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Advances in Mathematics 194 (2005) 1–33
ADVANCES IN Mathematics
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On the Heegaard Floer homology of branched double-covers

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Received 5 November 2003; accepted 28 May 2004

Communicated by Dr. Tomasz Mrowka
Available online 5 October 2004

Abstract

Let $L \subset S^3$ be a link. We study the Heegaard Floer homology of the branched double-cover $\Sigma(L)$ of S^3, branched along L. When L is an alternating link, $\widehat{HF}$ of its branched double-cover has a particularly simple form, determined entirely by the determinant of the link. For the general case, we derive a spectral sequence whose E^2 term is a suitable variant of Khovanov's homology for the link L, converging to the Heegaard Floer homology of $\Sigma(L)$.

Keywords: Heegaard Floer homology; Branched double-covers; Khovanov's homology; Alternating links

1. Introduction

Given a link $L \subset S^3$, we can form its branched double cover, a new three-manifold which we denote by $\Sigma(L)$. In this paper, we study the Heegaard Floer homology of this three-manifold $\widehat{HF}(\Sigma(L))$ (c.f. [21]).

The starting point for these investigations is a skein exact sequence which this link invariant $L \mapsto \widehat{HF}(\Sigma(L))$ satisfies. Specifically, fix a projection of L, and let L_0 and L_1 denote the two resolutions of L at a crossing for the projection, as illustrated in Fig. 1.

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1 Supported by NSF grant number DMS 0234311 and a Sloan Research Fellowship.
2 Supported by NSF grant number DMS 0107792 and a Packard Fellowship.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2004.05.008

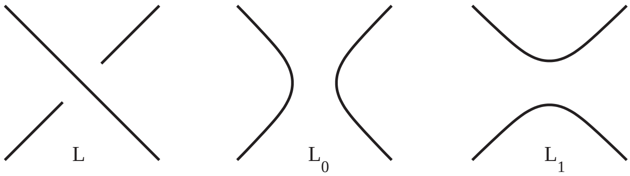

Fig. 1. Skein moves. Given a link with a crossing as labeled in L above, we have two "resolutions" L_0 and L_1, obtained by replacing the crossing by the two simplifications pictured above.

It is a quick consequence of the surgery long exact sequence for $\widehat{HF}$ that for any link $L \subset S^3$, the groups $\widehat{HF}(L)$, $\widehat{HF}(L_0)$, and $\widehat{HF}(L_1)$ fit into a long exact sequence

$$\ldots \longrightarrow \widehat{HF}(\Sigma(L_0)) \longrightarrow \widehat{HF}(\Sigma(L_1)) \longrightarrow \widehat{HF}(\Sigma(L)) \longrightarrow \ldots . \quad (1)$$

This skein exact sequence leads readily to a complete calculation of $\widehat{HF}(\Sigma(L))$, where L is any alternating link as explained in Section 2. In particular, it is shown there that if L is a link which admits a connected, alternating projection, then the rank of $\widehat{HF}(\Sigma(L))$ agrees with the number of elements in $H^2(\Sigma(L);\mathbb{Z})$, i.e. that $\Sigma(L)$ is what might be called an "ungraded Heegaard Floer homology lens space" or, in the terminology of [20], an L-space.

When Y is an arbitrary three-manifold, $\widehat{HF}(Y)$ has the structure of a $\mathbb{Z}/2\mathbb{Z}$-graded Abelian group, and that is the structure we will be concerned with throughout most of this paper. But in general, $\widehat{HF}(Y)$ also comes with a natural splitting into summands indexed by Spin^c structures on Y [21]. Indeed, when Y is a rational homology three-sphere, the groups are further endowed with an absolute $\mathbb{Q}$-grading [22].

By further elaborating on the calculations for $\Sigma(L)$ when L is alternating, we are able to determine this extra structure explicitly from the alternating diagram for L, as explained in Section 3. As explained in [22] (compare also [8]), this structure gives constraints on the intersection forms of negative-definite four-manifolds which bound $\Sigma(L)$.

Turning back to the case of a general link L, it is suggestive to compare the exact sequence (1) with the work of Khovanov, c.f. [14] (for the reader's convenience, we briefly review the construction in Section 5). Specifically, Khovanov introduces an invariant for links in S^3 whose Euler characteristic, in a suitable sense, is the Jones polynomial (c.f. [12], see also [13]). By construction, his invariants satisfy a "skein exact sequence" inspired by the skein relation for the Jones polynomial. In particular, just like $\widehat{HF}(\Sigma(L))$, Khovanov's invariants fit into a long exact sequence relating the invariant for a link and its two resolutions:

$$\ldots \longrightarrow Kh(r(L_0)) \longrightarrow Kh(r(L_1)) \longrightarrow Kh(r(L)) \longrightarrow \ldots , \quad (2)$$

where here $r(L)$ denotes the mirror of L. (Note that our conventions on L_0 and L_1 are opposite to Khovanov's; this is why we use the mirror.) But unlike $\widehat{HF}(\Sigma(L))$, Khovanov's theory comes with extra gradings (which the maps in the exact sequence respect), which allow one to extract the Jones polynomial from the Betti numbers.

The connection between the two link invariants is provided by the following result.

Theorem 1.1. *Let $L \subset S^3$ be a link. There is a spectral sequence whose E^2 term consists of Khovanov's reduced homology of the mirror of L with coefficients in $\mathbb{Z}/2\mathbb{Z}$, and which converges to $\widehat{HF}(\Sigma(L); \mathbb{Z}/2\mathbb{Z})$.*

See Section 6 for a precise statement (c.f. Theorem 6.3), and also the proof. Note that in the above statement, we use here a "reduced" version of Khovanov's homology, which he introduced in [16], with coefficents in $\mathbb{Z}/2\mathbb{Z}$. Correspondingly, we also take Heegaard Floer homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

We have the following quick corollary (whose proof is spelled out in Section 6):

Corollary 1.2. *Let $L \subset S^3$ be a link, and let $rk\,\widetilde{Kh}(L)$ denote the rank of its reduced Khovanov homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Then, we have the inequalities*

$$\det(L) \leqslant rk_{\mathbb{Z}/2\mathbb{Z}} \widehat{HF}(\Sigma(L); \mathbb{Z}/2\mathbb{Z}) \leqslant rk_{\mathbb{Z}/2\mathbb{Z}}\, \widetilde{Kh}(L),$$

where here $\det(L)$ *denotes the determinant of the link.*

Theorem 1.1 is seen as a consequence of a "link surgeries spectral sequence" established in Section 4, which holds in a more general setting (c.f. Theorem 4.1). To place this result in context, recall that if $K \subset Y$ is a framed knot in a three-manifold, in [18], it is shown that if Y_0, Y_1 denote the result of surgeries on Y along K (here, as usual, Y_0 denotes surgery along K in Y with respect to the given framing, while Y_1 denotes surgery along K in Y with respect to the framing obtained by adding a meridian to the given framing), then there is a long exact sequence relating $\widehat{HF}(Y)$, $\widehat{HF}(Y_0)$, and $\widehat{HF}(Y_1)$, compare also [7]. When the knot is replaced by a multi-component link, the corresponding object is a spectral sequence relating the various surgeries on the various components of the link. This spectral sequence, in turn, is established with the help of the associativity properties of the pseudo-holomorphic polygon construction, see also [5,9,26].

To establish Theorem 1.1 we specialize the link surgeries spectral sequence of Section 4 to the case arising from the branched double cover of a link projection. Given a projection of L, $\Sigma(L)$ comes equipped with a link, whose components correspond to crossings in the projection, framed so that surgeries on these components give branched double-covers of the resolutions of L (this is the topological input for establishing Eq. (1)). With this said, the key observation leading to Theorem 1.1 is the following. Consider the branched double cover of a collection of unlinks in the plane, connected by cobordisms induced from the connected sums among (and within) the circles. Applying $\widehat{HF}$ (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) to these objects and morphisms (as required in the E^1 term coming from the link surgery spectral sequence), one recaptures the $(1+1)$-dimensional topological quantum field theory which underpins Khovanov's invariants. Armed with this observation, Theorem 1.1 follows quickly.

1.1. Further remarks and speculation

It is interesting to note that the results from Section 3 on non-split alternating links, can be interpreted as saying that the spectral sequence of Theorem 1.1 collapses at the E^2 stage. (Note that it is clear from the more precise statement that if L is an n-crossing link, then the spectral sequence always collapses after the E^n stage.)

A rather striking example where there are non-trivial differentials beyond the E^2 stage is illustrated for torus knots. For example, let $T_{p,q}$ denotes the (p,q) torus knot. When both p and q are odd, $\Sigma(T_{p,q})$ is the Brieskorn homology sphere with multiplicities 2, p, and q. In particular, $\Sigma(T_{3,5})$ is the Poincaré homology sphere, so $\widehat{HF}(\Sigma(T_{3,5}; \mathbb{Z}/2\mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z}$ (c.f. [22] or [23]), while its reduced Khovanov homology clearly has larger rank, as its Jones polynomial has three non-zero coefficients.

Results from this paper raise a number of further questions, which further link Khovanov's essentially combinatorial theory with problems involving holomorphic disks.

As a first point, it is quite plausible that the link surgeries spectral sequence can be made to work with $\mathbb{Z}$, rather than only $\mathbb{Z}/2\mathbb{Z}$ coefficients. This suggests an E^2 term whose $\mathbb{Z}/2\mathbb{Z}$ reduction agrees with Khovanov's reduced theory, but which differs from the sign conventions as defined by Khovanov. It would be interesting to construct such a theory, and to pin down the new sign conventions of this theory, not only from the point of view of applications to Heegaard Floer homology (i.e. to give information about $\widehat{HF}$ over $\mathbb{Z}$ of the branched cover), but also from the point of view of Khovanov's theory, as it would give a link invariant with $\mathbb{Z}$-coefficients whose Euler characteristic is the normalized Jones polynomial.

In another direction, it is reasonable to expect that the induced filtered quasi-isomorphism type associated to the branched double cover spectral sequence from Theorem 1.1 is also a link invariant, i.e. that is independent of the projection used in its definition. This would, in principle, give a countable sequence of link invariants, starting with Khovanov's homology, and ending with $\widehat{HF}$ of the branched double cover.

This also raises the question of finding a combinatorial description of the higher differentials for the spectral sequence. Although finding a combinatorial description of the Heegaard Floer homology in general is a very interesting, if difficult problem, it is perhaps easier when one specializes to the case of branched double covers of links in the three-sphere.

Another question concerns naturality properties of Khovanov's homology. On the one hand, it is known that a knot cobordism X from L_1 to L_2 induces a (combinatorially defined) map between Khovanov homologies (c.f. [11,15]). Now, the branched double-cover of X inside $[1,2] \times S^3$ is a four-manifold $\Sigma(X)$ which gives a cobordism from $\Sigma(L_1)$ to $\Sigma(L_2)$, and correspondingly induces a map on $\widehat{HF}$ (c.f. [19]), defined by counting holomorphic triangles. This map, in general, can be quite difficult to compute. It is reasonable to expect that there is a well-defined map between the filtered complexes which give rise to Theorem 1.1 and hence between spectral sequences which at the E^2 stage induces Khovanov's map, and at the E^∞ stage induces the map on $\widehat{HF}$ induced by $\Sigma(X)$.

1.2. Organization

The skein exact sequence for $\widehat{HF}(\Sigma(L))$ is established in Section 2; the results for alternating links (with a sample calculation) are explained in Section 3. The link surgeries spectral sequence is established in Section 4 (note that this general result applies not only to branched double covers, considered in the rest of the paper). In Section 5, we review Khovanov's link invariant (with $\mathbb{Z}/2\mathbb{Z}$ coefficients), setting up the notation for Section 6, where we establish the precise form of Theorem 1.1.

2. Skein moves and branched double covers

Let K be a framed knot in a three-manifold Y (i.e. a knot with a choice of longitude λ). Let $Y_0 = Y_0(K)$ denote the three-manifold obtained from λ-framed surgery on Y along K, and let $Y_1 = Y_1(K)$ denote the three-manifold obtained from $\lambda + \mu$-framed surgery on Y along K (where here μ denotes the canonical meridian for the knot K). We call the ordered triple (Y, Y_0, Y_1) a *triad* of three-manifolds.

This relationship between Y, Y_0, and Y_1 is symmetric under a cyclic permutation of the three three-manifolds. Indeed, it is not difficult to see that (Y, Y_0, Y_1) fit into a triad if and only if there is a single oriented three-manifold M with torus boundary, and three simple, closed curves γ, γ_0, and γ_1 in ∂M with

$$\#(\gamma \cap \gamma_0) = \#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma) = -1 \tag{3}$$

(where here the algebraic intersection number is calculated in ∂M, oriented as the boundary of M), so that Y resp. Y_0 resp. Y_1 are obtained from M by attaching a solid torus along the boundary with meridian γ resp. γ_0 resp. γ_1.

In [18], we established a long exact sequence connecting $\widehat{HF}$ for any three three-manifolds which fit into a triad:

$$\dots \longrightarrow \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y_0) \longrightarrow \widehat{HF}(Y_1) \longrightarrow \dots .$$

The skein exact sequence for $\widehat{HF}(\Sigma(L))$ (Eq. (1)) follows readily:

Proposition 2.1. *Fix a crossing for a projection of a link $L \subset S^3$, and let L_0 and L_1 be the two resolutions of that crossing as in Fig. 1. Then the three-manifolds $(\Sigma(L), \Sigma(L_0), \Sigma(L_1))$ form a triad. In particular, there is an induced long exact sequence*

$$\dots \longrightarrow \widehat{HF}(\Sigma(L_0)) \longrightarrow \widehat{HF}(\Sigma(L_1)) \longrightarrow \widehat{HF}(\Sigma(L)) \longrightarrow \dots .$$

Proof. Fix a sphere S meeting the link L in four points, containing a ball B which contains two arcs of L, and in whose complement L, L_0, and L_1 agree. Clearly, letting M be the branched double-cover of $S^3 - B$ branched along $L - (L \cap B)$, we see that $\Sigma(L)$, $\Sigma(L_0)$, and $\Sigma(L_1)$ are all obtained from M by attaching the branched double cover of B branched along two arcs.

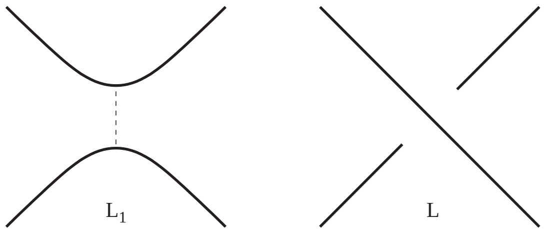

Fig. 2. Obtaining $\Sigma(L)$ from $\Sigma(L_1)$. The three-manifold $\Sigma(L)$ (corresponding to the branched double cover of a link with the a crossing as illustrated on the right) is obtained from $\Sigma(L_1)$ by surgery on the knot obtained as a branched double cover of the dashed arc indicated in the picture on the left.

Now, it is easy to see that the branched double-cover of B branched along two standard, unknotted arcs is a solid torus. Indeed, a meridian for this solid torus can be realized by pushing either of the two arcs out to the boundary, and taking its branched double-cover. Thus, letting γ, γ_0, and γ_1 denote curves obtained by pushing arcs out into the boundary torus for L and its resolutions L_0 and L_1, it is straightforward to verify that these curves satisfy Eq. (3).

Thus, $(\Sigma(L), \Sigma(L_0), \Sigma(L_1))$ form a triad of three-manifolds. The exact sequence now is a direct consequence of the aforementioned surgery long exact sequence ([18, Theorem 9.12]; see Theorem 4.5 below for another proof). $\square$

In particular, we have seen that $\Sigma(L)$ is obtained as surgery on a knot in $\Sigma(L_1)$. This knot can be explicitly seen as the branched double cover of a standard arc inside the three-ball B containing the two resolved arcs in L_1. In turn, this arc can be pictured in a knot projection of L_1 as an arc A which meets L_1 in exactly two points, both of which are on the boundary of A, and which connect the two resolved strands in L_1, as pictured in Fig. 2.

3. Alternating links

Let Y be an oriented three-manifold. Let $|H^2(Y;\mathbb{Z})|$ denote the number of elements in $H^2(Y;\mathbb{Z})$ provided that $b_1(Y)=0$, and let $|H^2(Y;\mathbb{Z})|=0$ if $b_1(Y)>0$. Now, if L is a link in S^3, the determinant of L is defined by $\det(L)=|\Delta_L(-1)|$, where here $\Delta_L(T)$ denotes the Alexander polynomial of L. It is well-known (see for example [17]) that $\det(L)=|H^2(\Sigma(L);\mathbb{Z})|$.

Recall that the Euler characteristic of $\widehat{HF}(Y)$ is given by $|H^2(Y;\mathbb{Z})|$ (c.f. [18, Proposition 5.1]); in particular, $|H^2(Y;\mathbb{Z})|\leqslant rk\widehat{HF}(Y)$. Three-manifolds with $b_1(Y)=0$ for which $|H^2(Y;\mathbb{Z})|=rk\widehat{HF}(Y)$ are called L-spaces (c.f. [20]). This special class of three-manifolds is closed under connected sums and includes all lens spaces and, more generally, all Seifert fibered spaces with finite fundamental group; other examples are given in [23,20]. We will prove that if L is a non-split, alternating link, then $\Sigma(L)$ is an L-space. Indeed, the class of links we work with here is wider. To this end, we have the following:

Definition 3.1. The set $\mathcal{Q}$ of *quasi-alternating links* is the smallest set of links which satisfies the following properties:

1. the unknot is in $\mathcal{Q}$
2. the set $\mathcal{Q}$ is closed under the following operation. Suppose L is any link which admits a projection with a crossing with the following properties:
   - both resolutions $L_0, L_1 \in \mathcal{Q}$,
   - $\det(L_0), \det(L_1) \neq 0$,
   - $\det(L) = \det(L_0) + \det(L_1)$;

   then $L \in \mathcal{Q}$.

Note that quasi-alternating in this sense is different from the notion of almost-alternating, which appears in the literature (c.f. [1]).

Lemma 3.2. *Every link which admits a connected, alternating projection is quasi-alternating*.

Proof. Recall that a complement of a knot projection in the plane admits a checkerboard coloring. The collection of black regions can be given the structure of a planar graph $\mathcal{B}(L)$, whose vertices correspond to black regions and edges correspond to vertices which are corners of pairs of black regions. It is a classical result [4] that if L admits an alternating projection, then the determinant of L is the total number of maximal subtrees of the black graph of L. To fix orientation conventions, when coloring an alternating link, we always use the coloring scheme indicated in Fig. 3.

We now induct on the determinant of the link. In the basic case where the determinant is one, it follows at once that there is only one maximal subtree, and hence that the knot is the unknot.

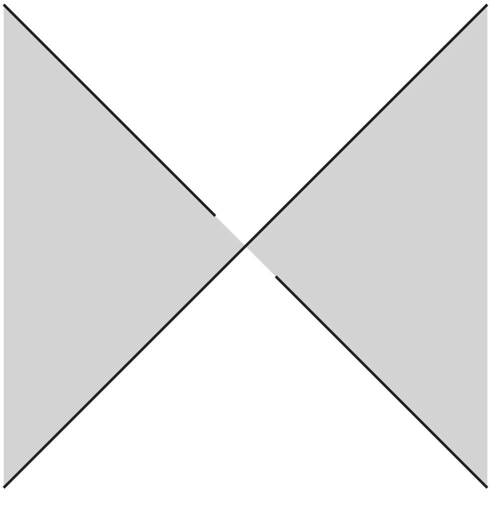

Fig. 3. Coloring conventions for alternating knots. We adopt the pictured convention when coloring an alternating projection.

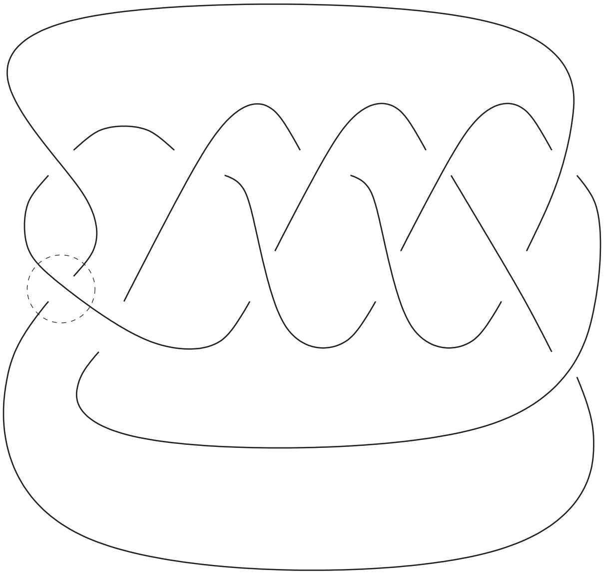

Fig. 4. A quasi-alternating, but not alternating, knot. The pictured knot 9_{47} is quasi-alternating: its determinant is 29, and if we resolve the indicated crossing either way, we obtain (non-split) alternating links with determinants 5 and 24.

For the inductive step, it is easy to see that for a reduced alternating projection of L, if we choose any crossing x, both resolutions L_0 and L_1 at x are connected, alternating projections of links. Moreover, it is easy to see that $\det(L) = \det(L_0) + \det(L_1)$: maximal subtrees of the black graph of L which contain, resp. do not contain, the edge corresponding to x are in one-to-one correspondence with the maximal subtrees of the black graph of L_i, resp L_j, where here $i, j \in \{0, 1\}$ and $i \neq j$. Thus, by the inductive hypothesis, the theorem has been established for both L_0 and L_1; and hence, the inductive step follows. □

Of course, there are quasi-alternating links which are not alternating. For a picture of one, see Fig. 4.

Proposition 3.3. *If L is a quasi-alternating link, $\Sigma(L)$ is an L-space, i.e.*

$$\widehat{HF}(\Sigma(L)) \cong \mathbb{Z}^{\det(L)}.$$

Proof. The proposition is now established by induction on the determinant of L. In the basic case where the determinant is one, it follows at once that there is only one maximal subtree, and hence that the knot is the unknot, so $\widehat{HF}(\Sigma(L)) = \widehat{HF}(S^3) \cong \mathbb{Z}$.

The bound $\det(L) \leqslant rk\,\widehat{HF}(\Sigma(L))$ combined with the long exact sequence stated in Proposition 2.1 readily provides the inductive step (c.f. [20, Proposition 2.1]). □

We describe now the absolute $\mathbb{Q}$-grading on $\widehat{HF}(\Sigma(L))$, when L is a non-split, alternating link.

Let L be a link with a connected, alternating projection, and choose a maximal subtree T of the black graph $\mathcal{B}(L)$, and let $\{e_i\}_{i=1}^m$ denote the edges in $Z_T = \mathcal{B}(L) - T$. Let V denote the lattice generated by these edges. We can equip V with a bilinear form

$$Q : V \otimes V \longrightarrow \mathbb{Z}$$

as follows (compare also Chapter 13 of [3]). Choose orientations for each edge $e_i \in Z_T$, let C_i denote the oriented circuit in $T \cup \{e_i\}$; and if X is any subgraph of $\mathcal{B}(L)$, let $E(X)$ denote the number of edges in X. Note that the orientation on e_i induces an orientation on the circuit C_i. Given a pair of distinct edges $e_i, e_j \in Z_T$ with the property that $C_i \cap C_j \neq \emptyset$, we let

$$Q(e_i \otimes e_j) = \varepsilon(i, j) \cdot E(C_i \cap C_j),$$

where here $\varepsilon(i, j)$ is given by

$$\varepsilon(i,j) = \begin{cases} +1 & \text{if the orientation on } C_i \cap C_j \text{ induced from } C_i \\ & \text{is opposite to the one induced from } C_j, \\ -1 & \text{otherwise.} \end{cases}$$

In particular, $Q(e_i \otimes e_i) = -E(C_i)$.

This quadratic form Q is the intersection form of a certain four-manifold which bounds K. (Indeed, Q is equivalent under a suitable change of basis to the usual Goeritz form of K, c.f. [10].)

A characteristic vector for M is a vector in the lattice $K \in V^*$ with $\langle K, v\rangle + Q(v, v) \equiv 0 \pmod 2$ for each $v \in V$. Two characteristic vectors K and K' are said to be equivalent if $K - K' = 2Q(v \otimes \cdot)$ for some $v \in V$.

Theorem 3.4. *There is an identification i equivalence classes of characteristic vectors for Q with* Spin^c *structures over $\Sigma(L)$. Moreover, given an equivalence class of characteristic vectors Ξ, $\widehat{HF}(\Sigma(L), i(\Xi)) \cong \mathbb{Z}$ is supported in dimension*

$$d(\Xi) = \frac{\max_{K \in \Xi} K^2 + b}{4},$$

where here K^2 is the length of K with respect to the inner product on V^* induced from Q, and b is the number of edges in Z_T (or, more invariantly, the rank of $H_1(\mathcal{B}(L); \mathbb{Z})$).

Remark 3.5. We emphasize that we are using the coloring conventions pictured in Fig. 3, which breaks the apparent symmetry between the “white” and “black” graphs. In fact, using the white graph in place of the black graph to construct the form analogous to Q, it is not difficult to see that we obtain the dimensions of the generators for $\widehat{HF}(-\Sigma(L))$, whose sign is opposite to those for $\widehat{HF}(\Sigma(L))$.

We break the proof into several pieces. First, we describe a four-manifold X_L which bounds $\Sigma(L)$. To construct X_L, fix a projection for the link L, and let n denote its number of crossings. If we form 1-resolutions at each intersection, we obtain a k-component unlink. The branched double cover of this manifold is $Y_0 = \#^{k-1}(S^2 \times S^1)$. Attaching one two-handle for each crossing to "unresolve" the crossing (as in Proposition 2.1), we obtain a cobordism from Y_0 to $\Sigma(L)$. Indeed, by filling Y_0 by the boundary connected sum of $k-1$ copies of $B^3 \times S^1$, we obtain a four-manifold X_L which bounds Y. (In fact, one can check that we are describing here a two-handle decomposition of the four-manifold from [10].)

Lemma 3.6. *If L is a non-split alternating link, the four-manifold X_L described above is negative-definite. Indeed, there is an identification of the form Q on the vector space V described above with the intersection form on the two-dimensional homology of X_L.*

Proof. As we have described it, X_L is built from one zero-handle, $k-1$ one-handles, and n two-handles. In fact, the tree T specifies $k-1$ two-handles which cancel the one-handles; i.e. after attaching the two-handles from the tree, we obtain the branched double cover of a single unknot, which is S^3. Now, X_L is obtained from the four-ball by surgery on a link in S^3 (the branched double cover of the unknot) whose components correspond to the remaining edges in $\mathcal{B}(L) - T$ (i.e. the link components are the branched double covers of the arcs with boundary in the unknot, which are associated to the edges in $\mathcal{B}(L) - T$). We claim that a choice of orientation on each edge e_i simultaneously orients all the components of this link, up to an overall sign.

To see this, we proceed as follows. Let U denote the unknot as specified by the tree T. Let ϕ be a vector field normal to U which is orthogonal to the kernel of the projection map used in describing the knot projection. This vector field ϕ, of course, specifies the blackboard framing of U. The vector field ϕ has two possible lifts in the branched double cover of the unknot (in the sense that there are two lifts in the branched double cover of the knot obtained by displacing U by ϕ). Choose one, and denote it $\widetilde{\phi}$ (while the other is denoted $\widetilde{\phi}'$). Then, the knot corresponding to e_i (thought of as an arc connecting x to y in the unknot) is oriented so that its tangent vector at x agrees with $\widetilde{\phi}_x$ (as opposed to $\widetilde{\phi}'_x$). We denote the oriented knot associated to e_i with its orientation by k_i. (Note that the other lift of the blackboard framing has the effect of reversing the induced orientations on all the knots k_i simultaneously.)

Next, we argue that the intersection form of X_L is negative-definite. We prove this by induction on the number of crossings. The basic case is obvious. Next, recall that

$$|H^2(\Sigma(L);\mathbb{Z})| = |H^2(\Sigma(L_0);\mathbb{Z})| + |H^2(\Sigma(L_1);\mathbb{Z})|,$$

so it readily follows that the two-handle from $\Sigma(L_1)$ to $\Sigma(L)$ (and also the one from $\Sigma(L)$ to $\Sigma(L_0)$) is negative-definite. Now, it is easily seen that X_L is obtained by attaching this negative-definite two-handle to X_{L_1}.

We show that the intersection form on $H_2(X_L;\mathbb{Z})$ is given by Q. To this end, observe that if we attach m of the remaining two handles $\{e_i\}_{i=1}^m$, to S^3, the number of elements in H^2 of the boundary three-manifold (with $\mathbb{Z}$ coefficients) is given by the determinant of the matrix $(Q(e_i \otimes e_j))_{i,j\in\{1,\dots,n\}}$, which in turn is obtained from the number of maximal subtrees in $T \cup \{e_i\}_{i=1,\dots,m}$.

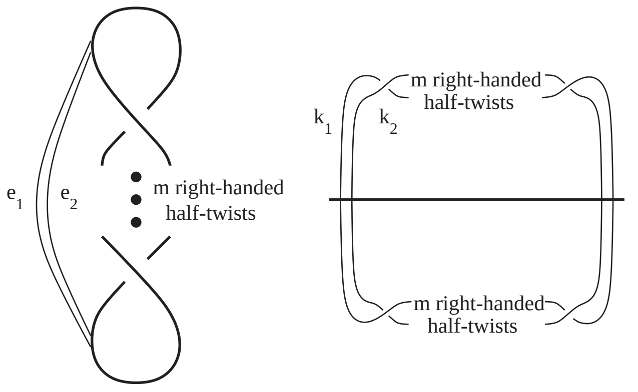

Fig. 5. Standard picture for two crossings. If e_1 and e_2 correspond to two closed circuits with $m = E(C_1 \cap C_2)$ edges in common, then the unknot corresponding to the tree T, together with the two arcs associated to e_1 and e_2, is isotopic to the picture on the left (which depicts a projection of the unknot with m right-handed half-twists in it, of which two have already been drawn, together with two arcs which meet the unknot in the specified manner). Passing to the branched double cover of the unknot (which in turn is best visualized by unwinding the twists on the unknot), at the expense of twisting the arcs corresponding to e_1 and e_2, and then taking the branched double cover of these edges to obtain knots k_1 and k_2, we obtain the picture shown on the right, where the solid line indicates the branched locus.

In particular, since X_L is negative-definite, it follows at once that if we choose a basis for $H_2(X_L;\mathbb{Z})$ given by the two-handles in X_L (with any set of orientations), then if $[e_i]$ the homology class corresponding to an edge e_i, then $\#[e_i]\cap[e_i] = -E(C_i)$ (since the number of maximal subtrees of a circuit is the length of the circuit, and the sign is forced by the negative-definiteness).

When $i \neq j$, $\#[e_i]\cap[e_j]$ is given by the linking number of k_i with k_j. This in turn is calculated in a model case: consider the unknot corresponding to the tree T, together with the two arcs corresponding e_i and e_j. This is easily seen to be isotopic to an unknot with two arcs attached, in a manner which has a standard projection depending only on the integer $E(C_i \cap C_j)$, as pictured in Fig. 5. Unwinding the unknot and taking the branched double cover, we see that the branched double covers of the original arcs become circles which are linked $|E(C_i \cap C_j)|$ times. It is then straightforward to see that the sign of this linking number is the one stated (once we choose a lift of the blackboard framing for the unknot). □

Proof of Theorem 3.4. With Lemma 3.6, the proof of the theorem now follows along the lines of Section 2 of [23]. We sketch here the main points. Let $Ch(X_L)$ denote the set of characteristic vectors for the intersection form $H^2(X_L;\mathbb{Z})$. We write $K \sim K'$ if there is an element $v \in H^2(X_L, \partial X_L)$ with the property that $K = K' + 2v$. Next (compare [23]), consider the subgroup

$$\widehat{\mathbb{H}}(X_L) \subset Hom(Ch(X_L), \mathbb{Z}),$$

consisting of maps ϕ with the properties that

- $\phi(K) = \phi(K')$ if $K \sim K'$ and $Q(K,K) = Q(K',K')$
- $\phi(K) = 0$ if there is some $K' \sim K$ with $Q(K',K') > Q(K,K)$.

Viewing X_L as a cobordism from $-\Sigma(L)$ to S^3, we obtain a naturally induced map (c.f. [19])

$$T_{X_L} : \widehat{HF}(-\Sigma(L)) \longrightarrow Hom(Ch(X_L), \mathbb{Z}),$$

in view of the fact that $\widehat{HF}(S^3) \cong \mathbb{Z}$.

Unless the diagram of L represents the unknot, we can always find a double-point p whose two resolutions are connected diagrams. This gives the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{HF}(-\Sigma(L_0)) & \longrightarrow & \widehat{HF}(-\Sigma(L)) & \longrightarrow & \widehat{HF}(-\Sigma(L_1)) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle T_{X_{L_0}\#\overline{\mathbb{CP}}^2}} & & \Big\downarrow{\scriptstyle T_{X_L}} & & \Big\downarrow{\scriptstyle T_{X_{L_1}}} & & \\ & & Hom(Ch(X_{L_0}\#\overline{\mathbb{CP}}^2), \mathbb{Z}) & \xrightarrow{\mathbb{A}} & Hom(Ch(X_L), \mathbb{Z}) & \xrightarrow{\mathbb{B}} & Hom(Ch(X_{L_1}), \mathbb{Z})), & & \end{array} \tag{4}$$

where the top row is exact, the squares commute, the maps $\mathbb{A}$ and $\mathbb{B}$ are given by

$$\mathbb{A}(\phi_0)(K) = \sum_{\{K_0 \in Ch(X_{L_0}\#\overline{\mathbb{CP}}^2) | K_0|_{H^2(X_L;\mathbb{Z})} = K\}} \phi_0(K_0)$$

$$\mathbb{B}(\phi)(K_1) = \sum_{\{K \in Ch(X_L) | K|_{H^2(X_{L_1};\mathbb{Z})} = K_1\}} \phi(K_1).$$

A straightforward induction on the number of crossings in the diagram shows that the image of T_{X_L} is contained in $\widehat{\mathbb{H}}(X_L)$. The sphere with square -1 contained in the composite cobordism from $\Sigma(L_0)$ to $\Sigma(L_1)$ through $\Sigma(L)$ is used to show that $\mathbb{B} \circ \mathbb{A} = 0$, and also that $\mathbb{A}$ is injective (more details can be found in Section 2.8 of [24]). Straightforward homological algebra then shows that T_{X_L} is an isomorphism, again, by induction on the number of crossings, together with Diagram 4, and an identification $\widehat{\mathbb{H}}(X_{L_0}) \cong \widehat{\mathbb{H}}(X_{L_0}\#\overline{\mathbb{CP}}^2)$. (For a more detailed argument establishing an analogous result, see also the proof of Lemma 2.10 of [23].)

Endow $\widehat{\mathbb{H}}(X_L)$ with a grading, by declaring an element to be homogeneous of degree d if it is supported on those $K \in Ch(X_L)$ with

$$-\left(\frac{K^2 + rkH^2(X_L)}{4}\right) = d.$$

Clearly, T_{X_L} carries $\widehat{HF}_d(-\Sigma(L))$ to $\widehat{\mathbb{H}}_d(X_L)$ (c.f. [22]). Since $\widehat{HF}_d(-\Sigma(L)) \cong \widehat{HF}^{-d}(\Sigma(L))$ (c.f. [22]), the result now follows. $\square$

Note that the long exact sequence can be pushed slightly further than we have done in the above discussion. For example, recall that if W is a cobordism between two

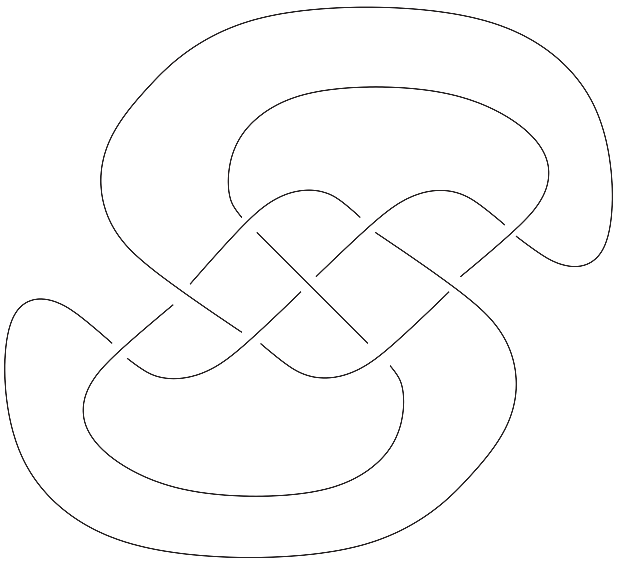

Fig. 6. The knot 9_{40}.

L-spaces with $b_2^+ > 0$, then the induced map on HF^+ is trivial (c.f. [19]). This gives at once the result that if L differs from an alternating (or indeed quasi-alternating) knot by at a single crossing, then all the elements of $HF^+_{\mathrm{red}}(\Sigma(L))$ have the same $\mathbb{Z}/2\mathbb{Z}$-grading. Indeed, the map induced by a two-handle from $\Sigma(L_0)$ to $\Sigma(L_1)$, where L_0 and L_1 are both quasi-alternating, is determined purely by homological information. This can be used to give information about the Heegaard Floer homology of $\Sigma(L)$ when its two resolutions are quasi-alternating. We do not pursue this any further here, contenting ourselves instead with a sample calculation illustrating Theorem 3.4.

3.1. An example: 9_{40}

To illustrate Theorem 3.4, we calculate $\widehat{HF}(\Sigma(L))$ where L is the alternating knot with nine crossings 9_{40}, pictured in Fig. 6.

The black graph of this knot is illustrated in Fig. 7. Using as our base tree T the solid edges pictured in the figure, and the orientations of the remaining edges indicated, the intersection form of X_L takes the form

$$Q_L = \begin{pmatrix} -3 & -2 & -1 & -1 \\ -2 & -5 & -2 & -3 \\ -1 & -2 & -4 & -3 \\ -1 & -3 & -3 & -5 \end{pmatrix}.$$

It is a straightforward if tedious matter to find the maximal lengths of the characteristic vectors for Q in its equivalence classes. Note that this is a finite search: it is easy to see that all maximal characteristic vectors have the property that $|\langle K, v\rangle| \leqslant |Q(v \otimes v)|$, and hence to determine the absolute gradings of the generators of $\widehat{HF}(\Sigma(L))$ (this

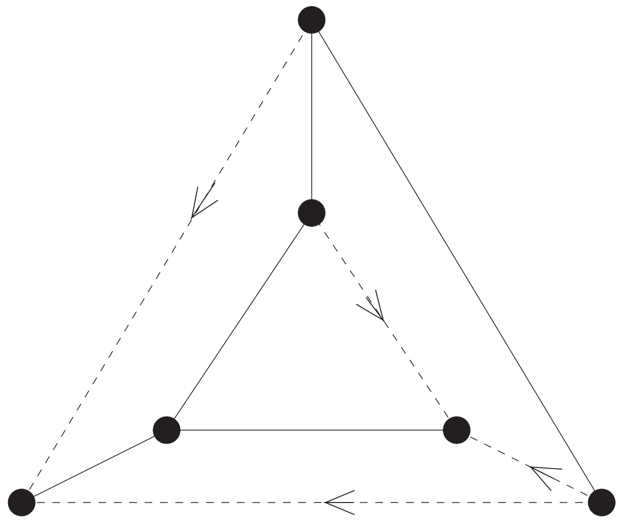

Fig. 7. Black graph for the knot 9_{40}. All edges (including those which are dashed) are included in the graph. The solid edges constitute the tree T used for the matrix given in the text. The dashed edges, when oriented, give rise to the matrix in the text.

and the further calculations in this section were all done with the help of Mathematica [27]). We display the results below. The numbers are ordered as suggested by the group structure of $H^2(\Sigma(L);\mathbb{Z}) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$; i.e. having chosen such an isomorphism, we have a naturally induced identification $\mathrm{Spin}^c(\Sigma(L)) \cong \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ (where we choose as the origin the spin structure on $\Sigma(L)$; since H_1 has no two-torsion, this structure is uniquely determined); i.e. the element in the ith row (counting from 0 to 4) and jth column (counting from 0 to 14) is the absolute grading of the element in the Spin^c structure corresponding to $(i, j) \in \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$.

$$
\begin{array}{ccccccccccccccc}
-\frac{1}{2} & -\frac{11}{30} & \frac{1}{30} & \frac{7}{10} & -\frac{11}{30} & \frac{5}{6} & \frac{3}{10} & \frac{1}{30} & \frac{1}{30} & \frac{3}{10} & \frac{5}{6} & -\frac{11}{30} & \frac{7}{10} & \frac{1}{30} & -\frac{11}{30} \\
-\frac{1}{10} & \frac{13}{30} & -\frac{23}{30} & \frac{3}{10} & -\frac{11}{30} & -\frac{23}{30} & -\frac{9}{10} & -\frac{23}{30} & -\frac{11}{30} & \frac{3}{10} & -\frac{23}{30} & \frac{13}{30} & -\frac{1}{10} & -\frac{11}{30} & -\frac{11}{30} \\
-\frac{9}{10} & \frac{1}{30} & -\frac{23}{30} & \frac{7}{10} & \frac{13}{30} & \frac{13}{30} & \frac{7}{10} & -\frac{23}{30} & \frac{1}{30} & -\frac{9}{10} & \frac{13}{30} & \frac{1}{30} & -\frac{1}{10} & \frac{1}{30} & \frac{13}{30} \\
-\frac{9}{10} & \frac{13}{30} & \frac{1}{30} & -\frac{1}{10} & \frac{1}{30} & \frac{13}{30} & -\frac{9}{10} & \frac{1}{30} & -\frac{23}{30} & \frac{7}{10} & \frac{13}{30} & \frac{13}{30} & \frac{7}{10} & -\frac{23}{30} & \frac{1}{30} \\
-\frac{1}{10} & -\frac{11}{30} & -\frac{11}{30} & -\frac{1}{10} & \frac{13}{30} & -\frac{23}{30} & \frac{3}{10} & -\frac{11}{30} & -\frac{23}{30} & -\frac{9}{10} & -\frac{23}{30} & -\frac{11}{30} & \frac{3}{10} & -\frac{23}{30} & \frac{13}{30}
\end{array}
$$

4. The link surgeries spectral sequence

In this section, we turn our attention away from branched double-covers, and consider the case of a general three-manifold Y. Our aim here is to describe a generalization of the surgery long exact sequence (with $\mathbb{Z}/2\mathbb{Z}$ coefficients) for the case of multi-component links in Y. In the course of making this generalization, we give a quick (and slightly stronger) proof of the long exact sequence based on associativity properties of the holomorphic polygon construction, combined with some homological algebra discussed in Section 4.1. But first, we introduce some notation.

Let $L = K_1 \cup \cdots \cup K_\ell$ be an ℓ-component, framed link in a three-manifold Y. A "multi-framing" is a vector $I = (m_1, \ldots, m_\ell)$, where each $m_i \in \{0, 1, \infty\}$. For a multi-framing, there is a three-manifold $Y(I)$, which is obtained from Y by performing m_i-framed surgery on the component K_i for $i = 1, \ldots, n$. As usual, when $m_i = \infty$, this means no surgery, $m_i = 0$ this means λ_i-framed surgery, and when $m_i = 1$, this is surgery with framing $\mu_i + \lambda_i$.

We give the set $\{0, 1, \infty\}^\ell$ the lexicographical ordering (with the understanding that $0 < 1 < \infty$). If $I \in \{0, 1, \infty\}^\ell$, we call I' an *immediate successor* of I where $I = (m_1, \ldots, m_\ell)$ and $I' = (m'_1, \ldots, m'_\ell)$ if there is some j so that for all $i \neq j$, $m_i = m'_i$, while $m_j < m'_j$, excluding the case where $m_j = 0$ and $m'_j = \infty$. Clearly, if I' is an immediate successor of I, there is a corresponding map on Floer homology

$$\widehat{G}_{I<I'} : \widehat{HF}(Y; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \widehat{HF}(Y'; \mathbb{Z}/2\mathbb{Z})$$

associated to the single two-handle addition, c.f. [19].

Consider a chain complex C filtered by the cube $\{0, 1\}^\ell$ with its reverse lexicographical ordering, in the sense that C is generated by subcomplexes $F_I \subset C$ with $F_I \subset F_J$ if $I > J$ (so that in particular $C = F_{0^\ell}$). There is a naturally induced $\mathbb{Z}$-filtration on C obtained by "flattening" the cube (c.f. [14]). Specifically, given $I \in \{0, 1\}^\ell$, let

$$|I| = \sum_{i \in I} i,$$

then for each $i \in \mathbb{Z}$, let $F_i \subset C$ be the subcomplex

$$F_i = \bigcup_{\{I \in \{0,1\}^\ell \,||I|=i\}} F_I.$$

In the corresponding Leray spectral sequence $\{E^r, d^r\}$, the E^1 term E^1_* can be further decomposed

$$E^1_i = \bigoplus_{\{I \in \{0,1\}^\ell ||I|=i\}} E^1_I,$$

where here

$$E^1_I = H(F_I / \bigcup_{J>I} F_J).$$

This spectral sequence converges to $H(C)$ (in the usual sense—it calculates the graded object associated to the filtration of $H(C)$ by the subobjects $H(F_I)$). Note, that this spectral sequence collapses at the $(\ell+1)$th stage; i.e. $d^r \equiv 0$ for all $r \geqslant \ell+1$, and hence

$$E^{\ell+1} = E^{\ell+2} = \cdots = E^\infty.$$

The reader is reminded that the subscript in the terms E^1_I for the spectral sequence takes values in the cube $\{0, 1\}^\ell$, and should not be confused with the usual bigrading on the Leray spectral sequence of a $\mathbb{Z}$-filtered complex $C = C_*$ with an internal $\mathbb{Z}$-grading. (Floer homology has an internal $\mathbb{Z}/2\mathbb{Z}$ grading, and indeed it is not difficult to work out how the terms in the spectral sequence behave with respect to this additional structure; however, we will have no need for this in the present applications.)

With these preliminaries in place, we can now state the link surgeries spectral sequence alluded to in the introduction.

Theorem 4.1. *Let Y be a closed, oriented three-manifold, equipped with an ℓ-component framed link $L = K_1 \cup \cdots \cup K_\ell$. Then, there is an induced cubical filtration on $\widehat{CF}(Y)$ whose corresponding Leray spectral sequence has E^1 term given by*

$$E^1 = \bigoplus_{I \in \{0,1\}^\ell} \widehat{HF}(Y(I); \mathbb{Z}/2\mathbb{Z})$$

and d^1 differential obtained by adding up all the $\widehat{G}_{I<I'}$ (where I' is an immediate successor of I). In particular, this spectral sequence (which collapses at the $(\ell+1)$th stage) converges to $\widehat{HF}(Y)$.

Although we have stated Theorem 4.1 for $\widehat{HF}$, the same result can be established for HF^+ (again with $\mathbb{Z}/2\mathbb{Z}$-coefficients), with some notational changes.

Before proceeding to the proof, we indulge in a purely homological-algebraic digression. The algebra here was inspired by a conversation with Paul Seidel, who communicated to us some version of Lemma 4.2.

4.1. Mapping cones

We begin with some terminology.

Let A_1 and A_2 be a pair of chain complexes of $\mathbb{Z}/2\mathbb{Z}$-vector spaces.[1] A chain map

$$\phi \colon A_1 \longrightarrow A_2$$

is called a *quasi-isomorphism* if the induced map on homology is an isomorphism. Two chain complexes A_1 and A_2 are said to be *quasi-isomorphic* if there is a third chain complex B and a pair of quasi-isomorphisms $\phi_1 \colon A_1 \longrightarrow B$ and $\phi_2 \colon A_2 \longrightarrow B$.

Recall that if we have a chain map between chain complexes $f_1 \colon A_1 \longrightarrow A_2$, we can form its mapping cone $M(f_1)$, whose underlying module is the direct sum $A_1 \oplus A_2$, endowed with the differential

$$\partial = \begin{pmatrix} \partial_1 & 0 \\ f_1 & \partial_2 \end{pmatrix},$$

where here ∂_i denotes the differential for the chain complex A_i. Recall that there is a short exact sequence of chain complexes

$$0 \longrightarrow A_2 \xrightarrow{\iota} M(f_1) \xrightarrow{\pi} A_1 \longrightarrow 0.$$

This induces a long exact sequence, for which the connecting homomorphism is the map on homology induced by f_1.

[1] The discussion from this section can be carried over to $\mathbb{Z}$ coefficients in a routine manner; we suppress these signs, however, since the application at hand uses $\mathbb{Z}/2\mathbb{Z}$ coefficients.

The mapping cylinder is natural in the following sense. Suppose that we have a diagram of chain complexes

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & A_2 \\ \psi_1 \big\downarrow & & \big\downarrow \psi_2 \\ B_1 & \xrightarrow{g_1} & B_2 \end{array}$$

which commutes up to homotopy, then there is an induced map

$$m(\psi_1, \psi_2) \colon M(f_1) \longrightarrow M(g_1)$$

which fits into the following diagram, where the rows are exact and the squares are homotopy-commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_2 & \longrightarrow & M(f_1) & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \psi_2 \big\downarrow & & m(\psi_1,\psi_2) \big\downarrow & & \big\downarrow \psi_1 & & \\ 0 & \longrightarrow & B_2 & \longrightarrow & M(g_1) & \longrightarrow & B_1 & \longrightarrow & 0. \end{array}$$

Lemma 4.2. *Let* $\{A_i\}_{i=1}^{\infty}$ *be a collection of chain maps and let*

$$\{f_i \colon A_i \longrightarrow A_{i+1}\}_{i \in \mathbb{Z}}$$

be a collection of chain maps satisfying the following two properties:

(1) $f_{i+1} \circ f_i$ *is chain homotopically trivial, by a chain homotopy*

$$H_i \colon A_i \longrightarrow A_{i+2}$$

(2) *the map*

$$\psi_i = f_{i+2} \circ H_i + H_{i+1} \circ f_i \colon A_i \longrightarrow A_{i+3}$$

is a quasi-isomorphism.

Then, $M(f_2)$ *is quasi-isomorphic to* A_4.

Proof. Hypothesis (1) proves that the map

$$\psi_i = f_{i+2} \circ H_i + H_{i+1} \circ f_i \colon A_i \longrightarrow A_{i+3}$$

is a chain map; and indeed that the square

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & A_{i+1} \\ \psi_i \big\downarrow & & \big\downarrow \psi_{i+1} \\ A_{i+3} & \xrightarrow{f_{i+3}} & A_{i+4} \end{array} \tag{5}$$

commutes up to homotopy.

Next, define $\alpha_i : M(f_i) \longrightarrow A_{i+2}$ by

$$\alpha_i(a_i, a_{i+1}) = H_i(a_i) + f_{i+1}(a_{i+1})$$

and $\beta_i : A_i \longrightarrow M(f_{i+1})$ by

$$\beta_i(a_i) = (f_i(a_i), H_i(a_i)).$$

Now

$$\alpha_{i+1} \circ \beta_i = \psi_i,$$

which is a quasi-isomorphism.

Moreover, consider the diagram:

$$\begin{array}{ccccccccc}
A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{\iota_3} & M(f_2) & \xrightarrow{\pi_3} & A_2 & \xrightarrow{f_2} & A_3 \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle \alpha_2}\downarrow & & {\scriptstyle \psi_2}\downarrow & & {\scriptstyle \psi_3}\downarrow \\
A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 & \xrightarrow{f_5} & A_6 \\
{\scriptstyle \psi_2}\downarrow & & {\scriptstyle \psi_3}\downarrow & & {\scriptstyle \beta_4}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\
A_5 & \xrightarrow{f_5} & A_6 & \xrightarrow{\iota_6} & M(f_5) & \xrightarrow{\pi_5} & A_5 & \xrightarrow{f_5} & A_6
\end{array} \tag{6}$$

The map $X : M(f_2) \longrightarrow A_5$ defined by

$$X(a_2, a_3) = H_3(a_3)$$

gives a chain homotopy between $\psi_2 \circ \pi_3$ and $f_4 \circ \alpha_2$, while the map $Y : A_3 \longrightarrow M(f_5)$ defined by

$$Y(a_3) = (H_3 a_3, 0)$$

gives a chain homotopy between $\beta_4 \circ f_3$ and $\iota_6 \circ \psi_3$. Thus, all the squares in Diagram (6) commute up to homotopy, and the maps induced on homology on the top and bottom rows are exact. From the five-lemma, it follows that the map induced on homology $\beta_4 \circ \alpha_2$ is also an isomorphism. Thus (in view of the fact that $\alpha_5 \circ \beta_4$ is a quasi-isomorphism), we conclude β_4 and hence α_2 is a quasi-isomorphism. □

It is useful to interpret Lemma 4.2 in the following terms. Under the hypotheses of that lemma, we can form an "iterated mapping cone" $M(f_1, f_2, f_3)$ whose underlying module is $A_1 \oplus A_2 \oplus A_3$, and whose differential is given by the matrix

$$\partial = \begin{pmatrix} \partial_1 & 0 & 0 \\ f_1 & \partial_2 & 0 \\ H_1 & f_2 & \partial_3 \end{pmatrix}. \tag{7}$$

Indeed, Hypothesis (1) guarantees that ∂ determines a differential on $M(f_1, f_2, f_3)$. Consider the short exact sequence

$$0 \longrightarrow A_3 \longrightarrow M(f_1, f_2, f_3) \longrightarrow M(f_1) \longrightarrow 0.$$

It is easy to see that its connecting homomorphism

$$H_*(M(f_1)) \longrightarrow H_*(A_3)$$

is the map on homology induced by the map α_1 in the above lemma. Thus, Lemma 4.2 can be interpreted as saying that $H_*(M(f_1, f_2, f_3)) = 0$.

4.2. Pseudo-holomorphic n-gons

With the above homological algebra in place, we proceed to the geometrical underpinnings of Theorem 4.1. We will make heavy use of the pseudo-holomorphic polygon construction, c.f. [5,9,26] and its relationship with Heegaard Floer homology, as explained in Section 8 of [21]. We recall this construction very briefly here.

Let Σ be a connected, closed, oriented two-manifold of genus g, and fix $(m+1)$ g-tuples of attaching circles $\{\boldsymbol{\eta}^i\}_{i=1}^m$. Specifically, for each fixed i, the set $\boldsymbol{\eta}^i = \{\eta^i_j\}_{j=1}^g$ is a collection of g pairwise disjoint, homologically linearly independent, embedded curves in Σ. We choose also a reference point z disjoint from all the η^i_j. In the terminology of [21], this data $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^m, z)$ is called a pointed Heegaard $(m+1)$-tuple or, less precisely, a pointed Heegaard multi-diagram. We have a map of groups

$$\widehat{f}_{\eta^0,\dots,\eta^m}\colon \bigotimes_{i=1}^m \widehat{CF}(Y_{\eta^{i-1},\eta^i}) \longrightarrow \widehat{CF}(Y_{\eta^0,\eta^m}),$$

where here Y_{η^i,η^j} denotes the three-manifold described by the Heegaard diagram $(\Sigma, \boldsymbol{\eta}^i, \boldsymbol{\eta}^j)$. This map is obtained by counting pseudo-holomorphic $(m+1)$-gons in $Sym^g(\Sigma)$ which are disjoint from the subvariety $\{z\} \times Sym^{g-1}(\Sigma)$.

More precisely, let $\mathbb{T}_{\eta^i}$ denote the g-dimensional torus $\eta^i_1 \times \cdots \times \eta^i_g$ in the g-fold symmetric product $Sym^g(\Sigma)$. A *Whitney* $(m+1)$*-gon* is a map u from the standard $(m+1)$-gon into $Sym^g(\Sigma)$ which maps the ith edge into $\mathbb{T}_{\eta^i}$ (where here the edges are labelled $0, \dots, m$). Fixing $\mathbf{x}_i \in \mathbb{T}_{\eta^{i-1}} \cap \mathbb{T}_{\eta^i}$ and $\mathbf{y} \in \mathbb{T}_{\eta^0} \cap \mathbb{T}_{\eta^m}$, we let $\pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y})$ denote the set of homotopy classes of Whitney $(m+1)$-gons which, for $i = 1, \dots, m$, map the vertex between the $(i-1)$st and the ith edge to $\mathbf{x}_i$ and the vertex between mth and 0th edges to $\mathbf{y}$.

For fixed $\varphi \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y})$, we let $\mathcal{M}(\varphi)$ denote the set of pseudo-holomorphic representatives for φ. With this notation in place, then, the map $\widehat{f}_{\eta^0,\dots,\eta^m}$ (when $m > 1$) is defined by

$$\widehat{f}_{\eta^0,\dots,\eta^m}(\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_m) = \sum_{\mathbf{y} \in \mathbb{T}_{\eta^0} \cap \mathbb{T}_{\eta^m}} \sum_{\{\varphi \in \pi_2(\mathbf{x}_1,\dots,\mathbf{x}_m,\mathbf{y}) | \mu(\varphi)=0, n_z(\varphi)=0\}} (\#\mathcal{M}(\varphi)) \cdot \mathbf{y},$$

where here $n_z(\varphi)$ denotes the intersection number of φ with the subvariety $\{z\} \times Sym^{g-1}(\Sigma) \subset Sym^g(\Sigma)$, and $\mu(\varphi)$ denotes the expected dimension of the moduli space $\mathcal{M}(\varphi)$ (i.e. the Maslov index of φ, c.f. [6,5]). In the special case where $m = 1$, we sum over homotopy classes with $\mu(\varphi) = 1$, and count points in the quotient space $\mathcal{M}(\varphi)/\mathbb{R}$. Thus, when $m = 1$, the map $\widehat{f}_{\eta^0,\eta^1}$ is simply the differential for the Heegaard Floer chain complex for Y_{η^0,η^1}, and when $m = 2$, $\widehat{f}_{\eta^0,\eta^1,\eta^2}$ is the chain map induced from the counts of pseudo-holomorphic triangles.

Strictly speaking, for the above map to be finite, we require that the Heegaard tuple $(\Sigma, \{\boldsymbol{\eta}^i\}_{i=1}^m, z)$ satisfy a suitable weak admissibility hypothesis. It is sufficient for our purposes to assume that any multi-periodic domain—i.e. two-chain P in Σ which gives a relation amongst homology classes chosen from the γ^i, which has $n_z(P) = 0$—has both positive and negative local multiplicities. This can always be arranged after isotopies, compare Section 4.2.2 of [21].

These maps are well-known to satisfy a generalized associativity property, c.f. [9,25,5]:

$$\sum_{0 \leqslant i < j \leqslant m} \widehat{f}_{\eta^i, \eta^{i+1}, \dots \eta^j} \circ \widehat{f}_{\eta^0, \dots \eta^{i-1}, \eta^i, \eta^j, \dots, \eta^m} = 0. \tag{8}$$

For example, when $m = 1$, the above associativity statement is equivalent to the statement that the square of the differential for $\widehat{CF}(Y_{\eta^0\eta^1})$ is trivial. When $m = 2$, associativity asserts that the maps induced by holomorphic triangles are chain maps, and when $m = 3$, it states that the triangle pairing is associative, up to chain homotopy (hence the name). Thinking of the tuples $\boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^m$ as corresponding to edges of an $(m+1)$-gon, we see that for any pair of edges, there is a corresponding degeneration of the $(m+1)$-gon as a juxtaposition of a pair of a- and b-gons, with $a + b - 3 = m$. The above sum is a sum over all such degenerations of the corresponding composition of maps.

We can construct Heegaard diagrams for the $Y(I)$ as follows. Given Y with the framed link L, we can construct a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$, where here $\beta_1, \dots, \beta_\ell$ are meridians for the links, $\gamma_1, \dots, \gamma_\ell$ are corresponding framing curves (and $\gamma_{\ell+1}, \dots, \gamma_g$ are exact Hamiltonian translates of $\beta_{\ell+1}, \dots, \beta_g$). (For more on the construction of this diagram, see Section 4 of [19]. In the terminology of that paper, the Heegaard triple we are considering is the Heegaard triple subordinate to some bouquet for the framed link L.) We also choose curves $\delta_1, \dots, \delta_\ell$ to be corresponding curves representing the framings obtained by adding meridians to the original framings (and $\delta_{\ell+1}, \dots, \delta_g$ are exact Hamiltonian translates of the $\beta_{\ell+1}, \dots, \beta_g$). Indeed, we choose these curves so that the triply-periodic domain relating β_i, γ_i, δ_i has both positive and negative local multiplicities. Given $I \in \{0, 1, \infty\}^\ell$, let $\boldsymbol{\eta}(I) = \{\eta_1, \dots, \eta_g\}$ denote the g-tuple of attaching circles, where here

$$\eta_i = \begin{cases} \beta_i & \text{if } m_i = \infty, \\ \gamma_i & \text{if } m_i = 0, \\ \delta_i & \text{if } m_i = 1. \end{cases}$$

Thus, a Heegaard diagram for $Y(I)$ is given by $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I), z)$. The required admissibility can be achieved by further winding the α_i-curves if necessary.

Given a sequence of multi-framings $I^0 < \dots < I^k$, there is an induced map

$$D_{I^0 < \dots < I^k} : \widehat{CF}(Y(I^0)) \longrightarrow \widehat{CF}(Y(I^k))$$

defined by

$$D_{I^0 < \dots < I^k}(\xi) = \widehat{f}_{\alpha, \eta(I^0), \dots, \eta(I^k)}(\xi \otimes \widehat{\Theta}_1 \otimes \dots \otimes \widehat{\Theta}_k), \tag{9}$$

where $\widehat{\Theta}_i$ are cycles representing the canonical top-dimensional generators for $\widehat{HF}$ of $Y_{\eta(I^{i-1}),\eta(I^i)}$ which is a connected sum of several copies of $S^2 \times S^1$.

We will be lax about distinguishing here between intersection points in $\mathbb{T}_{\eta(I^i)} \cap \mathbb{T}_{\eta(I^{i+1})}$ and generators of the homology groups $\widehat{HF}(\#^k(S^2 \times S^1); \mathbb{Z}/2\mathbb{Z})$. In fact, if we choose our Hamiltonian translates carefully (i.e. the perturbations of the β_i used in the construction of the Heegaard multi-diagram), we can arrange that for the induced Heegaard diagram for this three-manifold, the differentials vanish, hence each homology generator is represented by a unique intersection point. Recall also that $\widehat{HF}(\#^k(S^2 \times S^1); \mathbb{Z}/2\mathbb{Z}) \cong \Lambda^* H^1(\#^k(S^2 \times S^1); \mathbb{Z}/2\mathbb{Z})$, c.f. Section 3.1 of [18], see also Proposition 6.1 below.)

Let $X = \bigoplus_{I \in \{0,1,\infty\}^\ell} \widehat{CF}(Y(I))$, endowed with the map

$$D: X \longrightarrow X,$$

defined by

$$D\xi = \sum_J \sum_{\{I=I^1<\cdots<I^k=J\}} D_{I^1<\cdots<I^k}(\xi),$$

where here the index set of the inner sum is the set of all increasing sequences connecting I to J, with the property that for all $i = 1, \dots, k-1$, I^{i+1} is an immediate successor of I^i.

Lemma 4.3. *Fix $I, J \in \{0, 1, \infty\}^\ell$. We have that*

$$\sum_{I=I^0<I^1<\cdots<I^k=J} \widehat{f}_{\eta(I^0),\dots,\eta(I^k)}(\widehat{\Theta}_1 \otimes \cdots \otimes \widehat{\Theta}_k) \equiv 0,$$

where again the sum is taken over sequences with the property that I^{i+1} is an immediate successor of I^i.

Proof. We consider the case where $k > 2$. In this case, there is a juxtaposition of triangles representing

$$\widehat{f}_{\eta(I^0),\eta(I^1)\eta(I^2)} \circ \widehat{f}_{\eta(I^0),\eta(I^2),\eta(I^3)} \circ \cdots \circ \widehat{f}_{\eta(I^0),\eta(I^i),\eta(I^{i+1})} \circ \cdots \circ \widehat{f}_{\eta(I^0),\eta(I^{k-1}),\eta(I^k)}.$$

This juxtaposition gives rise to a $(k+1)$-gon $\varphi \in \pi_2(\widehat{\Theta}_1, \dots, \widehat{\Theta}_k, \widehat{\Theta})$, with $\mathcal{D}(\varphi) \geqslant 0$ and $n_z(\varphi) = 0$, where here $\Theta \in \mathbb{T}_{\eta(I^0)} \cap \mathbb{T}_{\eta(I^k)}$. By additivity of the Maslov index, this $k+1$-gon has $\mu(\varphi) = k-2$. It is not difficult that for the chosen Heegaard multi-diagram, there are no $k+1$-gons $\varphi' \in \pi_2(\widehat{\Theta}_1, \dots, \widehat{\Theta}_k, \widehat{\Theta}')$ (where $\widehat{\Theta}'$ is any element of $\mathbb{T}_{\eta(I^0)} \cap \mathbb{T}_{\eta(I^k)}$) with $\mu(\varphi') = 0$ and $\mathcal{D}(\varphi') \geqslant 0$, This follows from the fact that $\mathcal{D}(\phi)$ has small support relative to the multi-periodic domains for the given diagram $(\Sigma, \boldsymbol{\eta}(I^0), \dots \boldsymbol{\eta}(I^k), z)$.

Consider now the case where $k = 2$. In this case, I and J differ in at least one place, and at most two.

If I and J differ in one place, a direct inspection of the Heegaard triple (which leads to the "blowup formula" in [19]) shows that the maps appear (and hence cancel) in pairs. This is spelled out in Proposition 9.5 of [18].

If they differ in two places, there are two choices for I^1 with $I = I^0 < I^1 < J = I^2$. For each possible I^1, it is the case that

$$\widehat{f}_{\eta(I^0),\eta(I^1),\eta(I^2)}(\widehat{\Theta}_1 \otimes \widehat{\Theta}_2) = \widehat{\Theta}_3.$$

One can see this by explicitly drawing the Heegaard triple, which splits into torus summands, as in [19]. (See also Proposition 6.1 below.)

In the case where $k = 1$, the stated relation is simply the one that $\widehat{\Theta}_1$ is a cycle. □

Proposition 4.4. *The map D from Eq.* (9) *satisfies* $D^2 = 0$.

Proof. This follows from the associativity formula (Eq. (8)) for the Heegaard tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\eta}(I^1), \ldots, \boldsymbol{\eta}(I^k), z)$, together with Lemma 4.3. Specifically, according to that lemma, the only degenerations in Eq. (8) which do not contribute 0 to the sum are the ones which involve $\boldsymbol{\alpha}$ in both polygons. Those, in turn, are the various components of D^2. □

In view of Proposition 4.4, we can think of X as a chain complex, endowed with the differential D. We can define some other associated complexes as follows. If $S \subset \{0, 1, \infty\}^\ell$ is a subset with the property that for each $I, J \in S$, for all $K \in \{0, 1, \infty\}^\ell$ with $I < K < J$, we also have that $K \in S$, then we let $X(S)$ denote the group $\bigoplus_{I \in S} \widehat{CF}(Y(I))$ endowed with the differential naturally induced by D.

With all the notational background, we are now ready to prove a strong form of the surgery long exact sequence for a single knot in a three-manifold Y.

Theorem 4.5. *Let K be a framed knot in a three-manifold Y, and let*

$$\widehat{f}\colon \widehat{CF}(Y_0(K); \mathbb{Z}/2\mathbb{Z}) \longrightarrow \widehat{CF}(Y_1(K); \mathbb{Z}/2\mathbb{Z})$$

denote the chain map induced by the cobordism. Then, the chain complex $\widehat{CF}(Y; \mathbb{Z}/2\mathbb{Z})$ *is quasi-isomorphic to the mapping cone of* $\widehat{f}$.

Proof. To start, let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$ denote the associated Heegaard quintuple. In particular, $Y_{\alpha,\beta}$, $Y_{\alpha,\gamma}$ $Y_{\alpha,\delta}$ describe Y, Y_0, and Y_1, respectively, and the remaining three-manifolds on the boundary describe $\#^{g-1}(S^2 \times S^1)$. Indeed, to fit precisely with the hypotheses of that lemma, we choose infinitely many copies of the g-tuples $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ (denoted $\boldsymbol{\beta}^{(i)}$, $\gamma^{(i)}$, $\boldsymbol{\delta}^{(i)}$ for $i \in \mathbb{Z}$), all of which are generic exact Hamiltonian perturbations of one another, in the interest of admissibility (in the sense of Section 4.2.2 of [21]).

In this case, the chain map we described earlier X splits (as a module) as $\widehat{CF}(Y_0) \oplus \widehat{CF}(Y_1) \oplus \widehat{CF}(Y)$, and its differential decomposes as

$$\partial = \begin{pmatrix} D_0 & 0 & 0 \\ D_{0<1} & D_1 & 0 \\ D_{0<1<\infty} & D_{1<\infty} & D_\infty \end{pmatrix}. \tag{10}$$

Letting $\widehat{CF}(Y)$, $\widehat{CF}(Y_0)$, and $\widehat{CF}(Y_1)$ play the roles of A_1, A_2, and A_3 respectively, the various components of the differential play the roles of the f_i and H_i (compare Eqs. (10) and (7)),

Indeed, A_{3i+1}, A_{3i+2} and A_{3i+3} all represent $\widehat{CF}(Y_0)$, $\widehat{CF}(Y_1)$ and $\widehat{CF}(Y)$, respectively, only now we use the various translates of the $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, and $\boldsymbol{\beta}$; in particular A_{3i+1} is the Floer complex $\widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\gamma}^{(i)})$.

Hypothesis (1) of Lemma 4.2 follows at once from the fact that D is a chain complex (Proposition 4.4).

It remains to verify Hypothesis (2) of Lemma 4.2.

Let θ_i be the chain homotopy equivalences induced by equivalences of Heegaard diagrams; e.g. θ_{3i+1} is the chain map $\widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\gamma}^{(i)}) \longrightarrow \widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\gamma}^{(i+1)})$ obtained by product with the canonical generator $\widehat{\Theta}_{\gamma^{(i)},\gamma^{(i+1)}}$.

We claim that

$$f_3 \circ H_1 + H_2 \circ f_1 : A_1 \longrightarrow A_4$$

is chain homotopic to θ_1, and the chain homotopy is given by

$$\mathbf{x} \mapsto f_{\alpha,\gamma,\delta,\beta,\gamma^{(1)}}(\mathbf{x} \otimes \widehat{\Theta}_{\gamma,\delta} \otimes \widehat{\Theta}_{\delta,\beta} \otimes \widehat{\Theta}_{\beta,\gamma^{(1)}}).$$

This in turn follows at once from associativity, together with the fact that

$$f_{\gamma,\delta,\beta,\gamma^{(1)}}(\widehat{\Theta}_{\gamma,\delta} \otimes \widehat{\Theta}_{\delta,\beta} \otimes \widehat{\Theta}_{\beta,\gamma^{(1)}}) = \widehat{\Theta}_{\gamma,\gamma^{(1)}}. \tag{11}$$

This latter equality follows from a direct inspection of the Heegaard diagram for the quadruple $(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\beta}, \boldsymbol{\gamma}^{(1)}, z)$. (i.e. the count of pseudo-holomorphic quadrilaterals), as illustrated in Figs. 8 and 9.

In Fig. 8, we consider the special case where the genus $g = 1$. In the picture, and in the following discussion, $\gamma_1^{(1)}$ is denoted γ_1'. The four corners of the shaded quadrilateral are the canonical generators $\widehat{\Theta}_{\gamma_1,\delta_1}$, $\widehat{\Theta}_{\delta_1,\beta_1}$, $\widehat{\Theta}_{\beta_1,\gamma_1'}$, and $\widehat{\Theta}_{\gamma_1',\gamma_1}$ (read in clockwise order). Indeed, it is straightforward to see (by passing to the universal cover), that the shaded quadrilateral represents the only homotopy class φ_1 of Whitney quadrilaterals with $n_z(\varphi_1) = 0$ and all of whose local multiplicities are non-negative. By the Riemann mapping theorem, now, this homotopy class φ_1 has a unique holomorphic representative u_1. (By contrast, we have also pictured here another Whitney quadrilateral with hatchings, whose local multiplicities are all 0, +1, and −1; +1 at the region where the hatchings go in one direction and −1 where they go in the other.)

For the general case ($g > 1$), we take the connected sum of the case illustrated in Fig. 8 with $g-1$ copies of the torus illustrated in Fig. 9. In this picture, we have illustrated the four curves γ_i, δ_i, β_i, γ_i' for $i > 1$, which are Hamiltonian translates of one another. Now, there is a homotopy class of quadrilateral $\varphi_i \in \pi_2(\widehat{\Theta}_{\gamma_i,\delta_i}, \widehat{\Theta}_{\delta_i,\beta_i}, \widehat{\Theta}_{\beta_i,\gamma_i'}, \widehat{\Theta}_{\gamma_i',\gamma_i})$, and a forgetful map $\mathcal{M}(\phi) \longrightarrow \mathcal{M}(\Box)$ which remembers only the conformal class of the domain (where here $\mathcal{M}(\Box)$ denotes the moduli space of conformal classes of disks with four marked boundary points, also referred to simply as quadrilaterals). Both moduli spaces are one-dimensional (the first moduli space is parameterized by the length of the cut into the region, while the second is parameterized by the ratio of the length to the width, after the quadrilateral is uniformized to a rectangle). By

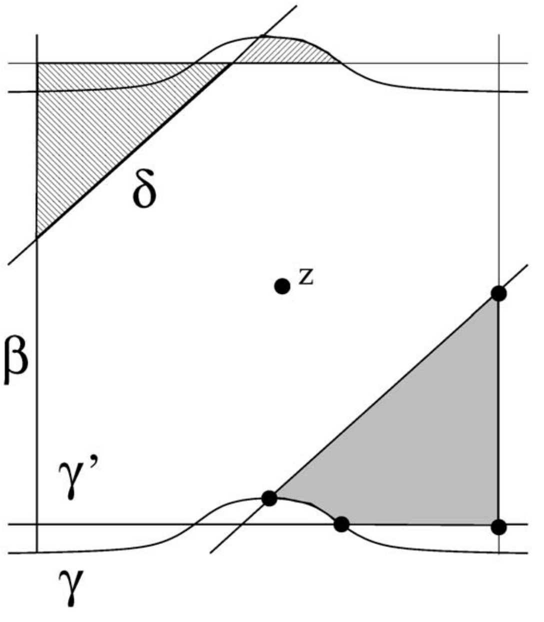

Fig. 8. A holomorphic quadrilateral. The shaded quadrilateral has a unique holomorphic representative (by the Riemann mapping theorem), while the one indicated with the hatching does not, as it has both positive and negative local multiplicities, as indicated by the two directions in the hatching.

Gromov's compactness theorem, the forgetful map is proper; and it is easy to see that it has degree one, and hence for some generic conformal class of quadrilateral, there is a unique pseudo-holomorphic quadrilateral whose domain has the specified conformal class. Now, letting u_1 (the pseudo-holomorphic representative of the homotopy class φ_1 described in the previous paragraph) determine the conformal class of the rectangle, we let u_i for $i > 1$ be the pseudo-holomorphic representatives for φ_i whose domain supports the same conformal class. Then $u_1 \times \cdots \times u_g \in \varphi_1 \times \cdots \times \varphi_g$ is easily seen to be the unique holomorphic quadrilateral in $\pi_2(\widehat{\Theta}_{\gamma,\delta}, \widehat{\Theta}_{\delta,\beta}, \widehat{\Theta}_{\beta,\gamma'}, \widehat{\Theta}_{\gamma',\gamma})$, hence proving Eq. (11) which, in turn, yields Hypothesis (2) of Lemma 4.2. The theorem now follows directly from Lemma 4.2. □

We now turn to Theorem 4.1.

Proof of Theorem 4.1. The theorem is established by induction on the number of components of the link. The case where the link has a single component is a direct consequence of Theorem 4.5.

We form the chain complex X as before. We claim first that $H_*(X) = 0$.

Let

$$S = \{0, 1\}^{\ell-1} \times \{0, 1, \infty\}.$$

The complex $X(S)$ can be filtered by the ordered set $\{0, 1\}^{\ell-1}$ so that its successive quotients are of the form $X(s \times \{0, 1, \infty\})$ (with $s \in \{0, 1\}^{\ell-1}$). By Theorem 4.5 (or,

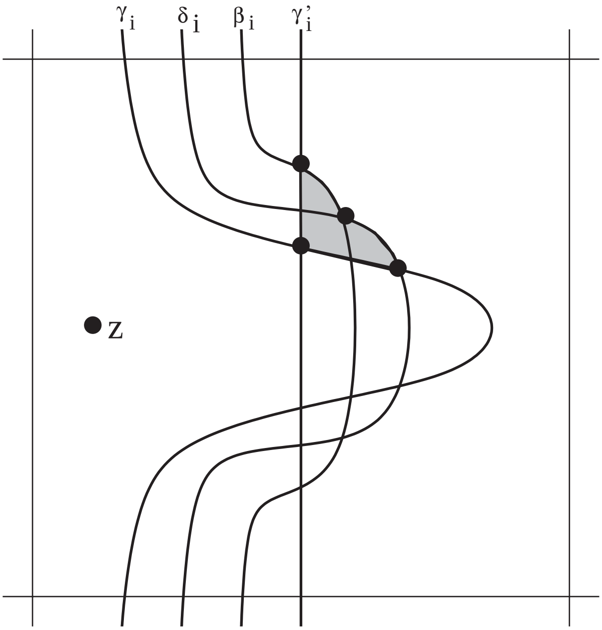

Fig. 9. Other factors of the holomorphic quadrilateral. We have illustrated here a Heegaard quadruple (in a genus one surface) whose four boundary components are $S^2 \times S^1$. In the homotopy class indicated by the shaded quadrilateral $\varphi_i \in \pi_2(\widehat{\Theta}_{\gamma,\delta}, \widehat{\Theta}_{\delta,\beta}, \widehat{\Theta}_{\beta,\gamma'}, \widehat{\Theta}_{\gamma',\gamma})$, there is a moduli space of pseudo-holomorphic quadrilaterals which is clearly one-dimensional, parameterized by a cut at the vertex where γ_i and δ_i meet. We take the connected sum of $g-1$ copies of this picture (at the reference point z) with the picture illustrated in Fig. 8 to obtain the general case of the quadrilateral considered in the proof of Theorem 4.5.

more precisely, the reformulation of the mapping cone lemma, Lemma 4.2, described after its proof), these successive quotients are acyclic, and hence so is $X(S)$. In particular, if we let $T = \{0,1\}^{\ell}$, we have a short exact sequence

$$0 \longrightarrow X(T) \xrightarrow{f} X(S) \xrightarrow{g} X(\{0,1\}^{\ell-1} \times \{\infty\}) \longrightarrow 0$$

from which it follows at once that the connecting homomorphism induces an isomorphism in homology

$$H_*(X(\{0,1\}^{\ell-1} \times \{\infty\})) \xrightarrow{\cong} H_*(X(\{0,1\}^{\ell})).$$

By our inductive hypothesis, it follows that $H_*(X\{0,1\}^{\ell-1} \times \{\infty\}) \cong \widehat{HF}(Y;\mathbb{Z}/2\mathbb{Z})$, completing the proof. □

5. Khovanov's invariants

We briefly describe here Khovanov's categorification of the Jones polynomial; for more details, see [14,2]. We make some simplifying assumptions here: we will use coefficients in $\mathbb{Z}/2\mathbb{Z}$ throughout, and we specialize to the case where $c = 0$ (in Khovanov's sense). Our notation and exposition are tailored to fit neatly into the context of the present paper. In particular, the groups we describe here are actually the Khovanov homology of the mirror of L.

Let $X = S_1 \cup \cdots \cup S_k$ be a collection of disjoint embedded, simple closed curves in the plane. Let $Z(X)$ denote the $\mathbb{Z}/2\mathbb{Z}$-vector space, formally generated by the components $[S_1], \ldots, [S_k]$, and let $V(X)$ denote exterior algebra

$$V(X) = \Lambda^* Z(X),$$

i.e. this is the quotient of the polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ generated by $[S_i]$, divided out by the relations $[S_i]^2 = 0$ for $i = 1, \ldots, k$. (When comparing the notation used in the discussion here with that of [14], observe that the element $[S_1] \wedge \cdots \wedge [S_{m_\ell}] \in V(X)$, where $\{m_j\}_{j=1}^{\ell}$ is a subsequence of $\{1, \ldots, k\}$ corresponds to the element

$$v_1^{\varepsilon_1} \otimes \cdots \otimes v_k^{\varepsilon_k},$$

where here $\varepsilon_i \in \{\pm\}$ is obtained by

$$\varepsilon_i = \begin{cases} - \ \text{if} \ i \in \{m_j\}_{j=1}^{\ell}, \\ + \ \text{otherwise} \end{cases}$$

in Khovanov's notation, c.f. [14].)

Next, consider a pair of pants, thought of as a morphism from $X = S_1 \cup \cdots \cup S_k \cup S_{k+1}$ to a new submanifold $X' = S_1 \cup \cdots \cup S_{k-1} \cup S_k'$ containing a component S_k' which is obtained by merging S_k and S_{k+1}. In this case, we have a natural identification

$$Z(X') = Z(X)/[S_k] \sim [S_{k+1}]$$

and correspondingly natural isomorphisms

$$\alpha\colon (S_{k+1} - S_k) \wedge V(X) \xrightarrow{\cong} V(X') \quad \text{and} \quad \beta\colon V(X') \xrightarrow{\cong} V(X)/(S_{k+1} - S_k) \wedge V(X).$$

We then define the multiplication

$$m\colon V(X) \longrightarrow V(X')$$

to be the composite

$$V(X) \xrightarrow{(S_{k+1}-S_k)\wedge\cdot} (S_{k+1} - S_k) \wedge V(X) \xrightarrow{\ \alpha\ } V(X').$$

By reversing the orientation of the "pair of pants", we have a morphism from X' to X, instead. In this case, we have a comultiplication

$$\Delta\colon V(X') \longrightarrow V(X)$$

induced by the composition

$$V(X') \xrightarrow{\ \beta\ } \frac{V(X)}{(S_{k+1}-S_k)\wedge V(X)} \xrightarrow{(S_{k+1}-S_k)\wedge} V(X).$$

Let L be a link, and fix a generic projection of L, $\mathcal{D}$, with ℓ double points. One can form resolutions indexed by subsets $I \in \{0, 1\}^{\ell}$ (using the conventions from Fig. 1).

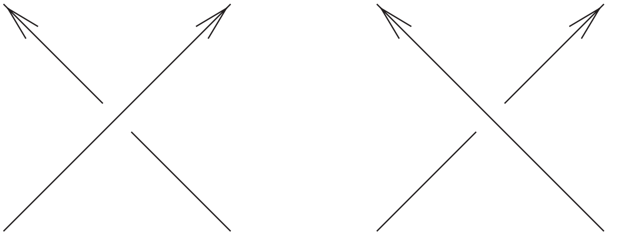

Fig. 10. Crossing conventions. Crossings of the first kind are positive, and those of the second kind are negative.

Specifically, for each I, $\mathcal{D}(I)$ is a disjoint union of circles in the plane. If I' is an immediate successor of I, then $\mathcal{D}(I')$ differs from $\mathcal{D}(I)$ by a single pair of pants. We have a map

$$\mathfrak{d}_{I<I'}\colon V(\mathcal{D}(I)) \longrightarrow V(\mathcal{D}(I')),$$

given by multiplication or co-multiplication, according to whether $\mathcal{D}(I')$ has one fewer or one more component than $\mathcal{D}(I)$.

Fix a diagram $\mathcal{D}$ for an oriented link L, and let $n_+(\mathcal{D})$ resp. $n_-(\mathcal{D})$ denote the number of positive resp. negative crossings for the link L, according to the usual conventions (c.f. Fig. 10). Consider next the graded Abelian group

$$CKh(\mathcal{D}, m) = \bigoplus_{\{I\in\{0,1\}^\ell \,|\, |I|+n_+(\mathcal{D})=m\}} V(\mathcal{D}(I)),$$

where $|I| = \sum_{i\in I} i$. (Note that the roles of $n_+(\mathcal{D})$ and $n_-(\mathcal{D})$ are the opposite to those in [14]: this is because we are describing here the Khovanov homology of the mirror of K.) This group is endowed with the differential

$$\mathfrak{d}\colon CKh(\mathcal{D}, m) \longrightarrow CKh(\mathcal{D}, m+1)$$

whose restriction to $V(\mathcal{D}(I)) \subset CKh(L, m)$ is the sum

$$\mathfrak{d} = \sum_{I<I'} \mathfrak{d}_{I<I'},$$

where the sum is taken over all immediate successors I' of I. In each dimension m, $CKh(\mathcal{D}, m)$ is endowed with an additional grading, the "q-grading", defined by the splitting

$$V(\mathcal{D}(I)) = \bigoplus_{n\in\mathbb{Z}} V_n(\mathcal{D}(I)),$$

where

$$V_n(\mathcal{D}(I)) = \Lambda^k Z^*(\mathcal{D}(I))$$

and

$$n = \dim Z(\mathcal{D}(I)) - 2k - n_-(\mathcal{D}) + 2n_+(\mathcal{D}) - m.$$

Correspondingly, we write

$$CKh(\mathcal{D}) = \bigoplus_{m,n\in\mathbb{Z}} CKh(\mathcal{D}, m, n).$$

Note that $\mathfrak{d}$ carries $CKh(\mathcal{D}, m, n)$ to $CKh(\mathcal{D}, m+1, n)$.

It is easy to see that $\mathfrak{d}^2 = 0$. Khovanov's homology of the mirror of L is the Abelian group

$$Kh(r(L)) = H_*(CKh_*(\mathcal{D}), \mathfrak{d}),$$

thought of as bi-graded Abelian group

$$Kh(r(L)) = \bigoplus_{m,n\in\mathbb{Z}} Kh(\mathcal{D}, m, n).$$

Note that the complex $CKh(\mathcal{D})$ depends on the projection of L. Khovanov shows, however, that the homology of this complex is independent of this choice, i.e. $Kh(L)$ is a link invariant. Moreover, he shows that these groups satisfy a skein exact sequence

$$.. \longrightarrow Kh(r(L)) \longrightarrow Kh(r(L_0)) \longrightarrow Kh(r(L_1)) \longrightarrow \ldots .$$

Khovanov's theory is related to the Jones polynomial by the formula

$$\widehat{J}(r(L)) = \sum_{m,n} (-1)^m (rkKh(L, m, n)) \cdot q^n,$$

where here $\widehat{J}(L) \in \mathbb{Z}[q, q^{-1}]$ is the un-normalized Jones polynomial of the link K, characterized by the formulas

$$\begin{aligned} \widehat{J}(\emptyset) &= 1, \\ \widehat{J}((\text{unknot}) \cup L) &\doteq (q + q^{-1}) \cdot \widehat{J}(L), \\ \widehat{J}(r(L)) &\doteq \widehat{J}(r(L_0)) - q \cdot \widehat{J}(r(L_1)), \end{aligned}$$

where in the last equation, L_0 and L_1 are is taken with respect to the two resolutions at any double-point of any projection of L, and where for $f, g \in \mathbb{Z}[q, q^{-1}]$, we write $f \doteq g$ if $f = q^j \cdot g$ for some $j \in \mathbb{Z}$.

In [16], Khovanov gives a modification of the above constructions to define a "reduced" theory $\widetilde{Kh}(L)$, which is related to the normalized Jones polynomial $J(L)$ defined by $(q+q^{-1}) \cdot J(L) = \widehat{J}(L)$. For the reduced theory, one marks a generic point in the projection of L, so that now in all the various resolutions, there is always a distinguished circle. The reduced Khovanov complex is the quotient of $\widetilde{CKh}(\mathcal{D})$ by the subcomplex of $CKh(\mathcal{D})$ given by

$$\bigoplus_{I\in\{0,1\}^\ell} [S_I] \wedge V(\mathcal{D}(I)),$$

where here S_I is the component in $\mathcal{D}(I)$ which contains the marked point. This gives a chain complex which splits into summands indexed by $I \in \{0,1\}^\ell$, and the corresponding summand is denoted

$$\widetilde{V}(\mathcal{D}(I)) = V(\mathcal{D}(I))/[S_I] \wedge V(\mathcal{D}(I)).$$

For this construction.

$$J(r(L)) = \sum_{m,n} (-1)^m (rk \ \ \widetilde{Kh}(L,m,n)) \cdot q^n,$$

c.f. [16].

We have described Khovanov's construction with $\mathbb{Z}/2\mathbb{Z}$ coefficients. In fact, Khovanov's original definition from [14] makes sense with coefficients in $\mathbb{Z}$. In this case, however the reduced homology (described in [16]) depends on the link, together with the distinguished component containing the marked point. If one takes coefficients in $\mathbb{Z}/2\mathbb{Z}$, as we have here, it is easy to see that the reduced theory is independent of this additional choice, hence giving a link invariant.

6. The spectral sequence for a branched double cover

Throughout this section, we fix our coefficient ring to be $\mathbb{Z}/2\mathbb{Z}$: i.e. if Y is a three-manifold, $\widehat{HF}(Y)$ will denote $\widehat{HF}$ of Y with coefficients in $\mathbb{Z}/2\mathbb{Z}$, and similarly, $H_*(Y)$ will denote singular homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

In comparing Khovanov's homology with $\widehat{HF}$, we rely on the following (fairly straightforward) result about $\widehat{HF}$, proved in earlier papers. For the statement, note that if Y is any three-manifold, then $\widehat{HF}(Y)$ is a module over the algebra $\Lambda^* H_1(Y)/\mathrm{Tors}$.

Proposition 6.1. *Let $Y \cong \#^k(S^2 \times S^1)$. Then, $\widehat{HF}(Y)$ is a rank one, free module over the ring $\Lambda^* H_1(Y)$, generated by some class $\Theta \in \widehat{HF}(Y)$. Moreover, if $K \subset Y$ is a curve which represents one of the circles in one of the $S^2 \times S^1$ summands, then the three-manifold $Y' = Y_0(K)$ is diffeomorphic to $\#^{k-1}(S^2 \times S^1)$, with a natural identification*

$$\pi\colon H_1(Y)/[K] \longrightarrow H_1(Y').$$

Under the cobordism W induced by the two-handle, the map

$$F_W : \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y')$$

is specified by

$$F_W(\xi \cdot \Theta) = \pi(\xi) \cdot \Theta',$$

where here Θ' is some fixed generator of $\widehat{HF}(Y')$, and ξ is any element of $\Lambda^* H_1(Y)$. Dually, if $K \subset Y$ is an unknot, then $Y'' = Y_0(K) \cong \#^{k+1}(S^2 \times S^1)$, with a natural inclusion

$$i : H_1(Y) \longrightarrow H_1(Y'').$$

Under the cobordism W' induced by the two-handle the map

$$F_{W'} : \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y'')$$

is specified by

$$F_{W'}(\xi \cdot \Theta) = \xi \wedge [K''] \cdot \Theta'',$$

where here $[K''] \in H_1(Y'')$ is a generator in the kernel of the map $H_1(Y'') \longrightarrow H_1(W')$.

Proof. The identification of $\widehat{HF}(Y)$ follows from a direct inspection of the Heegaard diagram, as explained in Section 3.1 of [18]. The fact that $F_W(\Theta)$ is a generator for $\widehat{HF}(Y')$ (which we denote by Θ') follows from a direct inspection of a Heegaard triple which naturally splits into genus one summands, c.f. [19]. (Alternately, one could use the surgery exact sequence which in this case reads

$$\ldots \longrightarrow \widehat{HF}(Y') \longrightarrow \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y') \longrightarrow \ldots$$

to deduce that the map from $\widehat{HF}(Y)$ to $\widehat{HF}(Y')$, which in this case is induced by the two-handle W equipped with its torsion Spin^c structure, is surjective.) The more general formula for F_W follows from naturality of the triangle maps under the H_1 action (c.f. [19]). The case of W' follows similarly. See [19,1]. $\square$

We can now link Khovanov's construction (using notation from Section 5) with $\widehat{HF}$:

Proposition 6.2. *Fix a projection $\mathcal{D}$ for K. There is an isomorphism for each I*

$$\Psi(I) : \widetilde{V}(\mathcal{D}(I)) \xrightarrow{\cong} \widehat{HF}(\Sigma(\mathcal{D}(I))),$$

which is natural under cobordisms, in the following sense. If I' is an immediate successor of I, then there is a naturally induced cobordism (induced from a single two-handle addition) from $\Sigma(\mathcal{D}(I))$ to $\Sigma(\mathcal{D}(I'))$, and hence an induced map

$$\widehat{G}_{I<I'} : \widehat{HF}(\Sigma(I)) \longrightarrow \widehat{HF}(\Sigma(I')).$$

Naturality of Ψ is captured in the following commutative diagram, which is valid whenever I' is an immediate successor of I:

$$\begin{array}{ccc} \widetilde{V}(\mathcal{D}(I)) & \xrightarrow{\mathfrak{d}_{I<I'}} & \widetilde{V}(\mathcal{D}(I')) \\ {\scriptstyle \psi(I)}\downarrow & & \downarrow{\scriptstyle \psi(I')} \\ \widehat{HF}(\Sigma(\mathcal{D}(I))) & \xrightarrow{G_{I<I'}} & \widehat{HF}(\Sigma(\mathcal{D}(I'))) \end{array} \qquad (12)$$

Proof. First, note that for each I, we can write $\mathcal{D}(I) = S_0 \cup \cdots \cup S_k$, where here the S_i are pairwise disjoint unknots, and $k \geqslant 0$. In this case, $\Sigma(\mathcal{D}(I)) \cong \#^k(S^2 \times S^1)$. Indeed, we give a basis $\{[\gamma_i]\}_{i=1}^k$ for $H_1(\Sigma(\mathcal{D}(I)))$ as follows. For $i > 0$, let $[\gamma_i] \in H_1(\Sigma(\mathcal{D}(I)))$ be the homology class of the curve obtained as the branched double cover of an arc from S_0 to S_i (recall that we are using here $\mathbb{Z}/2\mathbb{Z}$ coefficients). This induces the identification

$$\widetilde{Z}(\mathcal{D}(I)) \cong H_1(\Sigma(\mathcal{D}(I))).$$

Combined with Proposition 6.1, we get a canonical identification

$$\widetilde{V}(\mathcal{D}(I)) \cong \widehat{HF}(\Sigma(\mathcal{D}(I))).$$

Commutativity of Diagram (12) is proved in four cases, each of which follows from Proposition 6.1. See Fig. 11 for an illustration.

Suppose that I' is obtained from I by merging two circles S_1 and S_2, neither of which is marked. Then, we claim that in the cobordism W, the curves γ_1 and γ_2 become homologous; indeed, both are homologous to the new curve γ_1'. Commutativity of the square now follows readily from Proposition 6.1 and the definition of $\mathfrak{D}_{I<I'}$.

Dually, when I' is obtained from I by splitting an unmarked circle T_1 into two circles S_1 and S_2, the curve $\gamma_1 - \gamma_2$ is null-homologous in the induced cobordism W'. Again, commutativity of the claimed square now follows readily from Proposition 6.1.

The two corresponding cases involving a marked circle follow similarly (indeed, they follow formally in the same manner, once we declare $[\gamma_0] = 0$). $\square$

With these preliminaries in place, we can now state and prove the following precise version of Theorem 1.1 in the introduction.

Theorem 6.3. *Given a projection $\mathcal{D}$ of a link L, there is a spectral sequence converging to $\widehat{HF}(\Sigma(L))$ whose (E^1, d^1) complex is isomorphic to Khovanov's reduced chain complex* (*for the mirror of L*); *i.e. there are isomorphisms ψ^m making the following diagram commute* (Fig. 11):

$$\begin{array}{ccc} E^1_{m-n_+(\mathcal{D})} & \xrightarrow{d^1_{m-n_+(\mathcal{D})}} & E^1_{m-n_+(\mathcal{D})+1} \\ \psi^m \downarrow & & \psi^{m+1} \downarrow \\ \widetilde{CKh}(\mathcal{D}(L), m) & \xrightarrow{\mathfrak{D}^m} & \widetilde{CKh}(\mathcal{D}(L), m+1). \end{array}$$

In particular, the E^2 term of this sequence is identified with Khovanov's homology of L.

Proof. As explained in Section 2, a diagram $\mathcal{D}$ for a link K with ℓ crossings gives rise to a link L in $Y = \Sigma(K)$ whose components correspond to the crossings of $\mathcal{D}$. Moreover, for each $I \subset \{0, 1\}^\ell$, the three-manifold obtained by performing surgeries along these components of L is the branched double cover of S^3 branched along the collection

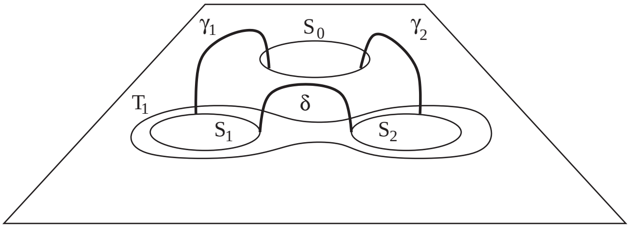

Fig. 11. Homological relations in the cobordisms. The plane of the (un)-link projection is indicated by the quadrilateral, on which we have the marked component S_0, two unlink components S_1 and S_2, and an alternative component T_1, obtained by merging S_1 and S_2. We have also illustrated the curves γ_1 and γ_2. The picture illustrates that γ_1 and γ_2 become homologous after S_1 and S_2 are merged. Dually, it illustrates that in the cobordism where T_1 is divided in two, it is the curve $\delta = \gamma_1 - \gamma_2$ which becomes null-homologous.

of unknots $\mathcal{D}(I)$. We now apply Theorem 4.1. The identification with Khovanov's (reduced) complex is now provided by Proposition 6.2. Note that the spectral sequence coming from Theorem 4.1 is filtered by a cube, rather than $\mathbb{Z}$. We pass to a $\mathbb{Z}$-filtered object by "flattening" the cube as usual. $\square$

Proof of Corollary 1.2. The inequalities follow from $\det(L) = |H^2(\Sigma(L);\mathbb{Z})|$ (c.f. [17]); but this agrees with $\chi(\widehat{HF}(\Sigma(L)))$ (c.f. [18, Proposition 5.1]). It follows at once that $\det(L) \leqslant rk\widehat{HF}(\Sigma(L))$. The other inequality follows from Theorem 6.3, together with the straightforward inequality for spectral sequences $rkE^{\infty} \leqslant rkE^2$. $\square$

Acknowledgments

Similarities between Khovanov's reduced theory and $\widehat{HF}$ of a branched double cover were first noticed by Jacob Rasmussen and Ciprian Manolescu. We would like to thank Jacob Rasmussen for some very helpful discussions on Khovanov's homology, Paul Seidel for some very helpful discussions on homological algebra, and Mikhail Khovanov for comments on an early version of this paper. We are also grateful to the referees for their many useful suggestions, and also, once again, to Ciprian Manolescu and Jacob Rasmussen for many very useful comments on an early draft of our paper.

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