the cardinality of a set S, and symmetric difference between two sets S_1, S_2 is denoted by $S_1 \triangle S_2$. If S is a set of sets, $\cap S$ denotes the set formed by intersecting all sets of S, and analogously $\cup S$ for union; $\min_{\subseteq} S$ denotes the subset of S containing only the minimal (with respect to set inclusion) sets in S, while $\max_{\subseteq} S$ denotes its maximal sets. Moreover, we use the symbol $\subset$ to denote *strict* containment, i.e. $S_1 \subset S_2$ if and only $S_1 \subseteq S_2$ and $\exists a \in S_2$ such that $a \notin S_1$.

Formula-based approaches operate on the formulae syntactically appearing in the knowledge base K. Let $C(K, A)$ be the set of the subsets of K that are consistent with the revising formula A:

$$C(K, A) = \{K' \subseteq K \mid K' \cup \{A\} \not\models \perp\}$$

and let $W(K,A)$ be the set of the maximal sets of $C(K,A)$:

$$W(K,A) = \max_{\subseteq} C(K,A).$$

The set $W(K,A)$ contains all the plausible subsets of K that we may retain when inserting A.

SBR. In [16,18], the revised knowledge base is defined as a set of theories:

$$K *_{\mathrm{SBR}} A \doteq \{K' \cup \{A\} \mid K' \in W(K,A)\}.$$

That is, the result of revising K is the set of all maximal subsets of K consistent with A, plus A. Logical consequence in the revised knowledge base is defined as logical consequence in each of the theories, i.e. $K *_{\mathrm{SBR}} A \models Q$ iff for all $K' \in W(K,A)$, $K' \cup \{A\} \models Q$. In other words, Fagin et al. [16] and Ginsberg [18] consider all sets in $W(K,A)$ equally plausible and inference is defined skeptically, i.e. Q must be a consequence of each set. For this reason, we term this method as *Skeptical Belief Revision* (SBR). Note that $K *_{\mathrm{SBR}} A$ can be equivalently rewritten as $\bigvee_{K' \in W(K,A)} K'$.

Note that formula-based approaches are sensitive to the syntactic form of the theory. That is, the revision with the same formula A of two logically equivalent theories K_1 and K_2, may yield different results, depending on the syntactic form of K_1 and K_2. We illustrate this fact through an example.

Example 1. Consider $K_1 = \{a, b\}$, $K_2 = \{a, a \rightarrow b\}$ and $A = \neg b$. Clearly, K_1 is equivalent to K_2. The only maximal subset of K_1 consistent with A is $\{a\}$, while there are two maximal consistent subsets of K_2, that are $\{a\}$ and $\{a \rightarrow b\}$.

Thus, $K_1 *_{\mathrm{SBR}} A = \{a, \neg b\}$ while $K_2 *_{\mathrm{SBR}} A = \{(a \wedge \neg b) \vee ((a \rightarrow b) \wedge \neg b)\}$ that is equivalent to $\{\neg b\}$.

Model-based approaches instead operate by selecting the models of A on the basis of some notion of proximity to the models of K. Model-based approaches assume K to be a single formula, if K is a set of formulae it is implicitly interpreted as the conjunction of all the elements. Many notions of proximity have been defined in the literature. We distinguish them between pointwise proximity and global proximity.

We first recall approaches in which proximity between models of A and models of K is computed pointwise with respect to each model of K. That is, they select models of K one-by-one and, for each one, choose the closest model of A. These approaches are considered as more suitable for knowledge update [24]. Let M be a model, we define $\mu(M,A)$ as the set containing the minimal differences (with respect to set inclusion) between each model of A and the given M; more formally:

$$\mu(M,A) \doteq \min_{\subseteq}\{M \triangle N \mid N \in \mathcal{M}(A)\}.$$

We also use the notation $k_{M,A}$ to denote the minimum cardinality of sets in $\mu(M,A)$, i.e. $k_{M,A} = \min(n \mid n = |S|,\ S \in \mu(M,A))$.

Winslett. In [35] the models of the updated knowledge base are defined as

$$\mathcal{M}(K *_{\mathrm{W}} A) \doteq \{N \in \mathcal{M}(A) \mid \exists M \in \mathcal{M}(K)\colon M \triangle N \in \mu(M, A)\}.$$

In other words, for each model of K he chooses the closest (with respect to set-containment) models of A.

Borgida. This operator $*_{\mathrm{B}}$, defined in [2], coincides with Winslett's one, except in the case when A is consistent with K, in which case Borgida's revised theory is simply $K \wedge A$.

Forbus. This approach [15] takes into account cardinality: The models of Forbus' updated theory are

$$\mathcal{M}(K *_{\mathrm{F}} A) \doteq \{N \in \mathcal{M}(A) \mid \exists M \in \mathcal{M}(K)\colon \mathrm{card}(M \triangle N) = k_{M,A}\}.$$

Note that by means of cardinality, Forbus can compare (and discard) models which are incomparable in Winslett's approach.

We now recall approaches where proximity between models of A and models of K is defined considering globally *all* models of K. In other words, these approaches consider at the same time all pairs of models $M \in \mathcal{M}(K)$ and $N \in \mathcal{M}(A)$ and find all the closest pairs. Let $\delta(K, A)$ denote the set of minimal differences between a model of A and one of K. More precisely:

$$\delta(K, A) \doteq \min_{\subseteq} \bigcup_{M \in \mathcal{M}(K)} \mu(M, A).$$

Similarly to the local approach, we use the notation $k_{K,A}$ to denote the minimum cardinality of sets in $\delta(K, A)$, i.e. $k_{K,A} = \min(n \mid n = |S|,\ S \in \delta(K, A))$.

Satoh. In [33], the models of the revised knowledge base are defined as

$$\mathcal{M}(K *_{\mathrm{S}} A) \doteq \{N \in \mathcal{M}(A) \mid \exists M \in \mathcal{M}(K)\colon N \triangle M \in \delta(K, A)\}.$$

That is, Satoh selects all closest pairs (by set-containment of the difference set) and then projects on the models of A.

Dalal. This approach is similar to Forbus', but global. In [10] the models of a revised theory are defined as

$$\mathcal{M}(K *_{\mathrm{D}} A) \doteq \{N \in \mathcal{M}(A) \mid \exists M \in \mathcal{M}(K)\colon \mathrm{card}(N \triangle M) = k_{K,A}\}.$$

That is, Dalal selects all closest pairs (by cardinality of the difference set) and then projects on the models of A.

Example 2. Let K and A be defined as

$$K = a \wedge b \wedge c,$$
$$A = (\neg a \wedge \neg b \wedge \neg d) \vee (\neg c \wedge b \wedge (a \equiv \neg d)).$$

Note that K has only two models, which are

$$J_1 = \{a, b, c, d\},$$

$$J_2 = \{a, b, c\},$$

while A has four models,

$$I_1 = \{a, b\},$$

$$I_2 = \{c\},$$

$$I_3 = \{b, d\},$$

$$I_4 = \emptyset.$$

The set differences between each model of K and each model of A are:

$I \triangle J$	$I_1 = \{a, b\}$	$I_2 = \{c\}$	$I_3 = \{b, d\}$	$I_4 = \emptyset$
$J_1 = \{a, b, c, d\}$	$\{c, d\}$	$\{a, b, d\}$	$\{a, c\}$	$\{a, b, c, d\}$
$J_2 = \{a, b, c\}$	$\{c\}$	$\{a, b\}$	$\{a, c, d\}$	$\{a, b, c\}$

Hence, the minimal differences between J_1 and models of A are $\mu(J_1, A) = \{\{c, d\}, \{a, b, d\}, \{a, c\}\}$; The minimal differences between J_2 and models of A are $\mu(J_2, A) = \{\{c\}, \{a, b\}\}$.

The cardinalities of set differences between each model of K and each model of A are:

$\mathrm{card}(I \triangle J)$	$I_1 = \{a, b\}$	$I_2 = \{c\}$	$I_3 = \{b, d\}$	$I_4 = \emptyset$
$J_1 = \{a, b, c, d\}$	2	3	2	4
$J_2 = \{a, b, c\}$	1	2	3	3

Winslett. The minimal differences in $\mu(J_1, A)$ correspond to the models I_1, I_2, I_3 of A, while those in $\mu(J_2, A)$ correspond to the models I_1, I_2 of A. Therefore, the models of $K *_{\mathrm{W}} A$ are $\{I_1, I_2, I_3\} \cup \{I_1, I_2\} = \{I_1, I_2, I_3\}$. The same result holds for Borgida's revision, since K and A are inconsistent.

Forbus. From the table with cardinalities: the minimal cardinality of differences between J_1 and each model of A is $k_{J_1,A} = 2$, corresponding to models I_1 and I_3; while $k_{J_2,A} = 1$, corresponding to I_1. Therefore, $K *_{\mathrm{F}} A$ has models $\{I_1, I_3\} \cup \{I_1\} = \{I_1, I_3\}$.

We now turn to global proximity approaches, where also entries in different rows of the above tables are compared for minimality.

Satoh. The minimal differences between any model of K and any model of A are $\delta(K, A) = \{\{c\}, \{a, b\}\}$. These minimal differences correspond to models I_1 and I_2 of A, which, therefore, are the models of $K *_S A$.

Dalal. The minimum cardinality of all set differences is $k_{K,A} = 1$, corresponding to I_1. As a result, $K *_D A$ selects the model I_1 only.

2.2. Circumscription

Circumscription has been originally introduced in [29]. Further extensions have been proposed by several authors. Here we stick to the semantic formulation of circumscription and restrict our interest to a propositional language. Following [26], we define:

Definition 3. Let T be a propositional formula, $V(T) = \{x_1, \ldots, x_n\}$ its alphabet, P, Q and Z disjoint sets of letters partitioning $V(T)$ (i.e. $P \cup Q \cup Z = V(T)$) and $M \in \mathcal{M}(T)$. M is called a (P, Z)*-minimal* model of T if there is no model N of T such that $N \cap Q = M \cap Q$ and $(N \cap P) \subset (M \cap P)$.

Definition 4. The circumscription of T with respect to the three sets of letters P, Q and Z, denoted as $\mathrm{CIRC}(T; P, Q, Z)$, is an higher order formula whose set of models is the set of all (P, Z)-minimal models of T, i.e. $M \models \mathrm{CIRC}(T; P, Q, Z)$ iff M is a (P, Z)-minimal model of T.

Informally, P is the set of letters we want to minimize, Q is the set of fixed letters, while letters in Z are allowed to vary. Given two interpretations M and N, we use the notation $M \leqslant_{(P,Z)} N$ to state that M is "(P, Z)-smaller" than or equal to N, that is $(M \cap Q) = (N \cap Q)$ and $(M \cap P) \subseteq (N \cap P)$. When we write $M <_{(P,Z)} N$ we mean that M is *strictly* "(P, Z)-smaller" than N, that is $(M \cap Q) = (N \cap Q)$ and $(M \cap P) \subset (N \cap P)$.

2.3. Cardinality-based circumscription

The minimality criterion of circumscription is based on set-containment. We now introduce, for propositional languages, a version of circumscription based on cardinality.

Definition 5. Let T be a propositional formula, $V(T) = \{x_1, \ldots, x_n\}$ its alphabet, P, Q and Z disjoint sets of letters partitioning $V(T)$ (i.e. $P \cup Q \cup Z = V(T)$) and $M \in \mathcal{M}(T)$. M is called a (P, Z)*-cardinality-minimal* model of T if there is no model N of T such that $N \cap Q = M \cap Q$ and $|N \cap P| < |M \cap P|$.

Definition 6. The cardinality-based circumscription of T with respect to the three sets of letters P, Q and Z, denoted as $\mathrm{NCIRC}(T; P, Q, Z)$, is an higher order formula whose set of models is the set of all (P, Z)-cardinality-minimal models of T, i.e. $M \models \mathrm{NCIRC}(T; P, Q, Z)$ iff M is a (P, Z)-cardinality-minimal model of T.

In other words, I am preferring models with the least number of true letters of the set P, rather than models with a least set of true letters. Given two interpretations M and N, we use the notation $M \preceq_{(P,Z)} N$ to state that M is "(P,Z)-cardinality-smaller" than or equal to N, that is $(M \cap Q) = (N \cap Q)$ and $|(M \cap P)| \leqslant |(N \cap P)|$. Again, $M \prec_{(P,Z)} N$ denotes strict ordering. In order to better clarify the difference between CIRC and NCIRC we provide an example:

Example 7. Let T be defined as

$$T = (a \vee \neg b) \wedge (a \vee c) \wedge (b \vee c) \wedge (\neg a \vee b \vee c).$$

Note that T has four models, which are

$$M_1 = \{a, b\},$$
$$M_2 = \{c\},$$
$$M_3 = \{a, c\},$$
$$M_4 = \{a, b, c\}.$$

If we minimize all letters ($P = \{a, b, c, d\}$, $Q = Z = \emptyset$), the (P,Z)-minimal models are M_1 and M_2. In fact, M_3 and M_4 are not minimal since $M_2 <_{(P,Z)} M_3 <_{(P,Z)} M_4$. On the other hand, there is only one (P,Z)-cardinality minimal model, since $M_2 \prec_{(P,Z)} M_1$. In fact, $M_2 \cap P$ has cardinality 1, which is strictly smaller than the cardinality of $M_1 \cap P$.

Cardinality-based circumscription is a natural variant of circumscription well-suited for all the applications where cardinality minimization is adopted as a preference criterion. One such domain is diagnosis, where cardinality minimization is often adopted as the choice criterion.

2.4. *Types of reductions*

In this paper we investigate whether circumscription and belief revision can be translated one into the other. To this end, we take into account various forms of reductions. Formally, a reduction from circumscription into belief revision is a pair of functions f_1 and f_2 that take as input a four-tuple T, P, Q, Z and produce two formulae $K = f_1(T, P, Q, Z)$ and $A = f_2(T, P, Q, Z)$. Clearly, we want that the reduction preserves the semantic content of the original knowledge base. Given a circumscriptive theory $\mathrm{CIRC}(T; P, Q, Z)$ and a revision operator $*$ we want that $K = f_1(T, P, Q, Z)$ and $A = f_2(T, P, Q, Z)$ are such that the following relation holds,

$$\{\gamma \mid \mathrm{CIRC}(T; P, Q, Z) \models \gamma\} = \{\gamma \mid K * A \models \gamma\}, \tag{2}$$

γ being any formula in which only symbols of T occur. We call this property *query-equivalence*, and we say that if $K * A$ satisfies the above criterion then the result of the reduction is *query-equivalent* to $\mathrm{CIRC}(T; P, Q, Z)$. Symmetrically, if given K and A we find a T, P, Q and Z such that the relation (2) holds (where γ is a formula using only symbols of K and A), we say that $\mathrm{CIRC}(T; P, Q, Z)$ is query-equivalent to $K * A$.

A tighter equivalence criterion we might look for is a form of equivalence characterized by the following requirement:

$$\mathrm{CIRC}(T; P, Q, Z) \equiv K * A. \tag{3}$$

We call this property *logical equivalence*, and we say that if a pair of formulae K and A is such that $K*A$ satisfies the above criterion it is *logically equivalent* to $\mathrm{CIRC}(T; P, Q, Z)$. Notice that if $K*A$ satisfies logical equivalence (3) it satisfies query-equivalence (2) as well, but not the other way around. Basically, query-equivalence (2) gives the possibility of introducing new propositional letters. This has definitely an impact on the possibility of translating circumscription into belief revision, as we will show. Symmetrically, if given K and A we find a T, P, Q and Z such that the relation (3) holds (where γ is a formula using only symbols of K and A), we say that $\mathrm{CIRC}(T; P, Q, Z)$ is logically equivalent to $K * A$.

The above properties do not take into account the computational cost of the reductions. This is far too general, in fact, unrestricted reductions are of little practical interest and, furthermore since both circumscription and the belief revision operators can represent all boolean functions, unrestricted reductions always exist. For this reason, we consider two restrictions on the cost of reductions: if the reduction can be computed in polynomial time (i.e. both f_1 and f_2 can be computed in polynomial time), we call it a *poly-time reduction*. On the other hand, if the size of the result of the reduction is polynomial in the size of the inputs (i.e. the result of both f_1 and f_2 has size polynomial), we call it a *poly-size reduction*. Note that poly-size reductions are more general than poly-time ones. In fact, all poly-time reductions are also poly-size, while the contrary does not necessarily hold. While poly-time reductions are of obvious interest and are the most widely used form of reduction used in theoretical computer science (see [20]), in this paper we also take into account reductions that cannot be accomplished in polynomial time, but only increase the size by a polynomial factor. Another important property of reductions is *modularity*. Adapting the definition of Imielinski in [21] to our setting, we say that a reduction from circumscription into belief revision is modular if, given T, P, Q, Z K and A such that $\mathrm{CIRC}(T; P, Q, Z)$ is logically (query-) equivalent to $K * A$, and a new formula γ, $\mathrm{CIRC}(T \cup \gamma; P, Q, Z)$ is logically equivalent to $K * (A \cup f(\gamma))$. In other words, the reduction from (T, P, Q, Z) into (K, A) is modular if adding a new formula to T, does not require to recompute K and A from scratch, but only adds on top of them.

2.5. Useful previous results

There a number of results that are used throughout the paper. These include known reductions between various forms of circumscription as well as computational results on both circumscription and belief revision.

The first important result is shown by de Kleer and Konolige in [12] where they prove the following corollary: [2]

[2] Adapted to a propositional language and rephrased using our terminology.

Corollary 8. *Let Q' be a new set of letters one-to-one with letters in Q. Then*

$$\mathrm{CIRC}(T \wedge (Q \equiv \neg Q'); P \cup Q \cup Q', \emptyset, Z)$$

is query-equivalent to $\mathrm{CIRC}(T; P, Q, Z)$.

This result has been further refined by Cadoli, Eiter and Gottlob who in [8] show how to eliminate varying predicates from a circumscription. However, their method allows to rewrite a circumscriptive theory *and a query* into a new circumscriptive theory without varying letters. It does not provide any straightforward way to rewrite a circumscriptive theory with varying letters into one without varying letters, independently of the queries that will be posed.

Note that the above corollary also holds for NCIRC. In fact we have:

Corollary 9. *Let Q' be a new set of letters one-to-one with letters in Q. Then*

$$\mathrm{NCIRC}(T \wedge (Q \equiv \neg Q'); P \cup Q \cup Q', \emptyset, Z)$$

is query-equivalent to $\mathrm{NCIRC}(T; P, Q, Z)$.

The issue of reducing belief revision into circumscription has been analyzed by Winslett in [34, 35], where she shows a reduction from her operator for belief revision into circumscription. The reduction we propose in Section 4 is slightly simpler than the one she presented, but is less general, since it only applies in the propositional case. Relations between circumscription and belief revision have also been pointed out by Satoh in [33].

In the paper we will also take advantage of the results on the complexity of inference and model-checking. The complexity of inference for circumscription has been studied by Eiter and Gottlob in [14] where they show that inference for circumscription is a Π_2^p-complete problem. Cadoli in [3] has shown that deciding whether an interpretation is a (P, Z)-minimal model of a theory is a coNP-complete problem.

The complexity of deciding $K * A \models Q$ (where $*$ is one of $\{*_{\mathrm{SBR}}, *_{\mathrm{W}}, *_{\mathrm{B}}, *_{\mathrm{F}}, *_{\mathrm{S}}, *_{\mathrm{D}}\}$, K, A and Q are the input) was studied in [13]: in Dalal's approach, the problem is $\mathrm{P}^{\mathrm{NP}[O(\log n)]}$-complete, while in all other approaches it is Π_2^p-complete. While the complexity of model checking, i.e. deciding $M \models K * A$ (where $*$ is one of $\{*_{\mathrm{SBR}}, *_{\mathrm{W}}, *_{\mathrm{B}}, *_{\mathrm{F}}, *_{\mathrm{S}}, *_{\mathrm{D}}\}$, K, A and M are the input) was studied in [28].

In order to show that, in some cases, there is no poly-size reduction we use the results on compactness of representations proved by Cadoli, Donini, Silvestri and the present authors in [4–7], where it is analyzed the relative compactness of various Knowledge Representation formalism to represent knowledge. A brief presentation of the technical tools used in those proofs is in the Appendix.

2.6. Notations

In order to make formulae more compact and easier to understand, we introduce a number of notations that we use in the rest of the paper.

In the paper we frequently use the notion of substitution of letters in a formula. The notation $F[x/y]$ denotes the formula F where every occurrence of the letter x is replaced by the formula y. This notation is generalized to ordered sets: $F[X/Y]$ denotes the formula F where all occurrences of letters in X are replaced by the corresponding elements in Y, where X is an ordered set of letters (in general, $X \subseteq V(F)$) and Y is an ordered set of formulae with the same cardinality. That is, $F[X/Y] = F[x_1/y_1, \ldots, x_k/y_k]$. For example, let $T = (x_1 \wedge (\neg x_3 \vee x_2))$ and $Y = \{y_1, y_2, y_3\}$. Then the formula $T[X/Y]$ is $(y_1 \wedge (\neg y_3 \vee y_2))$.

In order to make the formulae more compact and readable, we overload the boolean connectives to apply to sets of letters. For example, given three disjoint sets of letters W, S and R with the same number of elements k, we use the notation $(\neg S)$ as a shorthand for the formula $\bigwedge\{\neg s_i \mid s_i \in S\}$, $(S \equiv R)$ to denote $\bigwedge\{s_i \equiv r_i \mid 1 \leqslant i \leqslant k\}$, $(S \equiv \neg R)$ to denote $\bigwedge\{s_i \equiv \neg r_i \mid 1 \leqslant i \leqslant k\}$ and $(W \equiv (S \equiv \neg R))$ for $\bigwedge\{w_i \equiv (s_i \equiv \neg r_i) \mid 1 \leqslant i \leqslant k\}$. For example, the formula

$$T \wedge (W \equiv (X \equiv \neg Y),$$

where $T = (x_1 \wedge (\neg x_3 \vee x_2))$ is a shorthand for

$$\begin{aligned} & x_1 \wedge (\neg x_3 \vee x_2) \wedge [w_1 \equiv (x_1 \equiv \neg y_1)] \\ & \quad \wedge [w_2 \equiv (x_2 \equiv \neg y_2)] \wedge [w_3 \equiv (x_3 \equiv \neg y_3)]. \end{aligned}$$

3. Global model-based operators

In this section we establish relations between circumscription and the operators introduced by Satoh and Dalal, that are based on global minimality.

3.1. Satoh's revision

The connections between circumscription and Satoh's revision [33] are very simple. This is due to the similarity of these operations: CIRC takes the models with a minimal set of positive atoms of P, whereas Satoh's revision selects the models of A with a minimal set of differences with models of K.

To translate $\mathrm{CIRC}(T; P, \emptyset, Z)$ into a Satoh's revision it is enough to revise the knowledge base with all atoms in P negated. More precisely, we have

Theorem 10. *$(\neg P) *_S T$ is logically equivalent to* $\mathrm{CIRC}(T; P, \emptyset, Z)$.

Proof. Let M be a (P, Z)-minimal model of T. By definition, for all models N of T we have that $(N \cap P) \not\subset (M \cap P)$. Therefore, M has minimal distance from the models of P and, therefore, it is a model of $\neg P *_S T$. Now, let M be a model of $\neg P *_S T$. By definition, for all models N of T and all models L of $\neg P$, we have that $(N \triangle L) \not\subset (M \triangle L)$. Therefore, M is a model of T and it has minimal distance from the models of P. Hence, it is a model of $\mathrm{CIRC}(T; P, \emptyset, Z)$. $\square$

Note that the above one is a modular poly-time reduction that satisfies both logical and query-equivalence, but it assumes that $Q = \emptyset$. By combining the above theorem and Corollary 8 we obtain:

Corollary 11. $(\neg P) *_S (T \wedge (Q \equiv \neg Q'))$ *is query-equivalent to* $\mathrm{CIRC}(T; P, Q, Z)$.

Here Q' is a set of new letters one-to-one with letters in Q. The above reductions are simple because Satoh's revision seems somewhat more powerful than CIRC. In fact, it has CIRC as a sub-case, where K is a set of literals. However, it can be shown that Satoh's revision can be translated, satisfying query-equivalence, into circumscription.

Given a knowledge base K and a revising formula A, let $X = V(K) \cup V(A)$, Y and W two new distinct sets of letters, one-to-one with letters in X, we define $P_S = W$, $Q_S = \emptyset$ and $Z_S = X \cup Y$ and T_S as follows:

$$T_S = K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y)). \tag{4}$$

In order to make it clear we show an example of this reduction.

Example 12.

$$K = x_1 \wedge x_2,$$
$$A = (\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3)).$$

Using the above equation we obtain

$$\begin{aligned} T_S = {} & [y_1 \wedge y_2] \\ & \wedge [(\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3))] \\ & \wedge [(\neg w_1 \equiv (x_1 \equiv y_1)) \wedge (\neg w_2 \equiv (x_2 \equiv y_2)) \wedge (\neg w_3 \equiv (x_3 \equiv y_3))]. \end{aligned}$$

Note that T_S admits a model M iff $M_X = (M \cap X)$ is a model of A and $M_Y = (M \cap Y)$ is a model of $K[X/Y]$. The knowledge of M_X and M_Y uniquely determines which letters of W will belong to $M_W = (M \cap W)$. In fact, $w_i \in M_W$ if and only if $x_i \in M_X$ and $y_i \notin M_Y$ or $x_i \notin M_X$ and $y_i \in M_Y$.

The set W here plays the role of the *ab* predicates used by McCarthy in [29]. In fact, a model will satisfy a w_i only if there is no way to assign the same truth value to x_i and y_i. Therefore, if we force M_W to contain a minimal number of letters, we will retain only the models of T_S where the differences between the assignments to X and Y are as few as possible. Thus we obtain

Theorem 13. $\mathrm{CIRC}(T_S; W, \emptyset, X \cup Y)$ *is query-equivalent to* $K *_S A$.

Proof. Let γ be a formula such that $V(\gamma) \subseteq V(K) \cup V(A) = X$. We first show that $\mathrm{CIRC}(T_S; W, \emptyset, X \cup Y) \models \gamma$ implies that $K *_S A \models \gamma$. Assume that $\mathrm{CIRC}(T_S; W, \emptyset, X \cup Y) \models \gamma$ and $K *_S A \not\models \gamma$. Thus, there exists a model M_X of $K *_S A$ such that $M_X \not\models \gamma$. Let

M'_X be a model of K such that $M'_X \triangle M_X \in \delta(K,A)$. We define $M_Y = \{y_i \mid x_i \in M'_X\}$ and $M_W = \{w_i \mid ((x_i \in M_X) \text{ and } (y_i \notin M_Y)) \text{ or } ((x_i \notin M_X) \text{ and } (y_i \in M_Y))\}$. Now, let $M = M_X \cup M_Y \cup M_W$. Obviously, it holds that $M \models T_S$ and $M \not\models \gamma$. If M is a $(W, X \cup Y)$-minimal model the thesis follows, so assume that there exists a model N of T_S such that $N <_{(P,Z)} M$. Since N is a model of T_S and it cannot contain more literals of W than M, we have that $N_W = (N \cap W) \subset M_W$. Hence, the distance between $N_X = N \cap X$ and $N_Y = N \cap Y$ is smaller than the distance between M_X and M_Y. Thus, M_X is not one of the models of A closest to the models of K. As a consequence, M_X is not a model of $K *_S A$ and contradiction arises.

We now show that $K *_S A \models \gamma$ implies that $\mathrm{CIRC}(T_S; W, \emptyset, X \cup Y) \models \gamma$. Assume that $K *_S A \models \gamma$ and $\mathrm{CIRC}(T_S; W, \emptyset, X \cup Y) \not\models \gamma$. Thus, there exists a model M of $\mathrm{CIRC}(T_S; W, \emptyset, X \cup Y)$ such that $M \not\models \gamma$. Let $M_X = M \cap X$, we show that $M_X \models K *_S A$. It immediately follows that $M_X \models A$, if M_X is one of the models of A closer to models of K the thesis follows, so assume to the contrary that there exists a $N_X \subseteq X$, different from M_X, such that $N_X \models A$ and the distance of M_X from the closest model of K is a strict superset of the distance of N_X from its closest model of K, i.e. $\mu(N_X, K) \subseteq \mu(M_X, K)$ and $\mu(N_X, K) \neq \mu(M_X, K)$. Let N'_X be a model of K such that $N'_X \triangle N_X \in \delta(K,A)$. Let $N_Y = \{y_i \mid x_i \in N'_X\}$, $N_W = \{w_i \mid ((x_i \in N_X) \text{ and } (y_i \notin N_Y)) \text{ or } ((x_i \notin N_X) \text{ and } (y_i \in N_Y))\}$ and $N = N_X \cup N_Y \cup N_W$. Obviously N is a model of $K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y))$, moreover, $N <_{(P,Z)} M$. Hence, M is not a $(W, X \cup Y)$-minimal model of $K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y))$, hence contradiction arises. □

Summing up, we have a modular poly-time reduction from CIRC into Satoh's revision that satisfies logical equivalence and a reverse (modular and poly-time) reduction that only satisfies query-equivalence. It is natural to ask whether there exists a (poly-time or poly-size) reduction from Satoh's revision into CIRC that preserves logical equivalence. Using the result proven in [27] on the complexity of model checking for Satoh's operator and the known complexity of model checking for CIRC (see [3]), we can show that:

Theorem 14. *Unless* $\Sigma_2^p = \mathrm{coNP}$, *there is no poly-time reduction from* $K *_S A$ *into* $\mathrm{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence. Unless* $\Sigma_4^p = \Pi_4^p$, *there is no poly-size reduction from* $K *_S A$ *into* $\mathrm{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence.*

Proof. In [27] we have shown that the time complexity of deciding whether $M \models K *_S A$ is Σ_2^p-complete and that the compilability level of this problem is nu-comp-Σ_2^p-complete. In [3] it is shown that deciding whether $M \models \mathrm{CIRC}(T; P, Q, Z)$ is coNP-complete and in [5] we proved that the compilability level of this problem is nu-comp-coNP-complete. As a consequence, if there exists a poly-time reduction from $K *_S A$ into $\mathrm{CIRC}(T; P, Q, Z)$ satisfying logical-equivalence we have that a Σ_2^p-complete can be reduced to a coNP-complete one, and thus, $\Sigma_2^p \subseteq \mathrm{coNP}$. If there exists a poly-size reduction it follows that nu-comp-$\Sigma_2^p \subseteq$ nu-comp-coNP (the definitions of these classes are in Appendix A). By [5, Theorem 9] this implies that $\Sigma_4^p = \Pi_4^p$. □

3.2. Dalal's revision

The same reductions between Satoh's revision and usual (set-containment-based) circumscription hold between Dalal's revision [10] and cardinality-based circumscription. Proofs of all the following theorems are in Appendix B.

Theorem 15. *$(\neg P) *_{\mathrm{D}} T$ is logically equivalent to* $\mathrm{NCIRC}(T; P, \emptyset, Z)$.

Note that the above reduction is modular, computable in polynomial time and satisfies both logical and query-equivalence, but it assumes that $Q = \emptyset$. By combining the above theorem and Corollary 9 we obtain:

Corollary 16. *$(\neg P) *_{\mathrm{D}} (T \wedge (Q \equiv \neg Q'))$ is query-equivalent to* $\mathrm{NCIRC}(T; P, Q, Z)$.

Here Q' is a set of new letters one-to-one with letters in Q. To reduce Dalal's revision into cardinality-based circumscription, we use the same relation adopted to reduce Satoh's revision into CIRC,

$$K *_{\mathrm{D}} A \Rightarrow \mathrm{NCIRC}(T_{\mathrm{D}}; W, \emptyset, X \cup Y),$$

where $X = V(T) \cup V(A)$ and $T_{\mathrm{D}} = K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y))$. Note that, not surprisingly, T_{D} coincides with T_{S}.

Theorem 17. $\mathrm{NCIRC}(T_{\mathrm{D}}; W, \emptyset, X \cup Y)$ *is query-equivalent to $K *_{\mathrm{D}} A$.*

We have shown a reduction from NCIRC into Dalal's revision that satisfies logical equivalence and a reverse reduction that only satisfies query-equivalence. We show that, unless there is a collapse in the polynomial hierarchy, there is no poly-size or polytime reduction from Dalal's revision into NCIRC that preserves logical equivalence.

Theorem 18. *Unless* NP = coNP, *there is no poly-time reduction from $K *_{\mathrm{D}} A$ into* $\mathrm{NCIRC}(T; P, Q, Z)$ *satisfying logical-equivalence. Unless $\Sigma_3^p = \Pi_3^p$, there is no poly-size reduction from $K *_{\mathrm{D}} A$ into* $\mathrm{NCIRC}(T; P, Q, Z)$ *satisfying logical-equivalence.*

We end this section with a complete example of application of our reductions. We continue Example 12 and reduce Satoh's and Dalal's revision to CIRC and NCIRC, respectively.

Example 19.

$$K = x_1 \wedge x_2,$$
$$A = (\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3)).$$

Using the reduction of Theorems 13 and 17 we obtain $W = \{w_1, w_2, w_3\}$, $Q = \emptyset$, $Z = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and

$$\begin{aligned} T_S = T_D = {} & [y_1 \wedge y_2] \\ & \wedge [(\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3))] \\ & \wedge [(\neg w_1 \equiv (x_1 \equiv y_1)) \wedge (\neg w_2 \equiv (x_2 \equiv y_2)) \wedge (\neg w_3 \equiv (x_3 \equiv y_3))]. \end{aligned}$$

This formula admits the models

$$\begin{aligned} M_1 &= \{y_1, y_2, w_1, w_2\}, \\ M_2 &= \{y_1, y_2, y_3, w_1, w_2, w_3\}, \\ M_3 &= \{x_1, y_1, y_2, w_2\}, \\ M_4 &= \{x_1, y_1, y_2, y_3, w_2, w_3\}, \\ M_5 &= \{x_3, y_1, y_2, w_1, w_2, w_3\}, \\ M_6 &= \{x_3, y_1, y_2, y_3, w_1, w_2\}. \end{aligned}$$

Note that the only (W, Z)-minimal and (W, Z)-cardinality-minimal model is M_3. In fact, $M_3 \cap X$ is the only model of $K *_S A$ and $K *_D A$.

4. Local model-based operators

In this section we establish relations between circumscription and the operators introduced by Winslett, Borgida and Forbus, that are based on local minimality.

4.1. Winslett's update

Winslett's update method modifies models of K one-by-one, replacing each one with the closest one within the models of A. Local proximity methods are better related to circumscription where all letters are minimized. Circumscription without varying and fixed letters (i.e. $Q = Z = \emptyset$) is immediately expressed as

$$\mathrm{CIRC}(T; P, \emptyset, \emptyset) \Rightarrow \neg P *_W T$$

Theorem 20. *$(\neg P) *_W T$ is logically equivalent to* $\mathrm{CIRC}(T; P, \emptyset, \emptyset)$.

In order to reduce Winslett's update into circumscription, we must ensure that to each distinct model of K correspond incomparable models in the circumscriptive theory. Let $X = V(K) \cup V(A)$, Y and W be new sets of distinct variables, each one-to-one with variables in X. The desired relation is obtained (only satisfying query-equivalence) using the following equation:

$$T_W = K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y)). \tag{5}$$

In fact, we have

Theorem 21. $\mathrm{CIRC}(T_W; W, Y, X)$ *is query-equivalent to* $K *_W A$.

Note that T_W is equal to T_S, but now the letters in Y are kept fixed, not minimized. We show a negative proof on the existence of a (poly-size or poly-time) reduction from Winslett's revision into CIRC that satisfies logical equivalence.

Theorem 22. *Unless* $\Sigma_2^p = \text{coNP}$, *there is no poly-time reduction from* $K *_W A$ *into* $\text{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence. Unless* $\Sigma_4^p = \Pi_4^p$, *there is no poly-size reduction from* $K *_W A$ *into* $\text{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence.*

4.2. Borgida's revision

Borgida's revision operator [2] is very similar to Winslett's one, the only difference being that the result of the first one has to be $K \wedge A$ when not contradictory. It is easy to show that $*_B$ and $*_W$ coincide when K has a single model M. In fact, if M is also a model of A, then $K *_W A = K \wedge A = K *_B A$. If M is not a model of A the equivalence follows from the definition. As a consequence, the same reduction used for $*_W$ also holds for $*_B$. That is,

$$\text{CIRC}(T; P, \emptyset, \emptyset) \Rightarrow \neg P *_B T.$$

In the other direction, one can find a direct transformation from Borgida's revision into circumscription, very much like Winslett's one. The fact that the result must be $K \wedge A$ can be taken into account by selecting the models of this formula as minimal.

$$K *_B A \Rightarrow \text{CIRC}(T_B; R \cup Z \cup W, \emptyset, X \cup Y),$$

where $X = V(K) \cup V(A)$, Y, W and R are new sets one-to-one with the elements of X and T_B is defined as follows:

$$T_B = [K \vee (Y \equiv R)] \wedge K[X/Y] \wedge A \wedge [\neg W \equiv (X \equiv Y)]. \tag{6}$$

In fact, we have

Theorem 23. $\text{CIRC}(T_B; W, R, X \cup Y)$ *is query-equivalent to* $K *_B A$.

As for the other operators, we show a negative proof on the existence of a (poly-size or poly-time) reduction from Winslett's revision into CIRC that satisfies logical equivalence.

Theorem 24. *Unless* $\Sigma_2^p = \text{coNP}$, *there is no poly-time reduction from* $K *_B A$ *into* $\text{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence. Unless* $\Sigma_4^p = \Pi_4^p$, *there is no poly-size reduction from* $K *_B A$ *into* $\text{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence.*

4.3. Forbus' update

We first observe how circumscription and NCIRC can be expressed using Forbus' update. Reduction of NCIRC to Forbus' operator is trivial:

$$\text{NCIRC}(T; P, \emptyset, \emptyset) \Rightarrow \neg P *_F T.$$

Theorem 25. *$(\neg P) *_{\mathrm{F}} T$ is logically equivalent to* $\mathrm{NCIRC}(T; P, \emptyset, \emptyset)$.

The reduction of Forbus' update to circumscription is very similar to Borgida's one. We have only to take in account that Forbus' update is based upon a minimization of the cardinality of the distances between models.

The reduction is:

$$K *_{\mathrm{F}} A \Rightarrow \mathrm{CIRC}(T_{\mathrm{F}}; V, Y, X \cup W),$$

where T_{F} is defined as

$$T_{\mathrm{F}} = K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y)) \wedge EQ(W, V) \wedge BEGIN(V). \tag{7}$$

The formula $EQ(W, V)$ is a polynomial-size formula that is true if and only if W and V have exactly the same number of positive literals. It can be constructed in several ways, if n is the cardinality of the two sets, the simpler formula representing this boolean function uses two n-bits adders and then forces the two results to become (bit-by-bit) equal. Finally, $BEGIN(V)$ states that the positive literals of V are its first ones:

$$BEGIN(V) = (v_n \rightarrow v_{n-1}) \wedge \cdots \wedge (v_2 \rightarrow v_1).$$

Any interpretation M of $V \cup W$ satisfying $EQ(W, V) \wedge BEGIN(V)$ is such that $|M \cap V| = |M \cap W|$ and $v_i \in M$ if and only if for all $1 \leqslant j < i$ we have $v_i \in M$. That is, an interpretation of the set V has all the true atoms "at the beginning".

In fact, we have

Theorem 26. $\mathrm{CIRC}(T_{\mathrm{F}}; V, Y, X \cup W)$ *is query-equivalent to* $K *_{\mathrm{F}} A$.

Summing up, we have a reduction from NCIRC into Forbus' revision that satisfies logical equivalence (when $Q = Z = \emptyset$) and a reduction from $*_{\mathrm{F}}$ to CIRC that only satisfies query-equivalence. We show a negative proof on the existence of a reduction from Forbus' revision into CIRC that satisfies logical equivalence.

Theorem 27. *Unless $\Sigma_2^p = \mathrm{coNP}$, there is no poly-time reduction from $K *_{\mathrm{F}} A$ into $\mathrm{CIRC}(T; P, Q, Z)$ satisfying logical-equivalence. Unless $\Sigma_4^p = \Pi_4^p$, there is no poly-size reduction from $K *_{\mathrm{F}} A$ into $\mathrm{CIRC}(T; P, Q, Z)$ satisfying logical-equivalence.*

We end this section with an example of application of our reductions. We reduce Winslett's, Borgida's and Forbus' operators to CIRC.

Example 28. We use the same formulae K and A used in Example 12.

$$K = x_1 \wedge x_2,$$
$$A = (\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3)).$$

For Winslett's revision we use the reduction of Theorem 21 and we obtain $P = \{w_1, w_2, w_3\}$, $Q = \{y_1, y_2, y_3\}$, $Z = \{x_1, x_2, x_3\}$ and T_W,

$$\begin{aligned} T_W = {} & [y_1 \wedge y_2] \\ & \wedge [(\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3))] \\ & \wedge [(\neg w_1 \equiv (x_1 \equiv y_1)) \wedge (\neg w_2 \equiv (x_2 \equiv y_2)) \wedge (\neg w_3 \equiv (x_3 \equiv y_3))]. \end{aligned}$$

This formula admits the models

$$\begin{aligned} M_1 &= \{y_1, y_2, w_1, w_2\}, \\ M_2 &= \{y_1, y_2, y_3, w_1, w_2, w_3\}, \\ M_3 &= \{x_1, y_1, y_2, w_2\}, \\ M_4 &= \{x_1, y_1, y_2, y_3, w_2, w_3\}, \\ M_5 &= \{x_3, y_1, y_2, w_1, w_2, w_3\}, \\ M_6 &= \{x_3, y_1, y_2, y_3, w_1, w_2\}. \end{aligned}$$

Since we can only compare models with the same assignment to the letter in Q, the (P,Z)-minimal models are M_3, M_4 and M_6. In fact, $M_3 \cap X = \{x_1\} = M_4 \cap X$ and $M_6 \cap X = \{x_3\}$ are the only models of $K *_W A$.

For Borgida's revision we use the reduction of Theorem 23 and we obtain $P = \{w_1, w_2, w_3\}$, $Q = \{r_1, r_2, r_3\}$, $Z = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and T_B,

$$\begin{aligned} T_B = {} & [(x_1 \wedge x_2) \vee ((y_1 \equiv r_1) \wedge (y_2 \equiv r_2) \wedge (y_3 \equiv r_3))] \\ & \wedge [y_1 \wedge y_2] \\ & \wedge [(\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3))] \\ & \wedge [(\neg w_1 \equiv (x_1 \equiv y_1)) \wedge (\neg w_2 \equiv (x_2 \equiv y_2)) \wedge (\neg w_3 \equiv (x_3 \equiv y_3))]. \end{aligned}$$

This formula admits the models

$$\begin{aligned} M_1 &= \{y_1, y_2, w_1, w_2, r_1, r_2\}, \\ M_2 &= \{y_1, y_2, y_3, w_1, w_2, w_3, r_1, r_2, r_3\}, \\ M_3 &= \{x_1, y_1, y_2, w_2, r_1, r_2\}, \\ M_4 &= \{x_1, y_1, y_2, y_3, w_2, w_3, r_1, r_2, r_3\}, \\ M_5 &= \{x_3, y_1, y_2, w_1, w_2, w_3, r_1, r_2\}, \\ M_6 &= \{x_3, y_1, y_2, y_3, w_1, w_2, r_1, r_2, r_3\}. \end{aligned}$$

Since we can only compare models with the same assignment to the letter in $Q = \{r_1, r_2, r_3\}$, the (P,Z)-minimal models are M_3, M_4 and M_6. In fact, $M_3 \cap X = \{x_1\} = M_4 \cap X$ and $M_6 \cap X = \{x_3\}$ are the only models of $K *_B A$.

For Forbus' revision we use the reduction of Theorem 26 and we obtain $P = \{v_1, v_2, v_3\}$, $Q = \{y_1, y_2, y_3\}$, $Z = \{x_1, x_2, x_3, w_1, w_2, w_3\}$ and T_F.

$$\begin{aligned} T_F = &[y_1 \wedge y_2] \\ &\wedge [(\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3))] \\ &\wedge [(\neg w_1 \equiv (x_1 \equiv y_1)) \wedge (\neg w_2 \equiv (x_2 \equiv y_2)) \wedge (\neg w_3 \equiv (x_3 \equiv y_3))] \\ &\wedge EQ(W, V) \wedge [(v_3 \rightarrow v_2) \wedge (v_2 \rightarrow v_1)]. \end{aligned}$$

This formula admits the models

$$\begin{aligned} M_1 &= \{y_1, y_2, w_1, w_2, v_1, v_2\}, \\ M_2 &= \{y_1, y_2, y_3, w_1, w_2, w_3, v_1, v_2, v_3\}, \\ M_3 &= \{x_1, y_1, y_2, w_2, r_1\}, \\ M_4 &= \{x_1, y_1, y_2, y_3, w_2, w_3, v_1, v_2\}, \\ M_5 &= \{x_3, y_1, y_2, w_1, w_2, w_3, r_1, r_2\}, \\ M_6 &= \{x_3, y_1, y_2, y_3, w_1, w_2, v_1, v_2\}. \end{aligned}$$

Since we can only compare models with the same assignment to the letters in $Q = \{y_1, y_2, y_3\}$, the (P, Z)-minimal models are M_3, M_4 and M_6. In fact, $M_3 \cap X = \{x_1\} = M_4 \cap X$ and $M_6 \cap X = \{x_3\}$ are the only models of $K *_F A$.

5. Formula-based operators

The only formula-based operator we have analyzed is SBR, introduced by Ginsberg and Fagin, Ullman and Vardi. This operator is quite similar to Satoh's principle of minimization. The main difference between them is that the latter minimizes distance given as set of literals, while the first one maximizes the number of preserved formulae of K.

Two simple reductions from SBR into circumscription, and vice versa, are the following ones,

$$\begin{aligned} \mathrm{CIRC}(T; P, \emptyset, Z) &\Rightarrow (\neg P) *_{\mathrm{SBR}} T, \\ K *_{\mathrm{SBR}} A &\Rightarrow \mathrm{CIRC}(T_{SBR}; Y, \emptyset, X), \end{aligned}$$

where $K = \{f_1 \ldots, f_m\}$, $X = V(K) \cup V(A)$, Y is a set of m new letters one-to-one with formulae of K and T_{SBR} is defined as

$$T_{\mathrm{SBR}} = A \wedge (y_1 \equiv \neg f_1) \wedge \cdots \wedge (y_m \equiv \neg f_m)$$

More precisely, we have:

Theorem 29. *$(\neg P) *_{\mathrm{SBR}} T$ is logically equivalent to $\mathrm{CIRC}(T; P, \emptyset, Z)$ and $\mathrm{CIRC}(T_{\mathrm{SBR}}; Y, \emptyset, X)$ is query-equivalent to $K *_{\mathrm{SBR}} A$.*

Using Corollary 8 we can also obtain a reduction from $\mathrm{CIRC}(T;P,Q,Z)$ into SBR that preserves query-equivalence. We close this section with an example of application of our reductions. We reduce SBR to CIRC.

Example 30. We use the same formulae K and A used in Example 12, but K is now a set of two formulae.

$$K = \{x_1, x_2\},$$
$$A = (\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3)).$$

Using the reduction of Theorem 29 we obtain $P = \{y_1, y_2\}$, $Q = \emptyset$, $Z = \{x_1, x_2, x_3\}$ and T_{SBR},

$$\begin{aligned} T_{\mathrm{SBR}} = {} & [(\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge (x_1 \equiv \neg x_3))] \\ & \wedge [(y_1 \equiv \neg x_1) \wedge (y_2 \equiv \neg x_2)]. \end{aligned}$$

This formula admits the models

$$M_1 = \{y_1, y_2\},$$
$$M_2 = \{x_1, y_2\},$$
$$M_3 = \{x_3, y_1, y_2\}.$$

Note that the only (P,Z)-minimal model is M_2. In fact, $W(K,A) = \{x_1\}$ and, therefore, $K *_{\mathrm{SBR}} A = (A \wedge x_1)$. Simplifying the formula we obtain $K *_{\mathrm{SBR}} A = \neg x_2 \wedge \neg x_3 \wedge x_1$, whose only model is $M_2 \cap X$.

6. AGM operators

In previous sections we showed how we can reduce specific belief revision operators to circumscription and vice versa. Here we present a general methodology to transform any belief revision operator. The most general form of belief revision is given by the well-known postulates for revision (AGM postulates presented in Section 2). We want to point out that, among the operators considered so far, only Dalal's one satisfies all AGM postulates. Therefore, the reduction presented in this section also applies to $*_{\mathrm{D}}$ as well. We remind that any operator $(*_{\mathrm{AGM}})$ satisfying the AGM postulates can be expressed as

$$\mathcal{M}(K *_{\mathrm{AGM}} A) = \min(\mathcal{M}(A), \leqslant_K),$$

where $\leqslant_K$ is a transitive, reflexive and total relation based on K.

This characterization suggests to select the minimal models of a formula by imposing that models must be minimal with respect to $\leqslant_K$. Any ordering over interpretations can be represented via a propositional formula $LEQ_K(\cdot,\cdot)$, such that $LEQ_K(X,Y)$ is true iff

$X \leqslant_K Y$. Using this formula, AGM revision operators can be reduced to circumscription via

$$K *_{\mathrm{AGM}} A \Rightarrow w \wedge \mathrm{CIRC}(T_{\mathrm{AGM}}; \{w\}, X, Y),$$

where T_{AGM} is defined as follows,

$$T_{\mathrm{AGM}} = A \wedge A[X/Y] \wedge (\neg w \equiv LESS_K(Y, X)). \tag{8}$$

where $LESS_K(Y, X)$ imposes the constraint that for any assignment M to the letters of Y and N to the letters of X it must hold that $M[Y/X] <_K N$. More precisely, $LESS_K(Y, X) \equiv LEQ_K(Y, X) \wedge \neg LEQ_K(X, Y)$. Note that K is missing in the circumscription, since it is implicit in LEQ_K.

More precisely, we have:

Theorem 31. *$w \wedge \mathrm{CIRC}(T_{\mathrm{AGM}}; \{w\}, X, Y)$ is query-equivalent to $K *_{\mathrm{AGM}} A$.*

We want to point out that these transformations are not necessarily polynomial. In fact, we do not know what is the size of the formula $LEQ(X, Y)$ with respect to the size of X and Y. It might very well be exponential. However, if the ordering relation $\leqslant_K$ can be decided in polynomial time, then the formula $LEQ_K(X, Y)$ has size polynomial.

7. Relations among belief revision operators

In Sections 3, 4, 5 and 6 we found relations between circumscription and belief revision operators. Here we focus on relations among the various revision operators.

In particular, we show that Satoh's and Ginsberg, Fagin, Ullman and Vardi's operators can be reduced one to the other and that Winslett's one can be reduced to both. Note that these operators belong to three different classes of operators, namely formula-based (Ginsberg, Fagin, Ullman and Vardi), model-based with global proximity (Satoh) and model-based with local proximity (Winslett). Therefore, our results make evident the similarities between all these operators, pointing out, at the same time, their differences.

Ginsberg, Fagin, Ullman and Vardi's operator can be reduced to Satoh's operator via:

$$K *_{\mathrm{SBR}} A \Rightarrow Y *_{\mathrm{S}} A',$$

where $K = \{f_1 \ldots, f_m\}$, Y is a set of m new letters one-to-one with formulae of K and

$$A' = A \wedge (y_1 \rightarrow f_1) \wedge \cdots \wedge (y_m \rightarrow f_m).$$

More formally we have:

Theorem 32. *$Y *_{\mathrm{S}} A'$ is query-equivalent to $K *_{\mathrm{SBR}} A$.*

The reverse reduction is

$$K *_S A \Rightarrow W *_{SBR} A'',$$

where Y and W are sets of new letters one to one with letters of X and

$$A'' = K[X/Y] \wedge A \wedge (W \rightarrow (X \equiv Y)).$$

More formally we have:

Theorem 33. *$W *_{SBR} A''$ is query-equivalent to $K *_S A$.*

More complex, but still computable in polynomial time, is the reduction of Winslett's operator into SBR. Denoting with F the following formula,

$$F = K[X/Y] \wedge A \wedge (\neg Y \vee \neg Z) \wedge (W \rightarrow (X \equiv Y)),$$

where Y, W and Z are sets of new letters one to one with letters of X. The reduction is now the following one:

$$K *_W A \Rightarrow (W \cup Y \cup Z) *_{SBR} F.$$

More formally we have:

Theorem 34. *$(W \cup Y \cup Z) *_{SBR} F$ is query-equivalent to $K *_W A$.*

Composing the reductions of $*_W$ to $*_{SBR}$ and $*_{SBR}$ to $*_S$ we have a reduction of $*_W$ to $*_S$. All these reductions only preserve query-equivalence, we show that there cannot be any (poly-size or poly-time) reduction from either Winslett's or Satoh's operator into SBR.

Theorem 35. *Unless Σ_2^p = coNP, there is no poly-time reduction from $K *_S A$ ($K *_W A$) into $K' *_{SBR} A'$ satisfying logical-equivalence. Unless $\Sigma_4^p = \Pi_4^p$, there is no poly-size reduction from $K *_S A$ ($K *_W A$) into $K' *_{SBR} A'$ satisfying logical-equivalence.*

8. Syntactically-restricted knowledge bases

In this section we focus on knowledge bases of a restricted syntactic form. Among the restricted cases, Horn knowledge bases are of particular interest for several reasons. First of all, since Horn clauses can represent if-then relations, they are expressive enough to represent many real situations. Moreover, reasoning with Horn knowledge bases is significantly simpler than reasoning with general ones (see [11]) and also revising them is, in general, simpler than revising general ones (see [13]).

While reductions from circumscription to belief revision preserve the syntactic form of the original theory, reductions from belief revision to circumscription do not preserve the syntactic form of the formulae. As an example, notice that the relation $X \equiv \neg Y$ cannot be expressed as a Horn formula.

As a consequence, it is easy to apply results on restricted cases of belief revision to circumscription, but the other way around is less likely to produce interesting results.

There are several reasons why the revision of Horn theories cannot be expressed as the circumscription of a Horn formula. First of all, results of Eiter and Gottlob show that reasoning with the revision of a Horn knowledge base is coNP-hard for all operators considered, while reasoning with Horn theories under circumscription is a polynomial task. As a consequence, reductions from belief revision to circumscription preserving the syntactic form cannot be done in polynomial time (assuming $\mathrm{P} \neq \mathrm{NP}$).

Secondly, the result of revising a Horn knowledge base with a Horn formula might be a non-Horn formula. For example, the result of $\{a,b\} * (\neg a \vee \neg b)$ is $a \equiv \neg b$ for all operators, and $a \equiv \neg b$ cannot be expressed as a Horn formula. On the other hand, the circumscription of a Horn theory is a Horn theory.

9. Analysis and discussion

In the previous sections we showed new relations relating belief revision operators and circumscription. These relations point out the close connections between the two fields. We want to point out that our relations can be easily extended to full first-order languages, whenever the belief revision operators are defined in this setting. Take, for example, the reduction from circumscription into Satoh's revision operator. The reduction also applies when K and A are arbitrary first-order sentences. Clearly, in this case X is a set of n predicates, each one with its arity, while W and Y are new sets of n predicates, one-to-one with the predicates of X. Moreover, the constraint $\neg W \equiv (X \equiv Y)$ must be expressed as

$$\bigwedge_{i=1}^{n} \forall z^k (\neg w_i(z^k) \equiv (x_i(z^k) \equiv y_i(z^k))),$$

where w_i, x_i and y_i are k-ary predicates and z^k is a vector of k variables.

Many side benefits can be obtained from the established relations. In this section we want to point out the most important benefits obtained.

9.1. Compact representation of NCIRC

In two recent papers [4,7] Cadoli, Donini and the present authors analyze the size of the explicit representation of circumscription and belief revision operators. More precisely, taking as an example belief revision, it is determined the size of the smallest propositional formula K_1 that is equivalent to $K * A$, where $*$ is one of the belief revision operators analyzed.

As it turns out, the size of the explicit representation of the result of revising a knowledge base is, in general, exponential with respect to $|K| + |A|$. Differences arise between the various operators. The result of revising a knowledge base using Dalal's revision operator admits a polynomial-sized explicit representation, if we allow new variables in the representation. More precisely, there exists a formula K_1 using the letters of K and A and possibly new ones, whose size is polynomial in $|K| + |A|$, such that, for any q using only variables of K and A we have that $K_1 \models q$ if and only if $K *_{\mathrm{D}} A \models q$.

We show that $\mathrm{NCIRC}(T; P, Q, Z)$ always admits an explicit representation whose size is polynomial with respect to $|T|$, via the proof given for Dalal's belief revision operator. $\mathrm{NCIRC}(T; P, Q, Z)$ is the set of models of T with a least number of elements. Given T and P, we first compute the least number k of true letters of the set P in the models of T. At this point, we constrain the formula to have only models that contain at most k letters of the set P. This forces the formula to only retain the (P, Z)-cardinality-minimal models of T. This can be accomplished by conjoining T with a formula imposing that at most k letters of the set P must be true. That is,

$$\mathrm{NCIRC}(T; P, Q, Z) = T \wedge \mathit{ATMOST}(k, P).$$

The formula $\mathit{ATMOST}(k, P)$ can be constructed using an n-bit adder (where $n = |P|$) for the letters of P and then constraining the result to coincide with the binary representation of k. This a formula has size $\mathrm{O}(n^3)$. Thus, the size of $T \wedge \mathit{ATMOST}(k, P)$ is polynomial in $|T|$.

9.2. Computational complexity analysis

A valuable byproduct of the reductions presented in this work is the ability of importing complexity results obtained in one field into the other one. For example, in the general case, inference using the belief revision operators introduced by Satoh, Borgida and Winslett has the same complexity of inference under circumscription. While this result is not novel, it has been proven in [13, 14], several other interesting results can be obtained. As an example, it is known that deciding whether a clause follows from the circumscription (with all letters minimized) of a theory composed of binary clauses (i.e. clauses with at most two literals) is a coNP-hard problem [9]. We can use this result to prove that inference in the revision of a knowledge base composed of binary clauses is a coNP-hard problem for most operators.

Corollary 36. *Let K and A be two CNF formula where all clauses have at most two literals, and Q be a clause. The problem of deciding whether $K * A \models Q$ is* coNP-*hard for* $* \in \{*_{\mathrm{S}}, *_{\mathrm{D}}, *_{\mathrm{W}}, *_{\mathrm{B}}, *_{\mathrm{SBR}}\}$.

10. Conclusions

We have presented a complete analysis of the relations between belief revision operators on one hand and circumscription and its cardinality-based variant on the other hand. Furthermore, we have pointed out the many benefits that the established correlations can deliver to the analysis of both fields.

Our results greatly extends Winslett's results on transforming her revision operator into circumscription presented in [34]. Even though Winslett's analysis could be further extended to deal with other operators, our results provide us with more direct and simple translations.

Acknowledgments

We want to thank Marco Cadoli for helpful discussions on the content of this paper.

Appendix A. Compilability classes

In this section we summarize some definitions and results proposed in [5] adapting them to the context and terminology of belief revision formalisms. In that paper we introduced a new complexity measure for decision problems, called *compilability*. Following the intuition that a knowledge base is known well before questions are posed to it, we divide a reasoning problem into two parts: one part is *fixed* or *accessible off-line* (the revised knowledge base), and the second one is *variable*, or *accessible on-line* (the model). Compilability aims at capturing the *on-line complexity* of solving a problem composed of such inputs, i.e. complexity with respect to the second input when the first one can be preprocessed in an arbitrary way. We introduce a new hierarchy of classes, the *non-uniform compilability classes*, denoted as nu-comp-C, where C is a generic uniform complexity class, such as NP, coNP, Σ_2^p, etc.

Definition A.1 (*nu-comp-C classes*). A language of pairs $S \subseteq \Sigma^* \times \Sigma^*$ belongs to nu-comp-C iff there exists a binary poly-size function f and a language of pairs S' such that for all $\langle x, y\rangle \in S$ it holds:

(i) $\langle f(x, |y|), y\rangle \in S'$ iff $\langle x, y\rangle \in S$,

(ii) $S' \in$ C.

Notice that the poly-size function f takes as input both x (the revised knowledge base) and the size of y (the interpretation). If we want to rewrite off-line x into a new string (formula), our definition requires that we know in advance the *size* of y. Since y is an interpretation of the letters of x, its size is bounded by $|x|$ and therefore it is known in advance.

For each C, the class nu-comp-C generalizes the non-uniform class C/poly—i.e. C/poly $\subset$ nu-comp-C—by allowing for a fixed part x. In Figs. A.1 and A.2 we compare the machines corresponding to C/poly and nu-comp-C.

We introduce now a reduction between problems.

Definition A.2 (*Non-uniform comp-reducibility*). Given two problems A and B, A is *non-uniformly comp-reducible* to B (denoted as $A \leqslant_{\text{nu-comp}} B$) iff there exist two poly-size binary functions f_1 and f_2, and a polynomial-time binary function g such that for every pair $\langle x, y\rangle$ it holds that $\langle x, y\rangle \in A$ if and only if $\langle f_1(x, |y|), g(f_2(x, |y|), y)\rangle \in B$.

The $\leqslant_{\text{nu-comp}}$ reductions can be pictorially represented as shown in Fig. A.3.

Such reductions satisfy all important properties of a reduction:

Theorem A.3. *The reductions* $\leqslant_{\text{nu-comp}}$ *satisfy transitivity and are compatible* (*in the sense of Johnson* [22, p. 79]) *with the class* nu-comp-C *for every complexity class* C.

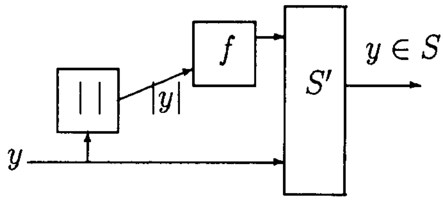

Fig. A.1. The C/poly machine.

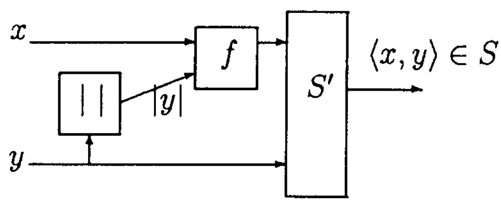

Fig. A.2. The nu-comp-C machine.

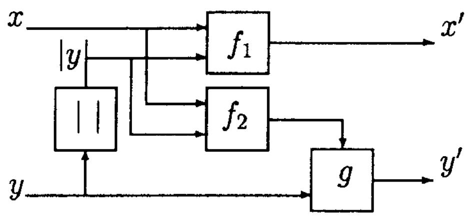

Fig. A.3. The $\leqslant_{\text{nu-comp}}$ reduction.

Therefore, it is possible to define the notions of *hardness* and *completeness* for nu-comp-C for every complexity class C.

Definition A.4 (*nu-comp-C-completeness*). Let S be a language of pairs and C a complexity class. S is nu-comp-C-*hard* iff for all problems $A \in$ nu-comp-C we have that $A \leqslant_{\text{nu-comp}} S$. Moreover, S is nu-comp-C-*complete* if S is in nu-comp-C and is nu-comp-C-*hard*.

We close the section by giving a rationale for the complexity classes we defined. Informally we may say that nu-comp-NPhard problems are "not compilable to P", as from the above considerations we know that if there exists a preprocessing of their fixed part that makes them on-line solvable in polynomial time, then the polynomial hierarchy collapses. The same holds for nu-comp-coNP-hard problems. In general, a problem which is nu-comp-C-complete for a class C containing P can be regarded as the "toughest" problem in C, even after arbitrary preprocessing of the fixed part. On the other hand, a problem in nu-comp-C is a problem that, after preprocessing of the fixed part, becomes a problem in C (i.e. it is "compilable to C").

Appendix B. Proofs of theorems

Theorem 15. *$(\neg P) *_{\mathrm{D}} T$ is logically equivalent to* NCIRC$(T; P, \emptyset, Z)$.

Proof. Let M be a (P, Z)-cardinality minimal model of T and $k = |M \cap P|$. By definition, for all models N of T we have that $|N \cap P| \geqslant |M \cap P|$. Therefore, M has cardinality minimal distance from the models of P and, therefore, it is a model of $\neg P *_{\mathrm{D}} T$. Now, let M be a model of $\neg P *_{\mathrm{D}} T$ and $k = |M \cap P|$. By definition, for all models N of T and all models L of $\neg P$, we have that $\mathrm{card}(N \triangle L) \geqslant k$. Therefore, M is a model of T and it has cardinality minimal distance from the models of P. Hence, it is a model of NCIRC$(T; P, \emptyset, Z)$. □

Theorem 17. NCIRC$(T_{\mathrm{D}}; W, \emptyset, X \cup Y)$ *is query-equivalent to* $K *_{\mathrm{D}} A$.

Proof. Let γ be a formula such that $V(\gamma) \subseteq V(K) \cup V(A) = X$. We first show that NCIRC$(T_{\mathrm{D}}; W, \emptyset, X \cup Y) \models \gamma$ implies that $K *_{\mathrm{D}} A \models \gamma$. Assume that NCIRC$(T_{\mathrm{D}}; W, \emptyset, X \cup Y) \models \gamma$ and $K *_{\mathrm{D}} A \not\models \gamma$. Thus, there exists a model M_X of $K *_{\mathrm{D}} A$ such that $M_X \not\models \gamma$. Let M'_X be a model of K such that $\mathrm{card}(M'_X \triangle M_X) \in k_{K,A}$. We define $M_Y = \{y_i \mid x_i \in M'_X\}$ and $M_W = \{w_i \mid ((x_i \in M_X) \text{ and } (y_i \notin M_Y)) \text{ or } ((x_i \notin M_X) \text{ and } (y_i \in M_Y))\}$. Now, let $M = M_X \cup M_Y \cup M_W$. Obviously, it holds that $M \models T_{\mathrm{D}}$ and $M \not\models \gamma$. If M is a $(W, X \cup Y)$-cardinality-minimal model the thesis follows, so assume that there exists a model N of T_{D} such that $N \prec_{(P,Z)} M$. Let $N_X = N \cap X$, $N_Y = N \cap Y$ and $N_W = N \cap W$. Since N is a model of T_{D} and it cannot contain more literals of W than M, we have that $|N_W| < |M_W|$. Hence, the distance between $N_X = N \cap X$ and $N_Y = N \cap Y$ is smaller than the distance between M_X and M_Y. Thus, M_X is not one of the models of A closest to the models of K. As a consequence, M_X is not a model of $K *_{\mathrm{D}} A$ and contradiction arises.

We now show that $K *_{\mathrm{D}} A \models \gamma$ implies that NCIRC$(T_{\mathrm{D}}; W, \emptyset, X \cup Y) \models \gamma$. Assume that $K *_{\mathrm{D}} A \models \gamma$ and NCIRC$(T_{\mathrm{D}}; W, \emptyset, X \cup Y) \not\models \gamma$. Thus, there exists a model M of NCIRC$(T_{\mathrm{D}}; W, \emptyset, X \cup Y)$ such that $M \not\models \gamma$. Let $M_X = M \cap X$, we show that $M_X \models K *_{\mathrm{D}} A$. It immediately follows that $M_X \models A$, if M_X is one of the models of A closer to models of K the thesis follows, so assume to the contrary that there exists a $N_X \subseteq X$, different from M_X, such that $N_X \models A$ and the distance of M_X from the closest model of K is strictly smaller than the distance of N_X from its closest model of K, i.e. $\mathrm{card}(\mu(N_X, K)) < \mathrm{card}(\mu(M_X, K))$. Let N'_X be a model of K such that $\mathrm{card}(N'_X \triangle N_X) \in k_{K,A}$. We define $N_Y = \{y_i \mid x_i \in N'_X\}$ and $N_W = \{w_i \mid ((x_i \in N_X) \text{ and } (y_i \notin N_Y)) \text{ or } ((x_i \notin N_X) \text{ and } (y_i \in N_Y))\}$. Obviously N is a model of $K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y))$, moreover, $N \prec_{(P,Z)} M$. Hence, M is not a $(W, X \cup Y)$-cardinality-minimal model of $K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y))$, hence contradiction arises. □

Theorem 18. *Unless* NP = coNP, *there is no poly-time reduction from $K *_{\mathrm{D}} A$ into* NCIRC$(T; P, Q, Z)$ *satisfying logical-equivalence. Unless $\Sigma_3^p = \Pi_3^p$, there is no poly-size reduction from $K *_{\mathrm{D}} A$ into* NCIRC$(T; P, Q, Z)$ *satisfying logical-equivalence.*

Proof. In [28] we have shown that deciding whether $M \models K *_{\rm D} A$ is $\mathrm{P}^{\mathrm{NP}[O(\log n)]}$-complete and that the compilability level of this problem nu-comp-NP-hard. It can be easily shown that deciding whether $M \models \mathrm{NCIRC}(T; P, Q, Z)$ is coNP-complete and that the compilability level is nu-comp-coNP. If there exists a poly-time reduction from $K *_{\rm D} A$ into $\mathrm{NCIRC}(T; P, Q, Z)$ satisfying logical-equivalence we have that a $\mathrm{P}^{\mathrm{NP}[O(\log n)]}$-complete can be reduced to a coNP-complete one. As a consequence, $\mathrm{P}^{\mathrm{NP}[O(\log n)]} \subseteq$ coNP, and this implies that NP = coNP. Moreover, the existence of a poly-size reduction implies that nu-comp-NP $\subseteq$ nu-comp-coNP, and this implies that $\Sigma_3^p = \Pi_3^p$. □

Theorem 20. $(\neg P) *_{\rm W} T$ *is logically equivalent to* $\mathrm{CIRC}(T; P, \emptyset, \emptyset)$.

Proof. By definition of Winslett's and Satoh's operators, it follows that $(K *_{\rm S} A) \equiv (K *_{\rm W} A)$ when K is a complete formula (i.e. it has only one model). Since there are no fixed and varying letters, i.e. $P = V(T)$, we have that $\neg P$ has only one model. By Theorem 10, the thesis follows. □

Theorem 21. $\mathrm{CIRC}(T_{\rm W}; W, Y, X)$ *is query-equivalent to* $K *_{\rm W} A$.

Proof. Let γ be a formula such that $V(\gamma) \subseteq V(K) \cup V(A) = X$. We first show that $\mathrm{CIRC}(T_{\rm W}; W, Y, X) \models \gamma$ implies that $K *_{\rm W} A \models \gamma$. Assume that $\mathrm{CIRC}(T_{\rm W}; W, Y, X) \models \gamma$ and $K *_{\rm W} A \not\models \gamma$. Thus, there exists a model M_X of $K *_{\rm W} A$ such that $M_X \not\models \gamma$. Let M'_X be a model of K such that $M'_X \triangle M_X \in \mu(M'_X, A)$. We define $M_Y = \{y_i \mid x_i \in M'_X\}$ and $M_W = \{w_i \mid ((x_i \in M_X) \text{ and } (y_i \notin M_Y)) \text{ or } ((x_i \notin M_X) \text{ and } (y_i \in M_Y))\}$. Now, let $M = M_X \cup M_Y \cup M_W$. Obviously, it holds that $M \models T_{\rm W}$ and $M \not\models \gamma$. If M is a $(W, X \cup Y)$-minimal model the thesis follows, so assume that there exists a model N of $T_{\rm W}$ such that $N <_{(P,Z)} M$. Since N must agree with M on the letters of Y (i.e. $N_Y = N \cap Y = M_Y$) and it cannot contain more literals of W than M, we have that $N_W = (N \cap W) \subset M_W$. Hence, the distance between $N_X = N \cap X$ and M_Y is smaller than the distance between M_X and M_Y. Thus, M_X is not one of the models of A closest to the model M_Y of $K[X/Y]$. As a consequence, M_X is not a model of $K *_{\rm W} A$ and contradiction arises.

We now show that $K *_{\rm W} A \models \gamma$ implies that $\mathrm{CIRC}(T_{\rm W}; W, Y, X) \models \gamma$. Assume that $K *_{\rm W} A \models \gamma$ and $\mathrm{CIRC}(T_{\rm W}; W, Y, X) \not\models \gamma$. Thus, there exists a model M of $\mathrm{CIRC}(T_{\rm W}; W, Y, X)$ such that $M \not\models \gamma$. Let $M_X = M \cap X$, $M_Y = M \cap Y$ and $M_W = M \cap W$. We show that $M_X \models K *_{\rm W} A$. It immediately follows that $M_X \models A$, if M_X is one of the models of A closer to the model M_Y of $K[X/Y]$ the thesis follows, so assume to the contrary that there exists a $N_X \subseteq X$, different from M_X, such that $N_X \models A$ and the distance of M_X from M_Y is a strict superset of the distance of N_X from M_Y. Let $N_Y = M_Y$, $N_W = \{w_i \mid ((x_i \in N_X) \text{ and } (y_i \notin N_Y)) \text{ or } ((x_i \notin N_X) \text{ and } (y_i \in N_Y))\}$ and $N = N_X \cup N_Y \cup N_W$. Obviously N is a model of $T_{\rm W}$ and $N_W \subset M_W$. Hence, M is not a P, Z-minimal model of $T_{\rm W}$, and contradiction arises. □

Theorem 22. *Unless* Σ_2^p = coNP, *there is no poly-time reduction from* $K *_{\rm W} A$ *into* $\mathrm{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence. Unless* $\Sigma_4^p = \Pi_4^p$, *there is no poly-size reduction from* $K *_{\rm W} A$ *into* $\mathrm{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence.*

Proof. In [28] we have shown that the time complexity of deciding whether $M \models K *_{\mathrm{W}} A$ is Σ_2^p-complete and that the compilability level of this problem is nu-comp-Σ_2^p-complete. In [3] it is shown that deciding whether $M \models \mathrm{CIRC}(T; P, Q, Z)$ is coNP-complete and in [5] we proved that the compilability level of this problem is nu-comp-coNP-complete. As a consequence, if there exists a poly-time reduction from $K *_{\mathrm{W}} A$ into $\mathrm{CIRC}(T; P, Q, Z)$ satisfying logical-equivalence we have that a Σ_2^p-complete can be reduced to a coNP-complete one, and thus, $\Sigma_2^p \subseteq$ coNP. If there exists a poly-size reduction it follows that nu-comp-$\Sigma_2^p \subseteq$ nu-comp-coNP. By [5, Theorem 9] this implies that $\Sigma_4^p = \Pi_4^p$. □

Theorem 23. $\mathrm{CIRC}(T_{\mathrm{B}}; W, R, X \cup Y)$ *is query-equivalent to* $K *_{\mathrm{B}} A$.

Proof. We divide the proof into two cases, depending on whether $K \wedge A$ is consistent or not. First note that the above formula T_{B} can be rewritten as the disjunction of two formulae T_{B}^1 and T_{B}^2, where $T_{\mathrm{B}}^1 = K \wedge K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y))$ and $T_{\mathrm{B}}^2 = (Y \equiv R) \wedge K[X/Y] \wedge A \wedge (\neg W \equiv (X \equiv Y))$. If $K \wedge A$ is inconsistent, T_{B}^1 is identically false and, therefore, $T_{\mathrm{B}} = T_{\mathrm{B}}^2$. Note that T_{B}^2 is $T_{\mathrm{W}} \wedge (Y \equiv R)$. But $(Y \equiv R)$ does not impose any constraint on the variables in X, therefore it is query-equivalent to T_{W} on X. Since we have already shown that T_{W} is query-equivalent to circumscription the thesis follows.

Now, assume that $K \wedge A$ is consistent, thus $K *_{\mathrm{B}} A = K \wedge A$. Let γ be a formula such that $V(\gamma) \subseteq V(K) \cup V(A) = X$. We first show that $\mathrm{CIRC}(T_{\mathrm{B}}; W, R, X \cup Y) \models \gamma$ implies that $K \wedge A \models \gamma$. Assume that $\mathrm{CIRC}(T_{\mathrm{B}}; W, R, X \cup Y) \models \gamma$ and $K \wedge A \not\models \gamma$. Thus, there exists a model M_X of $K \wedge A$ such that $M_X \not\models \gamma$. We define $M_Y = \{y_i \mid x_i \in M_X\}$, $M_W = \emptyset$ and $M_R = \{r_i \mid y_i \in M_Y\}$. Now let $M = M_X \cup M_Y \cup M_W \cup M_R$. Obviously, M is a $(W, X \cup Y)$-minimal model of T_{B}. In fact, it satisfies T_{B} and, since $W = \emptyset$, there cannot be any smaller model. Therefore, $\mathrm{CIRC}(T_{\mathrm{B}}; W, R, X \cup Y) \not\models \gamma$, contradicting the hypothesis. It remains to show that $K \wedge A \models \gamma$ implies $\mathrm{CIRC}(T_{\mathrm{B}}; W, R, X \cup Y) \models \gamma$. Assume that $K \wedge A \models \gamma$ and $\mathrm{CIRC}(T_{\mathrm{B}}; W, R, X \cup Y) \not\models \gamma$. Thus, there exists a $(W, X \cup Y)$-minimal model M of T_{B} that does not satisfy γ. Let $M_X = M \cap X$, $M_Y = M \cap Y$, $M_W = M \cap W$ and $M_R = M \cap R$. We show that $M_X \models K \wedge A$. If $M \models T_{\mathrm{B}}^1$ the thesis immediately follows, so assume that $M \not\models T_{\mathrm{B}}^1$ and $M \models T_{\mathrm{B}}^2$. Since we know that $K \wedge A$ is consistent, let N_X be a model of $K \wedge A$, $N_Y = \{y_i \mid x_i \in N_X\}$, $N_R = N_W = \emptyset$. Now we define $N = N_X \cup N_Y \cup N_R \cup N_W$. Note that $N \models T_{\mathrm{B}}^1$, and therefore, $N \models T_{\mathrm{B}}$. Furthermore, $N <_{(W, X \cup Y)} M$, thus contradicting the hypothesis. □

Theorem 24. *Unless* $\Sigma_2^p =$ coNP, *there is no poly-time reduction from* $K *_{\mathrm{B}} A$ *into* $\mathrm{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence. Unless* $\Sigma_4^p = \Pi_4^p$, *there is no poly-size reduction from* $K *_{\mathrm{B}} A$ *into* $\mathrm{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence.*

Proof. In [28] we have shown that the time complexity of deciding whether $M \models K *_{\mathrm{B}} A$ is Σ_2^p-complete and that the compilability level of this problem is nu-comp-Σ_2^p-complete. In [3] it is shown that deciding whether $M \models \mathrm{CIRC}(T; P, Q, Z)$ is coNP-complete and in [5] we proved that the compilability level of this problem is nu-comp-coNP-complete. As a consequence, if there exists a poly-time reduction from $K *_{\mathrm{B}} A$ into

$\text{CIRC}(T; P, Q, Z)$ satisfying logical-equivalence we have that a Σ_2^p-complete can be reduced to a coNP-complete one, and thus, $\Sigma_2^p \subseteq$ coNP. If there exists a poly-size reduction it follows that nu-comp-$\Sigma_2^p \subseteq$ nu-comp-coNP. By [5, Theorem 9] this implies that $\Sigma_4^p = \Pi_4^p$. □

Theorem 25. *$(\neg P) *_{\mathrm{F}} T$ is logically equivalent to* $\text{NCIRC}(T; P, \emptyset, \emptyset)$.

Proof. By definition of Forbus' and Dalal's operators, it follows that $K *_{\mathrm{D}} A \equiv K *_{\mathrm{F}} A$ when K is a complete formula (i.e. it has only one model). Since there are no fixed and varying letters, i.e. $P = V(T)$, we have that $\neg P$ has only one model. By Theorem 15 the thesis follows. □

Theorem 26. $\text{CIRC}(T_{\mathrm{F}}; V, Y, X \cup W)$ *is query-equivalent to* $K *_{\mathrm{F}} A$.

Proof. Let γ be a formula such that $V(\gamma) \subseteq V(K) \cup V(A) = X$. We first show that $\text{CIRC}(T_{\mathrm{F}}; V, Y, X \cup W) \models \gamma$ implies that $K *_{\mathrm{F}} A \models \gamma$. Assume that $\text{CIRC}(T_{\mathrm{F}}; V, Y, X \cup W) \models \gamma$ and $K *_{\mathrm{F}} A \not\models \gamma$. Thus, there exists a model M_X of $K *_{\mathrm{F}} A$ such that $M_X \not\models \gamma$. Let M'_X be a model of K such that $\text{card}(M'_X \triangle M_X) \in k_{M'_X, A})$. We define $M_Y = \{y_i \mid x_i \in M'_X\}$ and $M_W = \{w_i \mid ((x_i \in M_X) \text{ and } (y_i \notin M_Y)) \text{ or } ((x_i \notin M_X) \text{and } (y_i \in M_Y))\}$ and $M_V = \{v_i \mid EQ(M_W, V) \wedge BEGIN(V)\}$. Now, let $M = M_X \cup M_Y \cup M_W \cup M_V$. Obviously, it holds that $M \models T_{\mathrm{F}}$ and $M \not\models \gamma$. If M is a $(V, X \cup W)$-minimal model the thesis follows, so assume that there exists a model N of T_{F} such that $N <_{(P,Z)} M$. Let $N_X = N \cap X$, $N_Y = N \cap Y$ and $N_W = N \cap W$. Since $N <_{(P,Z)} M$ we have $N_Y = M_Y$ and $N_V \subset M_V$. But this also implies that $|N_W| < |M_W|$ and that the distance between $N_X = N \cap X$ and M_Y is (cardinality) smaller than the distance between M_X and M_Y. Thus, M_X is not one of the models of A closest to the model M_Y of $K[X/Y]$. As a consequence, M_X is not a model of $K *_{\mathrm{F}} A$ and contradiction arises.

We now show that $K *_{\mathrm{F}} A \models \gamma$ implies that $\text{CIRC}(T_{\mathrm{F}}; V, Y, X \cup W) \models \gamma$. Assume that $K *_{\mathrm{F}} A \models \gamma$ and $\text{CIRC}(T_{\mathrm{F}}; V, Y, X \cup W) \not\models \gamma$. Thus, there exists a model M of $\text{CIRC}(T_{\mathrm{F}}; V, Y, X \cup W)$ such that $M \not\models \gamma$. Let $M_X = M \cap X$, $M_Y = M \cap Y$, $M_W = M \cap W$ and $M_V = M \cap V$. We show that $M_X \models K *_{\mathrm{F}} A$. It immediately follows that $M_X \models A$, if M_X is one of the models of A closer to the model M_Y of $K[X/Y]$ the thesis follows, so assume to the contrary that there exists a $N_X \subseteq X$, different from M_X, such that $N_X \models A$ and $|N_X \triangle M_Y| < |M_X \triangle M_Y|$. Let $N_Y = M_Y$, $N_W = \{w_i | ((x_i \in N_X) \text{ and } (y_i \in N_Y)) \text{ or } ((x_i \notin N_X) \text{ and } (y_i \notin N_Y))\}$, $N_V = \{v_i \mid EQ(N_W, V) \wedge BEGIN(V)\}$ and $N = N_X \cup N_Y \cup N_W \cup N_V$. Obviously N is a model of T_{F} and $N_V \subset M_V$. Hence, M is not a $(V, X \cup W)$-minimal model of T_{F}, hence contradiction arises. □

Theorem 27. *Unless $\Sigma_2^p =$ coNP, there is no poly-time reduction from $K *_{\mathrm{F}} A$ into* $\text{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence. Unless $\Sigma_4^p = \Pi_4^p$, there is no poly-size reduction from $K *_{\mathrm{F}} A$ into* $\text{CIRC}(T; P, Q, Z)$ *satisfying logical-equivalence.*

Proof. In [28] we have shown that the time complexity of deciding whether $M \models K *_{\mathrm{F}} A$ is Σ_2^p-complete and that the compilability level of this problem is nu-comp-Σ_2^p-complete. In [3] it is shown that deciding whether $M \models \text{CIRC}(T; P, Q, Z)$ is coNP-complete and in [5] we proved that the compilability level of this problem is nu-comp-coNP-complete. As a consequence, if there exists a poly-time reduction from $K *_F A$ into $\mathrm{CIRC}(T; P, Q, Z)$ satisfying logical-equivalence we have that a Σ_2^p-complete can be reduced to a coNP-complete one, and thus, $\Sigma_2^p \subseteq$ coNP. If there exists a poly-size reduction it follows that nu-comp-$\Sigma_2^p \subseteq$ nu-comp-coNP. By [5, Theorem 9] this implies that $\Sigma_4^p = \Pi_4^p$. $\square$

Theorem 29. *$(\neg P) *_{\mathrm{SBR}} T$ is logically equivalent to $\mathrm{CIRC}(T; P, \emptyset, Z)$ and $\mathrm{CIRC}(T_{\mathrm{SBR}}; Y, \emptyset, X)$ is query-equivalent to $K *_{\mathrm{SBR}} A$.*

Proof. Eiter and Gottlob have shown in [13] that $K *_{\mathrm{SBR}} A = K *_{\mathrm{S}} A$ if K is a consistent set of literals. Therefore, the first reduction trivially holds.

Coming to the second reduction, we remind that SBR finds the maximal subsets K' of K such that K' and A are not contradictory, whereas the circumscription of a formula takes only the models with a maximal set of false variables. Let Q be a formula on the alphabet X such that $K *_{\mathrm{SBR}} A \not\models Q$. Therefore, there exists a maximal subset F of $\{f_1, \ldots, f_m\}$ consistent with A such that $F \wedge A \not\models Q$. Let M be a model of $F \wedge A$. We denote with $Y_F = \{y_i \mid f_i \notin F\}$ and $N = M \cup Y_F$. It immediately occurs that $N \not\models Q$. We now show that N is a minimal model of T_{SBR}. Assume N not minimal, then there exists a model N' of $A \wedge (y_1 \equiv \neg f_1) \wedge \cdots \wedge (y_m \equiv \neg f_m)$ such that $N' \cap Y \subset N \cap Y$. Let $F' = \{f_i \mid y_i \notin Y\}$, clearly $N' \models A \wedge F'$ and $F' \supset F$. As a consequence, F is not a maximal subset of $\{f_1, \ldots, f_m\}$ consistent with A, thus contradicting the assumptions.

The other direction is similar. Let Q be a formula on the alphabet X such that $\mathrm{CIRC}(T_{\mathrm{SBR}}; Y, \emptyset, X) \not\models Q$. Therefore, there exists a minimal model N of T_{SBR} such that $N \not\models Q$. Let $F = \{f_i \mid y_i \notin N\}$ and $M = N \cap X$. Obviously, $M \not\models Q$. We now show that F is a maximal subset of $\{f_1, \ldots, f_m\}$ consistent with A. Assume F is not maximal, then there exists an $F' \supset F$ consistent with A. Let $N' = M \cup \{y_i \mid f_i \notin F'\}$. Clearly, N' is smaller of N and $N' \models T_{\mathrm{SBR}}$. As a consequence, N is not a minimal model of T_{SBR}, thus contradicting the assumptions. $\square$

Theorem 31. *$w \wedge \mathrm{CIRC}(T_{\mathrm{AGM}}; \{w\}, X, Y)$ is query-equivalent to $K *_{\mathrm{AGM}} A$.*

Proof. Let γ be a formula such that $V(\gamma) \subseteq V(K) \cup V(A) = X$. We first show that $K *_{\mathrm{AGM}} A \models \gamma$ implies that $w \wedge \mathrm{CIRC}(T_{\mathrm{AGM}}; \{w\}, X, Y) \models \gamma$. Assume that $K *_{\mathrm{AGM}} A \models \gamma$ and $w \wedge \mathrm{CIRC}(T_{\mathrm{AGM}}; \{w\}, X, Y) \not\models \gamma$. Thus, there exists a model M of $w \wedge \mathrm{CIRC}(T_{\mathrm{AGM}}; \{w\}, X, Y)$ such that $M \not\models \gamma$. Let $M_X = M \cap X$, we show that $M_X \models K *_{\mathrm{AGM}} A$. It immediately follows that $M_X \models A$, so if M_X is $\leqslant_K$-minimal the thesis follows, so assume, on the contrary, that there exists a model N_X such that $N_X <_K M_X$. Let $N_Y = \{y_i \mid x_i \in N_X\}$ and $N = N_X \cup N_Y$. Since N_X is a model of A and $N_X <_K M_X$, we have that $N_Y \models A[X/Y]$ and $N \models T_{\mathrm{AGM}}$. Note that $N <_{(P,Z)} M$, since $w \in M$ while $w \notin M$. As a consequence, M is not a $(\{w\}, Y)$-minimal model of T_{AGM}, thus contradicting the hypothesis.

We now show that $w \wedge \mathrm{CIRC}(T_{\mathrm{AGM}}; \{w\}, X, Y) \models \gamma$ implies that $K *_{\mathrm{AGM}} A \models \gamma$. Assume that $w \wedge \mathrm{CIRC}(T_{\mathrm{AGM}}; \{w\}, X, Y) \models \gamma$ and $K *_{\mathrm{AGM}} A \not\models \gamma$. Thus, there exists a model M_X of $K *_{\mathrm{AGM}} A$ such that $M_X \not\models \gamma$. Let $M_Y = \{y_i \mid x_i \in M_X\}$ and $M =$

$M_X \cup M_Y \cup \{w\}$. Clearly, $M \not\models \gamma$, $M \models w$ and $M \models T_{AGM}$. We need to show that M is $(\{w\}, Y)$-minimal. Assume, on the contrary, that there exists a model N of T_{AGM} such that $N <_{(P,Z)} M$. By definition of $(\{w\}, Y)$-minimality we have $w \notin N$ and $N_X = M_X$. Since $N \models \neg w$ we also have that $N \models LESS_K(X, Y)$. As a consequence, we contradict the assumption that the model M_X is $\leqslant_K$-minimal, since the model $N = M_X \cup N_Y$ satisfies $LESS_K(X, Y)$ and therefore, the model $L = \{x_i \mid y_i \in N_Y\}$ is such that $L <_K M_X$. $\square$

Theorem 32. *$Y *_S A'$ is query-equivalent to $K *_{SBR} A$.*

Proof. Let γ be a formula such that $V(\gamma) \subseteq V(K) \cup V(A) = X$. We first show that $K *_{SBR} A \models \gamma$ implies that $Y *_S A' \models \gamma$. Assume that $K *_{SBR} A \models \gamma$ and $Y *_S A' \not\models \gamma$. Thus, there exists a model M of $Y *_S A'$ such that $M \not\models \gamma$. Let $M_X = M \cap X$, we show that $M_X \models K *_{SBR} A$. Let $F = \{f_i \in K \mid M \models f_i\}$, we must show that $F \cup \{A\} \in W(K, A)$. Assume, on the contrary, that there exists a set $F' \subset K$ such that $F \subset F'$ and $F' \cup \{A\}$ is consistent. Let N_X be a model of $F' \cup \{A\}$, $N_Y = \{y_i \mid f_i \in F'\}$ and $N = N_X \cup N_Y$. By construction, $N \models A'$ and $(N_Y \triangle Y) \subset (M_Y \triangle Y)$. Therefore, M_Y is not a model of $Y *_S A'$, thus contradicting the hypothesis.

We now show that $Y *_S A' \models \gamma$ implies that $K *_{SBR} A \models \gamma$. Assume that $Y *_S A' \not\models \gamma$ and $K *_{SBR} A \not\models \gamma$. Thus, there exists a model M_X of $K *_{SBR} A$ such that $M_X \not\models \gamma$. Let $F = \{f_i \in K \mid M \models f_i\}$, $M_Y = \{y_i \mid f_i \in F\}$ and $M = M_X \cup M_Y$. Clearly, $M \not\models \gamma$, we need to show that $M \models Y *_S A'$. Assume, on the contrary, that there exists a model N of $Y *_S A'$ such that $(N \triangle Y) \subset (M \triangle Y)$. Let $F' = \{f_i \mid y_i \in N\}$. Clearly, $N \models F'$ and $N \models A$. Moreover, since the formulae in A and F' only uses the letters in X, $N_X \models F' \cup \{A\}$. Therefore, F is not a maximally consistent subset of K, thus contradicting the hypothesis. $\square$

Theorem 33. *$W *_{SBR} A''$ is query-equivalent to $K *_S A$.*

Proof. Let γ be a formula such that $V(\gamma) \subseteq V(K) \cup V(A) = X$. We first show that $K *_S A \models \gamma$ implies that $W *_{SBR} A'' \models \gamma$. Assume that $K *_S A \models \gamma$ and $W *_{SBR} A'' \not\models \gamma$. Thus, there exists a model M of $W *_{SBR} A''$ such that $M \not\models \gamma$. Let $M_X = M \cap X$, $M_Y = M \cap Y$ and $M_W = M \cap W$, we show that $M_X \models K *_S A$. Clearly, $M_X \models A$, we must show that there exists a model M'_X of K such that $(M'_X \triangle M_X) \in \delta(K, A)$. Assume, to the contrary, that there exists a model N_X of A and a model N'_X of K such that $(N'_X \triangle N_X) \subset (M'_X \triangle M_X)$. let $N_Y = \{y_i \mid x_i \in N'_X\}$, $N_W = \{w_i \mid x_i \in N_X$ and $y_i \in N_Y$or $x_i \notin N_X$ and $y_i \notin N_Y\}$ and $N = N_X \cup N_Y \cup N_W$. Obviously, $N \models A''$ and $M_W \subset N_W$. Hence, M is not a model of $W *_{SBR} A''$, thus contradicting the hypothesis.

We now show that $W *_{SBR} A'' \models \gamma$ implies that $K *_S A \models \gamma$. Assume that $W *_{SBR} A'' \models \gamma$ and $K *_S A \not\models \gamma$. Thus, there exists a model M_X of $K *_S A$ such that $M \not\models \gamma$. Let M'_X be a model of K such that $(M'_X \triangle M_X) \in \delta(K, A)$, $M_Y = \{y_i \mid x_i \in M'_X\}$, $M_W = \{w_i \mid x_i \in M_X$ and $y_i \in M_Y$ or $x_i \notin M_X$ and $y_i \notin M_Y\}$ and $M = M_X \cup M_Y \cup M_W$. Clearly, $M \not\models \gamma$, we need to show that $M \models W *_{SBR} A''$. Assume, on the contrary, that there exists a model N of $W *_{SBR} A''$ such that $M_W \subset N_W$. That is, there is a larger set of the letters in W that is consistent with A''. Let $N_X = N \cap X$ and $N'_X = \{x_i \mid y_i \in N\}$. By construction, we have that $(N'_X \triangle N_X) \subset (M'_X \triangle M_X)$. Therefore, M is not a model of $K *_S A$, thus contradicting the hypothesis. □

Theorem 34. *$(W \cup Y \cup Z) *_{SBR} F$ is query-equivalent to $K *_W A$.*

Proof. Let γ be a formula such that $V(\gamma) \subseteq V(K) \cup V(A) = X$. We first show that $K *_W A \models \gamma$ implies that $(W \cup Y \cup Z) *_{SBR} F \models \gamma$. Assume that $K *_W A \models \gamma$ and $(W \cup Y \cup Z) *_{SBR} F \not\models \gamma$. Thus, there exists a model M of $(W \cup Y \cup Z) *_{SBR} F$ such that $M \not\models \gamma$. Let $M_X = M \cap X$, $M'_X = \{x_i \mid y_i \in M\}$ we show that $(M'_X \triangle M_X) \in \mu(M'_X, A)$ and, therefore $M_X \models K *_W A$. Assume, on the contrary, that there exists a model N_X of A such that $(M'_X \triangle N_X) \subset (M'_X \triangle M_X)$. We define $N_Y = M_Y$, $N_Z = M_Z$, $N_W = \{w_i \mid x_i \in N_X \text{ and } y_i \in N_Y \text{ or } x_i \notin N_X \text{ and } y_i \notin N_Y\}$ and $N = N_X \cup N_Y \cup N_Z \cup N_W$. Obviously, $N \models F$ and $(M_W \cup M_Y \cup M_Z) \subset (N_W \cup N_Y \cup N_Z)$. Hence, M is not a model of $(W \cup Y \cup Z) *_{SBR} F$, thus contradicting the hypothesis.

We now show that $(W \cup Y \cup Z) *_{SBR} F \models \gamma$ implies that $K *_W A \models \gamma$. Assume that $(W \cup Y \cup Z) *_{SBR} F \models \gamma$ and $K *_W A \not\models \gamma$. Thus, there exists a model M_X of $K *_W A$ such that $M \not\models \gamma$. Let M'_X be a model of K such that $(M'_X \triangle M_X) \in \mu(M'_X, A)$, $M_Y = \{y_i \mid x_i \in M'_X\}$, $M_Z = \{z_i \mid y_i \notin M_Y\}$, $M_W = \{w_i \mid x_i \in M_X \text{ and } y_i \in M_Y \text{ or } x_i \notin M_X \text{ and } y_i \notin M_Y\}$ and $N = M_X \cup M_Y \cup M_W$. Obviously, $M \models F$, we need to show that $M \models (W \cup Y \cup Z) *_{SBR} F$. Assume, on the contrary, that there exists a model N of $(W \cup Y \cup Z) *_{SBR} F$ such that $(M \cap (W \cup Y \cup Z)) \subset (N \cap (W \cup Y \cup Z))$. Let $N'_X = \{y_i \mid x_i \in N_Y\}$. Since N cannot contain more literals of Y or Z without violating the constraint $\neg Y \vee \neg Z$, it follows that $(M \cap W) \subset (N \cap W)$ and, therefore, $(N'_X \triangle N_X) \subset (M'_X \triangle M_X)$. Thus, M_X is not a model of $K *_W A$, thus contradicting the hypothesis. □

Theorem 35. *Unless $\Sigma_2^p = \text{coNP}$, there is no poly-time reduction from $K *_S A$ $(K *_W A)$ into $K' *_{SBR} A'$ satisfying logical-equivalence. Unless $\Sigma_4^p = \Pi_4^p$, there is no poly-size reduction from $K *_S A$ $(K *_W A)$ into $K' *_{SBR} A'$ satisfying logical-equivalence.*

Proof. In [28] we have shown that the time complexity of deciding whether $M \models K *_S A$ and $M \models K *_W A$ is Σ_2^p-complete and that the compilability level of this problem is nu-comp-Σ_2^p-complete, while the time complexity of deciding whether $M \models K *_{SBR} A$ is coNP-complete and that the compilability level of this problem is nu-comp-coNP-complete. As a consequence, if there exists a poly-time reduction from $K *_S A$ $(K *_W A)$ into $K' *_{SBR} A'$ satisfying logical-equivalence we have that a Σ_2^p-complete can be reduced to a coNP-complete one, and thus, $\Sigma_2^p \subseteq \text{coNP}$. If there exists a poly-size reduction it follows that nu-comp-$\Sigma_2^p \subseteq$ nu-comp-coNP. By [5, Theorem 9] this implies that $\Sigma_4^p = \Pi_4^p$. □

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