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ON SOME SEQUENCING PROBLEMS IN FINITE GROUPS

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A finite group is called Z-sequenceable if its non-identity elements can be listed $x_1, x_2, \ldots, x_n$ so that $x_i x_{i+1} = x_{i+1} x_i$ for $i = 1, 2, \ldots, n-1$. Various necessary and sufficient conditions are determined for such sequencings to exist. In particular, it is proved that if $n \geq 3$, then the symmetric group S_n is not Z-sequenceable.

1. Introduction

Let G be a group and suppose that $\Gamma(G)$ is the undirected graph whose vertices are the non-identity elements of G, with elements x and y joined by an edge if and only if $xy = yx$. Nakanishi [6] considered classes of groups for which $\Gamma(G)$ is connected, and showed that the symmetric group $S_n (n \geq 3)$ is connected if and only if n and $n-1$ are composite, while the alternating group $A_n (n \geq 4)$ is connected if and only if n, $n-1$, and $n-2$ are composite.

In this paper, we determine some necessary and sufficient conditions on G in order that $\Gamma(G)$ possess Euler and Hamiltonian paths and circuits. For example, we prove that if $n \geq 3$, $\Gamma(S_n)$ has no Hamiltonian path, while for $n \geq 4$, $\Gamma(A_n)$ has no such path.

If G is countable, the existence of a Hamiltonian path in $\Gamma(G)$ is equivalent to the existence of a sequencing $\{x_n\}$ of the non-identity elements of G such that $x_i x_{i+1} = x_{i+1} x_i$ for all i. (If G is infinite, this sequence might be two-way infinite.) If such a sequencing exists, we shall say that G is Z-sequenceable. Furthermore, we call the finite group G strongly Z-sequenceable if G has a Z-sequencing $x_1, x_2, \ldots, x_m$ such that $x_m x_1 = x_1 x_m$. This is clearly equivalent to the existence of a Hamiltonian circuit in $\Gamma(G)$.

We note that the sequencings of Gordon [3] and Friedlander [2] can also be interpreted as Hamiltonian paths, while the sequencings of Ringel [7], dealing with map-coloring problems, can similarly be viewed as Hamiltonian circuits.

It is clear that all finite abelian groups are strongly Z-sequenceable, while it can be shown that the group $S_3 \times \mathbf{Z}_4$, of order 24, is Z-sequenceable, but not strongly Z-sequenceable. Finally, we remark that if G is Z-sequenceable, then certainly $\Gamma(G)$ is connected. In what follows, we will assume all groups finite unless otherwise stated.

2. Z-Sequenceable groups

We first show

Theorem 2.1. *If $n \geqslant 3$, the dihedral group D_n is not Z-sequenceable.*

Proof. Write $D_n = \langle a, b \rangle$ subject to the relations $a^n = b^2 = 1$, $bab = a^{-1}$. Recall that $Z(D_n) = \{1\}$ or $\mathbf{Z}_2$ according as n is odd or even, and that if $x \in D_n$ is a non-central involution, then the centralizer of x is

$$C(x) = \begin{cases} \langle x \rangle & \text{if } n \text{ is odd,} \\ \langle a^{n/2}, x \rangle & \text{if } n \text{ is even.} \end{cases}$$

Assume on the contrary that D_n is Z-sequenceable. Let x and y be any two non-commuting involutions, where for definiteness x precedes y in some Z-sequencing of G. Let z be the first element following x that does not commute with x. This forces the element immediately to the left of z to be central. Thus any two non-commuting involutions are separated by a central element, and since $D_n (n \neq 4)$ contains at least three pairwise non-commuting involutions, we have a contradiction, and the result follows. (It is easily seen by inspection that D_4 is not Z-sequenceable.)

Theorem 2.2. *If $|G/Z(G)| - 1 \leqslant |Z(G)|$, then G is Z-sequenceable, while if $|G/Z(G)| \leqslant |Z(G)|$, then G is strongly Z-sequenceable.*

Proof. Since elements in the same coset of $Z(G)$ commute, we can obtain a Z-sequencing of G by successively listing the elements of each coset, separating elements of distinct cosets by central elements.

For example, the group $G = D_4 \times \mathbf{Z}_2$, of order 16, is strongly Z-sequenceable since $|Z(G)| = 4$. The statement of the theorem is as strong as possible since D_4 is not Z-sequenceable, and $|D_4/Z(D_4)| - 2 = |Z(D_4)|$.

If $g_1, g_2, \ldots, g_n$ is a Z-sequencing of G, then the sequencing $1, g_1, g_2, \ldots, g_n$ is called an augmented Z-sequencing of G.

Theorem 2.3. *If G and H are Z-sequenceable, then so is $G \times H$.*

Proof. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be augmented Z-sequencings of G and

H respectively (so $x_1 = y_1 = 1$). Then an augmented Z-sequencing of $G \times H$ is given by $z_{11}, z_{12}, \ldots, z_{mn}$, where $z_{ij} = x_i y_j$. By deleting $z_{11} = 1$, we obtain a Z-sequencing of $G \times H$.

Before studying the existence of Hamiltonian paths in $\Gamma(S_n)$ and $\Gamma(A_n)$, we need several lemmas.

Lemma 2.4. *Let G be a group and $S_i = \langle a_i \rangle$, $i = 1, 2, \ldots, r$, a collection of distinct cyclic subgroups such that $C(S_i) = S_i$ for all i. Let $S = \bigcup_{i=1}^{r} S_i$, and set $N = \{x \in S \mid \langle x \rangle \neq S_i$ for any $i\}$. Then if $|N| < r$, G is not Z-sequenceable.*

Proof. Assume on the contrary that G has a Z-sequencing Σ. Fix i and j, and suppose without loss of generality that a_i precedes a_j in Σ. Let y be the first element following a_i that is not in S_i. (As $a_j \notin S_i$, clearly such a y exists.) Then the element to the immediate left of y must be in N, since elements of $S_i - N$ commute only with elements of S_i. Thus there is a non-identity element of N between a_i and a_j for all i, j, a contradiction, and the result follows.

Lemma 2.5. *If $n \geq 1$, then $n! \geq \mathrm{e}(n/\mathrm{e})^n$.*

Proof. From the graph of $y = \log x$, we have $\log 2 + \log 3 + \cdots + \log n \geq \int_1^n \log x \, \mathrm{d}x$. But $\int_1^n \log x \, \mathrm{d}x = n \log n - (n-1)$. Thus $\log(2 \cdot 3 \cdot \ldots \cdot n) \geq n \log n - (n-1)$, and exponentiating gives $n! \geq \mathrm{e}(n/\mathrm{e})^n$.

Lemma 2.6.

(i) *The number of elements of S_n which are powers of n-cycles but not themselves n-cycles is*

$$\sum_{\substack{d \mid n \\ d < n}} \frac{n!}{(n/d)!\, d^{n/d}}.$$

(ii) *The number of elements of S_n $(n \geq 3)$ which are powers of $(n-1)$-cycles, but not themselves $(n-1)$-cycles is*

$$\sum_{\substack{d \mid n-1 \\ d < n-1}} \frac{n!}{((n-1)/d)!\, d^{(n-1)/d}} - (n-1).$$

Proof. (i) Each element in question is a product of n/d d-cycles for some proper divisor d of n. But for fixed d, it is well-known [8; 299] that the number of such elements is

$$\frac{n!}{d^{n/d}(n/d)!}.$$

The result follows by summing over the proper divisors of n.

(ii) As in (i), each element in question is a product of $(n-1)/d$ d-cycles for some proper divisor d of $n-1$. The argument is the same, except that we must subtract $n-1$ to account for duplication arising when $d=1$.

Theorem 2.7. *If $n \geqslant 3$, then S_n is not Z-sequenceable.*

Proof. *Case* 1: n odd. The number of n-cycles in S_n is $(n-1)!$, and these n-cycles generate $(n-1)!/\varphi(n)$ distinct cyclic subgroups, where φ is the Euler function. Recall that an n-cycle commutes only with its powers. By Lemma 2.6, the number of elements in S_n which are powers of n-cycles, but not themselves n-cycles is

$$\chi = \sum_{\substack{d \mid n \\ d<n}} \frac{n!}{(n/d)!\, d^{n/d}}.$$

By Lemma 2.4, it suffices to prove that $\chi < (n-1)!/\varphi(n)$. Now by Lemma 2.5 (replacing n by n/d), we have $(n/d)!\, d^{n/d} \geqslant \mathrm{e}(n/\mathrm{e})^{n/d}$, hence

$$\chi \leqslant \frac{n!}{\mathrm{e}} \sum_{\substack{d \mid n \\ d<n}} \left(\frac{\mathrm{e}}{n}\right)^{n/d}.$$

Since n is odd, this latter expression is less than

$$\frac{n!}{\mathrm{e}}\left(\frac{\mathrm{e}}{n}\right)^{3} \frac{1}{1-(\mathrm{e}/n)} = \frac{n!\,\mathrm{e}^2}{n^2(n-\mathrm{e})}.$$

It suffices then to show that $\varphi(n) \leqslant n(n-\mathrm{e})/\mathrm{e}^2$. If $n>10$, $(n-\mathrm{e})/\mathrm{e}^2>1$, so $\varphi(n) \leqslant n(n-\mathrm{e})/\mathrm{e}^2$. If $n=5$, 7 or 9, it is straightforward to verify directly that $\chi < (n-1)!/\varphi(n)$, while if $n=3$, the theorem is true by inspection.

Case 2: n even. The number of $(n-1)$-cycles in S_n is $n(n-2)!$, and these $(n-1)$-cycles generate $n(n-2)!/\varphi(n-1)$ distinct cyclic subgroups. Recall that an $(n-1)$-cycle commutes only with its powers. By Lemma 2.6, the number of elements in S_n which are powers of $(n-1)$-cycles, but not themselves $(n-1)$- cycles is

$$\sigma = \sum_{\substack{d \mid n-1 \\ d<n-1}} \frac{n!}{((n-1)/d)!\, d^{(n-1)/d}} - (n-1).$$

So by Lemma 2.4, it suffices to show that $\sigma < n(n-2)!/\varphi(n-1)$. This certainly holds if $\sigma + (n-1) < n(n-2)!/\varphi(n-1)$, or equivalently, if

$$\sum_{\substack{d \mid n-1 \\ d<n-1}} \frac{(n-1)!}{((n-1)/d)!\, d^{(n-1)/d}} < \frac{(n-2)!}{\varphi(n-1)}.$$

But this follows immediately from Case 1, since $n-1$ is odd.

Corollary. *If $n \geqslant 4$, then A_n is not Z-sequenceable.*

Proof. If n is odd, A_n contains all n-cycles, and the proof of Case 1 applies. If n is even, A_n contains all $(n-1)$-cycles, and the result follows from Case 2.

Using Theorems 2.2 and 2.3, we can construct infinitely many classes of finite Z-sequenceable groups. Moreover, it can be shown that if G is countably infinite, and $C(x) \cap C(y)$ is infinite for all $x, y \in G$, then G is Z-sequenceable. In particular, a countably infinite FC-group (a group in which each element has finitely many conjugates) is Z-sequenceable, as is the group S_∞ of all permutations of a countably infinite set moving only finitely many elements. There are many ways to extend the concept of Z-sequenceability to uncountable groups G. For example, we can call G Z-sequenceable if its non-identity elements can be well-ordered so that each element commutes with its successor.

We conclude this section by mentioning several unanswered questions:

(i) Is there a finite Z-sequenceable group G with $Z(G) = 1$?

(ii) For a countably infinite group G, are the concepts of one-way and two-way Z-sequenceability distinct? That is, is there a G which a one-way Z-sequencing, but not a two-way sequencing (and vice versa).

(iii) Can the direct product of two non-Z-sequenceable groups be Z-sequenceable?

3. Euler paths and circuits

We turn now to the existence of Euler paths and circuits. Recall that an undirected graph Γ has an Euler path (circuit) if and only if it is connected and has no (for a circuit) or exactly two vertices of odd degree. We thus have the following.

Theorem 3.1. *If $|G| > 2$, and $\Gamma(G)$ is connected, then $\Gamma(G)$ has an Euler circuit if and only if $|C(x)|$ is even for all $x \in G$.*

If G has non-trivial center, then clearly $\Gamma(G)$ is connected. It follows that if $|Z(G)|$ is even, then $\Gamma(G)$ has an Euler circuit. In particular, if G is abelian and $|G| > 2$, then $\Gamma(G)$ has an Euler circuit if and only if $|G|$ is even. On the other hand, if $n \geq 3$, then neither S_n nor A_n has an Euler circuit.

In order to determine the existence of Euler paths, we need two group-theoretical lemmas.

Lemma 3.2. (i) *If G is non-abelian, and τ is an automorphism of order 2, then τ fixes some non-identity element.*

(ii) *If H is a p-group, and $\{1\} \neq K \lhd | H$, then $K \cap Z(H) \neq \{1\}$.*

(iii) *If H is a group of odd order, then no non-identity element is conjugate to its inverse.*

(iv) *Suppose H is a subgroup of G, $x \in G$. If $[x, H] \subseteq Z(H)$, then $C_H(x) \lhd | H$. (Here $[x, H]$ denotes the subgroup of G generated by all $x^{-1}h^{-1}xh$, $h \in H$.)*

The proofs are straightforward and will be omitted. (See, e.g., [4, p. 336].)

Lemma 3.3. *Let G be a finite group having exactly two non-identity elements with odd-order centralizer. Then either $G = \mathbf{Z}_3$ or $G = S_3$.*

Proof. Assume $G \neq \mathbf{Z}_3$; then clearly $|G|$ is even. Suppose $a \in G$, and $H = C_G(a)$ has odd order. Since any generator of $\langle a \rangle$ also has odd-order centralizer, we see that ord $a = 3$, and $\langle a \rangle \triangleleft | G$. If $1 \neq b \in H$ has order m and $3 \nmid m$, then $1 \neq (ab)^m = a^m \in \langle a \rangle$, so $C(ab) \subseteq C(a)$. Thus $|C(ab)|$ is odd, a contradiction. Hence H is a 3-group. By the N/C Theorem [8, p. 50], $N_G\langle a \rangle / C_G\langle a \rangle \subseteq \mathrm{Aut}\langle a \rangle = \mathbf{Z}_2$, and since $|G|$ is even, we deduce that $|G| = 2 \cdot 3^r$ for some $r \geq 1$.

Let $x \in G$ have order 2, so each element of $G - H$ is of the form xc, $c \in H$. By hypothesis, for all $b \in H - \langle a \rangle$, there exists $c \in H$ such that $(xc)^{-1} b (xc) = b$; that is, $x^{-1} bx = cbc^{-1}$. Thus conjugation by x stabilizes the conjugacy classes of $H - \langle a \rangle$. It also follows immediately that $\langle a \rangle = Z(H)$.

If there exists $b \in H$ such that $[x, b] = x^{-1} b^{-1} xb \notin \langle a \rangle$, then $x^{-1}[x, b] x = [x, b]^{-1}$ is conjugate in H to $[x, b]$, contradicting Lemma 3.2 (iii). Thus for all $b \in H$, $[x, b] \in \langle a \rangle$, so by Lemma 3.2 (iv), $C_H(x) \triangleleft | H$. But $C_H(x)$ is the set of fixed points of the automorphism of H induced by x, so by Lemma 3.2 (i), (ii), we conclude that H is abelian. Since $\langle a \rangle = Z(H)$, we have $H = \mathbf{Z}_3$, and $G = S_3$.

Since $\Gamma(S_3)$ is not connected, Lemma 3.3 gives us the following:

Theorem 3.4. *If G is finite, then $\Gamma(G)$ possesses an Euler path that is not a circuit if and only if $G = \mathbf{Z}_3$.*

4. Anti-Z-sequenceable groups

If G is a group (finite or infinite), we denote by $\bar{\Gamma}(G)$ the complement of $\Gamma(G)$. Thus two vertices x and y of $\bar{\Gamma}(G)$ are joined by an edge if and only if $xy \neq yx$. Since G is never the union of two proper subgroups, it follows that if x and y are non-central elements, then $C(x) \cup C(y) \neq G$. Hence x and y are connected by a path in $\bar{\Gamma}(G)$.

We will say that a countable group G is AZ-sequenceable if there is a sequencing $\{x_n\}$ of its non-central elements such that $x_i x_{i+1} \neq x_{i+1} x_i$ for all i. (As before, if G is infinite, this sequence might be two-way infinite.) If G is finite, and its elements have such a sequencing $x_1, x_2, \ldots, x_m$ with the further property that $x_1 x_m \neq x_m x_1$, we say that G is strongly AZ-sequenceable. (An abelian group is vacuously strongly AZ-sequenceable.) Our aim is to show that every finite group is strongly AZ-sequenceable. For this purpose, we need the following result of Dirac [1].

Lemma 4.1. *If $\mathscr{G}$ is a linear graph of order m, and the degree at each vertex is at least $m/2$, then $\mathscr{G}$ has a Hamiltonian circuit.*

If G has order n, and $x \in G$ is non-central, then $|C(x)| \leq n/2$. Thus the subgraph of $\bar{\Gamma}(G)$ determined by the non-central elements satisfies the hypotheses of the lemma, and it follows that every finite group is strongly AZ-sequenceable.

We turn now to the question of maximal cliques in $\Gamma(G)$ and $\bar{\Gamma}(G)$. Since a set of pairwise commuting elements of G always generates an abelian subgroup, it follows that if G is non-abelian, a maximal clique in $\Gamma(G)$ has order $\leq |G|/2$. The dihedral groups D_n of order $2n$ $(n \geq 3)$ provide examples for which equality is attained.

The corresponding result for $\bar{\Gamma}(G)$ has recently been obtained by Mason, who proved the conjecture of Erdös and Straus that any clique in $\bar{\Gamma}(G)$ has cardinality $\leq \frac{1}{2}|G| + 1$.

5. C-sequenceable groups

In addition to the problem of determining the existence of Z-sequencings for finite groups, there are several other sequencing questions which give rise to difficult problems. Let us say that a finite group $G = \{1, x_1, x_2, \ldots, x_n\}$ is C-sequenceable if its non-identity elements can be listed $x_1, x_2, \ldots, x_n$ so that $\langle x_i, x_{i+1} \rangle$ is cyclic for $i = 1, 2, \ldots, n-1$. Denote by $\Delta(G)$ the undirected graph whose vertices are the non-identity elements of G, with two vertices x and y joined by an edge if and only if x and y generate a cyclic subgroup. The C-sequenceability of G is equivalent to the existence of a Hamiltonian path in $\Delta(G)$.

If $\langle x, y \rangle = \langle z \rangle$, then clearly both x and y are powers of z. This leads us to

Lemma 5.1. *Let $G = \{1, x_1, x_2, \ldots, x_n\}$ be a group, and $S \neq \{1\}$ a non-empty proper subset of G such that if $x \in S$, then $\langle x \rangle \subseteq S$, and if $y \notin S$, then $\langle y \rangle \cap S = \{1\}$. Then G is not C-sequenceable.*

Proof. Assume on the contrary that G is C-sequenceable; let $x_1, x_2, \ldots, x_n$ be such a sequencing. Since S is proper, there exist two adjacent elements x_i and x_{i+1}, exactly one of which is in S. Set $\langle x_i, x_{i+1} \rangle = \langle z \rangle$. Then $z^\alpha = x_i$, $z^\beta = x_{i+1}$ for suitable α and β. This contradicts the hypotheses, and the result follows.

Using this lemma, we can completely characterize C-sequenceable abelian p-groups.

Theorem 5.2. *If G is a finite abelian p-group, then G is C-sequenceable if and only if G is cyclic.*

Proof. If G is cyclic, it is clearly C-sequenceable. So suppose G is not cyclic, and write

$$\begin{aligned} G &= \mathbf{Z}_p^{n_1} \times \mathbf{Z}_p^{n_2} \times \cdots \times \mathbf{Z}_p^{n_r} \\ &= \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_r \rangle, \quad r \geqslant 2. \end{aligned}$$

Then each element x of G can be written uniquely in the form $x = a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_r^{\alpha_r}$, $0 \leqslant \alpha_i < p^{n_i}$. Let $S = \{x \in G \mid \operatorname{ord} a_i^{\alpha_i} = \operatorname{ord} a_j^{\alpha_j} \text{ for all } i, j\}$. As S is a p-group, we have $S \neq \{1\}$. It is easily seen that if $x \in S$, then $x^\alpha \in S$ for all α, and that if $y \in G - S$, then $y^\beta \in S$ only if $y^\beta = 1$. Thus by Lemma 5.1, G is not C-sequenceable.

The problem of determining which abelian groups are C-sequenceable appears very difficult. For example, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}, p_1 > p_2 > \cdots > p_r$, then $\mathbf{Z}_n \times \mathbf{Z}_n$ is C-sequenceable if and only if $p_2^{2\alpha_2} p_3^{2\alpha_3} \ldots p_r^{2\alpha_r} > p_1$. The first non-abelian group that is C-sequenceable is $D_4 \times \mathbf{Z}_2$, of order 16.

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