Journal of the Egyptian Mathematical Society (2015) 23, 391–396

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Estimation under Burr type *X* distribution based on doubly type II censored sample of dual generalized order statistics

Abd EL-Baset A. Ahmad [a], Magdy E. El-Adll [b,c,*], Tahani A. ALOafi [d]

[a] *Dept. of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt*
[b] *Dept. of Mathematics and Statistics, Faculty of Science, Helwan University, Ain Helwan, Cairo, Egypt[1]*
[c] *Dept. of Mathematics, Faculty of Science, Taibah University, Madinah, Saudi Arabia[2]*
[d] *Dept. of Mathematics and Statistics, Faculty of Science, Taif University, Taif, Saudi Arabia*

Received 2 October 2013; revised 25 March 2014; accepted 31 March 2014
Available online 5 May 2014

KEYWORDS

Burr type *X* distribution;
Dual generalized order statistics;
Maximum likelihood method;
Bayesian estimation;
Monte Carlo Simulation;
Monte Carlo Integration

Abstract In this paper, maximum likelihood and Bayes estimates of the parameters for Burr type *X* distribution based on doubly type II censored sample of dual generalized order statistics are obtained. Two different cases are considered. In the first case, the shape parameter is estimated when the scale parameter is known while in the second case the estimators for scale and shape parameters are obtained when both parameters are assumed to be unknown. For Bayesian estimation, Monte Carlo Integration is used to improve the approximation of resulting integrals. Simulation studies are conducted to demonstrate the efficiency of the proposed methods through two special cases.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 60E05; 62E15; 62F10; 62F15; 62G30

1. Introduction

Burr type *X* distribution is a member of the family of Burr distributions which was appeared since 1942. It is known also as generalized Rayleigh distribution. This distribution has increasing importance in several areas of applications such as lifetime tests, health, agriculture, biology, and other sciences.

The distribution function (df) and the probability density function (pdf) of Burr type *X* distribution with shape parameter α and scale parameter β are given, respectively, by

$$F(x;\alpha,\beta) = \left(1 - e^{-\beta x^2}\right)^{\alpha}, \quad x > 0, \quad (\alpha > 0, \beta > 0) \qquad (1.1)$$

[1] Permanent address.
[2] Current address.
* Corresponding author at: Dept. of Mathematics, Faculty of Science, Taibah University, Madinah, Saudi Arabia. Tel.: +00 966562908111/201099344955.
E-mail addresses: abahmad10@hotmail.com (A.A. Ahmad), meladll2@yahoo.com (M.E. El-Adll).

Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

1110-256X © 2014 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society.
http://dx.doi.org/10.1016/j.joems.2014.03.011

and

$$f(x;\alpha,\beta)=2\alpha\beta xe^{-\beta x^2}\left(1-e^{-\beta x^2}\right)^{\alpha-1},\quad x>0,\quad(\alpha>0,\beta>0). \tag{1.2}$$

The Burr type X distribution has received tremendous attention, this is due to its flexibility, power in fitting many types of observed data and the increasing number of published papers concern this distribution. The Bayes estimates of the reliability and failure rate functions of the Burr type X model have been studied by Jaheen [1]. Inferences of the function $R=P(Y<X)$ where X and Y are two independent no-identical Burr type X random variables have been studied by Ahmad et al. [2]. Inference and prediction for the Burr type X distribution based on records was studied by Ali Mousa [3]. Jaheen and Al-Matrafi [4] obtained Bayesian prediction bounds from the scaled Burr type X model, order statistics from the Burr type X model have been discussed by Raqab [5]. For more details of properties and applications for Burr type X distribution, see Jaheen [6], Kundu and Raqab [7], Raqab and Kundu [8], Aludaat et al. [9] and Al-Nachawati and Abu-Youssef [10].

The concept of generalized order statistics (gos) was introduced by Kamps [11], as a general framework for models of ordered random variables. Moreover, many other models of ordered random variables, such as, upper order statistics, k-record values, progressively Type II censoring order statistics, Pfeifer records, sequential order statistics are seen to be particular cases of gos. These models can be effectively applied, e.g., in reliability theory. However, random variables that are decreasingly ordered cannot be integrated into this framework. Consequently, this model is inappropriate to study, e.g. reversed ordered order statistic and lower record values models. Burkschat et al. [12] introduced the concept of dual generalized order statistics (dgos). The dgos models enable us to study decreasingly ordered random variables like reversed order statistics, lower k-records and lower Pfeirfer records, through a common approach. For more details of gos and dgos models see Barakat et al. [13], El-Adll et al. [14], Mahmoud and Ghazal [15] Ahsanullah [16], Barakat and El-Adll [17], Ahmed [18], Khan et al. [19].

By analogy with Kamps [11], Burkschat et al. [12] defined the dual generalized order statistics, $X_d(1,n,\tilde{m},k)$, $X_d(2,n,\tilde{m},k),\ldots,X_d(n,n,\tilde{m},k)$, based on an arbitrary continuous df F, by their joint density function,

$$f_{X_d(1,n,\tilde{m},k),X_d(2,n,\tilde{m},k),\ldots,X_d(n,n,\tilde{m},k)} = c_{n-1}\left(\prod_{j=1}^{n-1}(F(x_j))^{\gamma_j-\gamma_{j+1}-1}f(x_j)\right)(F(x_n))^{\gamma_n-1}f(x_n), \tag{1.3}$$

on the cone $\{(x_1,\ldots,x_n):F^{-1}(1)>x_1\geqslant x_2\geqslant\cdots\geqslant x_n>F^{-1}(0)\}\subset\mathbb{R}^n$, where $\gamma_1,\ldots,\gamma_n$ are positive parameters defined by $\gamma_n=k>0,\gamma_r=k+n-r+\sum_{j=r}^{n-1}m_j,\ r=1,2,\ldots,n-1,m_1,m_2,\ldots,m_{n-1}\in\mathbb{R}$ and $c_{n-1}=\left(\prod_{j=1}^{n}\gamma_j\right)$.

Consequently, $X_d(1,n,\tilde{m},k)\geqslant X_d(2,n,\tilde{m},k)\geqslant\cdots\geqslant X_d(n,n,\tilde{m},k)$ holds almost surely. As gos cover several models of increasingly ordered random variables, dgos represent a unification of models of decreasingly ordered random variables, e.g., lower record models.

Ahmad [18] obtained the joint pdf of the non-adjacent dgos, $X_d(r_1,n,m,k),\ldots,X_d(r_l,n,m,k)$, for $1\leqslant r_1<r_2<\cdots<r_l\leqslant n$, $r_0=0,r_{l+1}=n+1$, of the form

$$f_{X_d(r_1,n,m,k),\ldots,X_d(r_\ell,n,m,k)}(x_1,\ldots,x_\ell)=\frac{c_{r_\ell-1}}{\prod_{i=0}^{\ell-1}(r_{i+1}-r_i-1)!}\prod_{i=1}^{\ell}f(x_i)[F(x_\ell)]^{\gamma_{r_\ell}-1}\left\{\prod_{i=0}^{\ell-1}[F(x_i)]^m[h_m(F(x_i))-h_m(F(x_{i+1}))]^{r_{i+1}-r_i-1}\right\}, \tag{1.4}$$

where $F^{-1}(1)>x_1\geqslant\cdots\geqslant x_\ell>F^{-1}(0)$, $c_{r_i-1}=\prod_{i=1}^{r_\ell}\gamma_i, x_i\equiv x_{r_i}$ and

$$h_m(x)=\begin{cases}\frac{1}{m+1}x^{m+1}, & m\neq-1,\\ -\ln x, & m=-1,\end{cases} \tag{1.5}$$

with $g_m(x)=h_m(1)-h_m(x)$.

The rest of this paper is organized as follows: In Section 2, the maximum likelihood estimates are presented, while Section 3, discussed the Bayesian estimation. Section 4 contains numerical computations and some comparisons between the methods for selected models through simulation experiments.

2. Maximum likelihood estimation

Let $X_d(s,n,m,k),\ldots,X_d(r,n,m,k),\quad k>0,1\leqslant s<r\leqslant n$ be dgos based on Burr X (α,β) distribution with df and pdf are given by (1.1) and (1.2) respectively. Setting $r_1=s,r_2=s+1,\ldots,r_\ell=s+\ell+1\equiv r$ in (1.4), the likelihood function (LF), $L(\alpha,\beta|\underline{x})$, can be written in the form

$$L(\alpha,\beta|\underline{x})=\begin{cases}c_r^{(s)}(\alpha\beta)^{r-s+1}\sum_{\ell=0}^{s-1}w_\ell^{(s)}\exp\{\alpha Q_{\ell,i}^{(m)}(\beta)\\ \quad-\sum_{i=s}^{r}[\ln A_i(\beta)+\beta x_i^2]\}, & m\neq-1,\\ d_r^{(s)}\alpha^r\beta^{r-s+1}\exp\{k\alpha\ln A_r(\beta)\\ \quad-\sum_{i=s}^{r}[\ln A_i(\beta)+\beta x_i^2]\\ \quad+(s-1)\ln[\ln A_s(\beta)]\}, & m=-1,\end{cases} \tag{2.1}$$

where

$$\left.\begin{aligned}&c_r^{(s)}=\frac{c_{r-1}\prod_{i=s}^{r}2x_i}{(s-1)!(m+1)^{s-1}},\\ &d_r^{(s)}=\frac{(-1)^{s-1}k^r\prod_{i=s}^{r}2x_i}{(s-1)!},\\ &A_t(\beta)=1-e^{-\beta x_t^2},\\ &\underline{x}=(x_s,\ldots,x_r),\\ &w_\ell^{(s)}=(-1)^\ell\binom{s-1}{\ell},\\ &Q_{\ell,i}^{(m)}(\beta)=(m+1)\left(\sum_{i=s}^{r}\ln A_i(\beta)+\ell\ln A_s(\beta)\right)+\gamma_{r+1}\ln A_r(\beta).\end{aligned}\right\} \tag{2.2}$$

If the parameter β is known, the maximum likelihood estimate (MLE) of the parameter α, can be obtained by solving the nonlinear equation

$$\begin{cases} \frac{d\eta}{d\alpha} = 0 = \frac{r-s+1}{\alpha} - \frac{(s-1)(m+1)[A_s(\beta)]^{(m+1)\alpha}\ln A_s(\beta)}{1-[A_s(\beta)]^{(m+1)\alpha}} \\ \quad +\gamma_{r+1}\ln A_r(\beta) + (m+1)\sum_{i=s}^{r}\ln A_i(\beta), \quad m \neq -1, \\ \frac{d\eta}{d\alpha} = 0 = \frac{r}{\alpha} + k\ln A_r(\beta), \quad m = -1, \end{cases} \tag{2.3}$$

where $\eta = \ln L(\alpha, \beta|\underline{x})$.

To obtain the MLEs $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$, when the two parameters α and β are unknown, we solve the following nonlinear equations numerically when $m \neq -1$,

$$\left.\begin{array}{l} \frac{\partial\eta}{\partial\alpha} = 0 = \frac{r-s+1}{\alpha} - \frac{(s-1)(m+1)[A_s(\beta)]^{(m+1)\alpha}\ln A_s(\beta)}{1-[A_s(\beta)]^{(m+1)\alpha}} \\ +\gamma_{r+1}\ln A_r(\beta) + (m+1)\sum_{i=s}^{r}\ln A_i(\beta), \\ \frac{\partial\eta}{\partial\beta} = 0 = \frac{r-s+1}{\beta} + [\alpha(m+1)-1]\sum_{i=s}^{r}E_i(\beta) + \alpha\gamma_{r+1}E_r(\beta) \\ -\alpha(m+1)(s-1)\frac{E_s(\beta)[A_s(\beta)]^{\alpha(m+1)}}{1-[A_s(\beta)]^{\alpha(m+1)}} - \sum_{i=s}^{r}x_i^2. \end{array}\right\} \tag{2.4}$$

For $m = -1$, we can obtain the MLEs $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ by solving the following nonlinear equations numerically

$$\left.\begin{array}{l} \frac{\partial\eta}{\partial\alpha} = 0 = \frac{r}{\alpha} + k\ln A_r(\beta), \\ \frac{\partial\eta}{\partial\beta} = 0 = \frac{r-s+1}{\beta} - \sum_{i=s}^{r}[E_i(\beta) + x_i^2] + k\alpha E_r(\beta) + \frac{(s-1)E_s(\beta)}{\ln A_s(\beta)}, \end{array}\right\} \tag{2.5}$$

where

$$E_t(\beta) = \frac{\partial \ln A_t(\beta)}{\partial\beta} = \frac{x_t^2 e^{-\beta x_t^2}}{A_t(\beta)}. \tag{2.6}$$

3. Bayesian estimation

In this section, Bayesian estimation of the parameters of Burr type X distribution is considered in two cases. For the first case we assume that the parameter β is known and in the second case the two parameters α and β are assumed to be unknown.

3.1. One parameter case (β is known)

When the parameter β is known, we use the gamma conjugate prior density for the parameter α with density function of the form

$$\pi(\alpha) = \frac{a^b}{\Gamma(b)}\alpha^{b-1}e^{-a\alpha}, \quad \alpha > 0, \quad (a > 0, b > 0). \tag{3.1}$$

It follows from (2.1) and (3.1), that the posterior density of α can be written as

$$\pi^\star(\alpha|\beta,\underline{x}) = \begin{cases} k_1\alpha^{r-s+b}\sum_{\ell=0}^{s-1}w_\ell^{(s)}\exp\left\{-\alpha[a - Q_{\ell,i}^{(m)}(\beta)]\right\}, & m \neq -1, \\ k_2\alpha^{r+b-1}\exp\{-\alpha[a - k\ln A_r(\beta)]\}, & m = -1, \end{cases} \tag{3.2}$$

where

$k_1^{-1} = \Gamma(r-s+b+1)\sum_{\ell=0}^{s-1}w_\ell^{(s)}[a - Q_{\ell,i}^{(m)}(\beta)]^{s-r-b-1}$ and

$k_2^{-1} = \Gamma(r+b)[a - k\ln A_r(\beta)]^{-(r+b)}$.

Under a squared error loss (SEL) function, the Bayes estimate of the parameter α is the posterior mean in the form

$$\hat{\alpha}_B = \begin{cases} (r-s+b+1)\frac{\sum_{\ell=0}^{s-1}w_\ell^{(s)}[a-Q_{\ell,i}^{(m)}(\beta)]^{s-r-b-2}}{\sum_{\ell=0}^{s-1}w_\ell^{(s)}[a-Q_{\ell,i}^{(m)}(\beta)]^{s-r-b-1}}, & m \neq -1, \\ \frac{r+b}{a-k\ln A_r(\beta)}, & m = -1. \end{cases} \tag{3.3}$$

3.2. Two unknown parameters case

When both of the two parameters α and β are unknown, the prior density function is given by

$$\pi(\alpha,\beta) = \pi_1(\alpha|\beta)\pi_2(\beta), \tag{3.4}$$

where

$$\pi_1(\alpha|\beta) = \frac{\beta^b}{\Gamma(b)a^b}\alpha^{b-1}e^{-\frac{\beta}{a}\alpha}, \quad \alpha > 0, \quad (a > 0, b > 0) \tag{3.5}$$

and

$$\pi_2(\beta) = \frac{d^c}{\Gamma(c)}\beta^{c-1}e^{-d\beta}, \quad \beta > 0, \quad (c > 0, d > 0). \tag{3.6}$$

Therefore, the joint probability density function of α and β takes the form

$$\pi(\alpha,\beta) = k_3\alpha^{b-1}\beta^{b+c-1}e^{-\left(\frac{\alpha}{a}+d\right)\beta}, \quad \alpha > 0, \ \beta > 0,$$

where

$$k_3^{-1} = \Gamma(b)\Gamma(c)a^b d^{-c}. \tag{3.7}$$

It follows from (2.1) and (3.2) that the joint posterior density function of α and β given the data is given by

$$\pi^\star(\alpha,\beta|\underline{x}) = \begin{cases} k_4\alpha^{r+b-s}\beta^{r+b+c-s}\sum_{\ell=0}^{s-1}w_\ell^{(s)}\exp\{\alpha Q_{\ell,i}^{(m)}(\beta) \\ \quad -\sum_{i=s}^{r}[\ln A_i(\beta)+\beta x_i^2] - \beta(\frac{\alpha}{a}+d)\}, & m \neq -1, \\ k_5\alpha^{r+b-1}\beta^{r+b+c-s}\exp\{k\alpha\ln A_r(\beta) \\ \quad -\sum_{i=s}^{r}[\ln A_i(\beta)+\beta x_i^2] \\ \quad +(s-1)\ln[\ln A_s(\beta)] - \beta(\frac{\alpha}{a}+d)\}, & m = -1, \end{cases}$$

where

$$k_4^{-1} = \int_0^\infty\int_0^\infty \alpha^{r+b-s}\beta^{r+b+c-s}\sum_{\ell=0}^{s-1}w_\ell^{(s)} \times \exp\left\{\alpha Q_{\ell,i}^{(m)}(\beta) - \sum_{i=s}^{r}[\ln A_i(\beta)+\beta x_i^2] - \beta\left(\frac{\alpha}{a}+d\right)\right\}d\alpha d\beta$$

$$= \Gamma(r+b-s+1)\sum_{\ell=0}^{s-1}w_\ell^{(s)}I_\ell(s),$$

$$I_\ell(s) = \int_0^\infty \frac{\beta^{r+b+c-s}\exp\left\{-\beta d - \sum_{i=s}^{r}[\ln A_i(\beta)+\beta x_i^2]\right\}}{\left[\frac{\beta}{a} - Q_{\ell,i}^{(m)}(\beta)\right]^{r+b-s+1}}d\beta, \tag{3.8}$$

$$k_5^{-1} = \int_0^\infty\int_0^\infty \alpha^{r+b-1}\beta^{r+b+c-s}\exp\left\{\alpha k\ln A_r(\beta) - \sum_{i=s}^{r}[\ln A_i(\beta) + \beta x_i^2] + (s-1)\ln[\ln A_s(\beta)] - \beta\left(\frac{\alpha}{a}+d\right)\right\}d\alpha\, d\beta = \Gamma(r+b)I_0$$

and

$$I_0 = \int_0^{\infty} \frac{\beta^{r+b+c-s} \exp\left\{-\sum_{i=s}^{r}[\ln A_i(\beta) + \beta x_i^2] + (s-1)\ln[\ln A_s(\beta)]\beta d\right\}}{\left[\frac{\beta}{a} - k \ln A_r(\beta)\right]^{r+b}} d\beta. \tag{3.9}$$

Assuming a SEL function, the Bayes estimate of any function U of the parameters α and β is the posterior mean of the form

$$\begin{aligned} E[U(\alpha,\beta)|\underline{x}] &= \int_0^{\infty}\int_0^{\infty} U(\alpha,\beta)\pi^{\star}(\alpha,\beta;\underline{x})d\alpha d\beta \\ &= \frac{\int_0^{\infty}\int_0^{\infty} U(\alpha,\beta)L(\alpha,\beta;\underline{x})\pi(\alpha,\beta)d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty} L(\alpha,\beta;\underline{x})\pi(\alpha,\beta)d\alpha d\beta}. \end{aligned} \tag{3.10}$$

The ratio of integrals in (3.10) does not seem to take a closed form, so we shall consider Monte Carlo Integration (MCI) method to approximate such integrals. MCI can be used to approximate posterior distributions required for a Bayesian analysis.

The integral (3.10) can be approximated by

$$\widehat{U} = E[U(\alpha,\beta)|\underline{x}] = \frac{\sum_{j=1}^{M} U(\alpha_j,\beta_j)L(\alpha_j,\beta_j|\underline{x})}{\sum_{j=1}^{M} L(\alpha_j,\beta_j|\underline{x})}, \tag{3.11}$$

where α_j and β_j generated from (3.5) and (3.6) and $L(\alpha_j,\beta_j)$ can be obtained from (2.1) as following

$$L(\alpha_j,\beta_j) = \begin{cases} c_r^{(s)}(\alpha_j\beta_j)^{r-s+1}\sum_{\ell=0}^{s-1} w_\ell^{(s)} \exp\left\{\alpha_j Q_{\ell,i}^{(m)}(\beta_j) \right. \\ \left. \quad -\sum_{i=s}^{r}[\ln A_i(\beta_j) + \beta_j x_i^2]\right\}, & m \neq -1, \\ d_r^{(s)}\alpha_j^r \beta_j^{r-s+1} \exp\left\{k\alpha_j \ln A_r(\beta_j) \right. \\ \quad -\sum_{i=s}^{r}[\ln A_i(\beta_j) + \beta_j x_i^2] \\ \left. \quad +(s-1)\ln(\ln A_s(\beta_j)\right\}, & m = -1. \end{cases} \tag{3.12}$$

4. Simulation study

In order to explain the efficient of the theoretical results of Sections 2 and 3, numerical results for two important special cases from dgos are presented through simulation experiments. The first special case is reversed ordered order statistics (r-oos) and the second is lower record values. For each special case we considered the one unknown parameter case and the two unknown parameters case. Finally, we introduced some algorithms to prepare the numerical computations by using Mathematica 9.

4.1. Reversed ordered order statistics

The r-oos can be obtained from dgos model by choosing $m = 0, k = 1$, i.e. $\gamma_i = n - i + 1$. Now, we consider one unknown parameter and two unknown parameters cases.

4.1.1. One unknown parameter case

The maximum likelihood estimator, $\hat{\alpha}_{ML}$, and the Bayes estimator, $\hat{\alpha}_B$, for the shape parameter α of Burr type X distribution when the scale parameter β is known based on r-oos can be obtained from (2.3) and (3.3), respectively, by setting $m = 0$ and $k = 1$. For this purpose, we introduced the following algorithm:

Algorithm 1.

1. Choose the values of n, β and the prior parameters a and b,
2. generate a random number α from gamma distribution with pdf (3.1),
3. generate a random sample of size n from Burr type X distribution with parameters α and β,
4. sort the random sample from largest to smallest,
5. get the ML estimate of the parameter α by solving the nonlinear Eq. (2.3),
6. get the Bayes estimate, under a SEL function of α is computed by using MCI, from the Eq. (3.3) with $m = 0, k = 1$ using the reversed ordered sample in step 4,
7. compute the squared deviations $(\hat{\alpha} - \alpha)^2$ for each estimate (ML or Bayes),
8. repeats steps 3, 4, 5 and 6, 1000 times and then compute the mean of estimates ($\bar{\hat{\alpha}}_{ML}$, $\bar{\hat{\alpha}}_B$) and the mean squared error ($MSE(\alpha_{ML})$, $MSE(\alpha_B)$), by averaging the estimates and the squared deviations. (See Table 1).

4.1.2. Two unknown parameters case

Recall we assume that the two parameters, α and β are unknown. In this subsection, we use the r-oos model to obtain ML and Bayes estimates of the parameters α and β. we get the following algorithm:

Algorithm 2.

1. Choose the values of n and the prior parameters (c, d), of the unknown parameter β,
2. generate a random number β from gamma distribution with pdf (3.6),
3. use step 2 to generate a random number α from gamma distribution with parameters a, b and β with density (3.5),
4. generate a random sample of size n from Burr type X distribution with parameters α and β,
5. sort the random sample from largest to smallest,
6. get the ML estimates of the parameter α and β by solving the nonlinear equations (2.4) with $m = 0$ and $k = 1$,
7. get the Bayes estimates, under a SEL function of α and β are computed by using MCI, from the Eqs. (3.11), (3.12) with $m = 0$, $k = 1$ and $U = \alpha_j$ to compute $\hat{\alpha}_B$ and $U = \beta_j$ to compute $\hat{\beta}_B$,
8. compute the squared deviations $(\hat{\alpha} - \alpha)^2$ and $(\hat{\beta} - \beta)^2$ for each estimate (ML or Bayes),
9. repeats steps 4, 5, 6, 7 and 8, 1000 times and then compute the mean of estimates ($\bar{\hat{\alpha}}_{ML}$, $\bar{\hat{\alpha}}_B$), ($\bar{\hat{\beta}}_{ML}$, $\bar{\hat{\beta}}_B$) and the mean squared error ($MSE(\alpha_{ML}), MSE(\alpha_B)$), ($MSE(\beta_{ML}), MSE(\beta_B)$), by averaging the estimates and the squared deviations.

The results are displayed in Table 2.

4.2. Lower record values

The lower records $X_{l(s)}, \ldots, X_{l(r)}$ can be obtained from the dgos model by setting $m = -1$ and $k = 1$. The ML and Bayes estimates of the parameters α and β in terms of lower records are compared in this subsection. The comparison is used through Monte Carlo Simulation study in two cases, the parameter β is known and α is unknown, and when the two parameters are unknown.

Table 1 The mean of estimates ($\bar{\hat{\alpha}}_{ML}$, $\bar{\hat{\alpha}}_B$) of the parameter α and the mean squared errors ($MSE(\alpha_{ML}), MSE(\alpha_B)$) when $\beta = 1, a = 0.2, b = 0.5, \alpha = 0.3687$, based on r-oos.

n	r	s	$\bar{\hat{\alpha}}_{ML}$	$MSE(\alpha_{ML})$	$\bar{\hat{\alpha}}_B$	$MSE(\alpha_B)$
12	12	1	0.40096	0.0174	0.4146	0.0193
		2	0.4314	0.0272	0.4462	0.0303
	8	1	0.3019	0.0260	0.3178	0.02595
		2	0.3284	0.0276	0.3459	0.0287
30	30	1	0.3806	0.0049	0.3859	0.0052
		2	0.3873	0.0054	0.3927	0.0058
	20	1	0.281	0.0159	0.2871	0.0151
		2	0.2920	0.0168	0.2984	0.0162
90	90	1	0.3726	0.0017	0.3743	0.0017
		2	0.3761	0.0018	0.3779	0.0018
	60	1	0.2765	0.01399	0.2785	0.0137
		2	0.2750	0.0141	0.2770	0.0138

Table 2 The mean of estimates ($\bar{\hat{\alpha}}_{ML}$, $\bar{\hat{\alpha}}_B$) and ($\bar{\hat{\beta}}_{ML}$, $\bar{\hat{\beta}}_B$) of the parameters α and β and the mean squared errors ($MSE(\alpha_{ML}), MSE(\alpha_B)$) and ($MSE(\beta_{ML}), MSE(\beta_B)$) when $b = 2, c = 3, d = 5, \beta = 1.0558, \alpha = 2.0476$.

n	r	s	$\bar{\hat{\alpha}}_{ML}$ (MSE)	$\bar{\hat{\beta}}_{ML}$ (MSE)	$\bar{\hat{\alpha}}_B$ (MSE)	$\bar{\hat{\beta}}_B$ (MSE)
30	23	1	2.5796	1.1614	2.2025	1.0435
			(1.6523)	(0.0889)	(0.2913)	(0.00215)
		2	2.5429	1.1531	2.1026	1.0059
			(1.4915)	(0.0904)	(0.1993)	(0.0199)
50	45	1	2.2275	1.1022	2.1489	1.0615
			(0.3634)	(0.0361)	(0.2421)	(0.0246)
		2	2.2489	1.1030	2.2138	1.0778
			(0.3895)	(0.0373)	(0.1953)	(0.0194)
90	60	1	2.2302	1.0928	2.1742	1.0562
			(0.3679)	(0.0249)	(0.1990)	(0.0155)
		2	2.2079	1.0886	2.1095	1.0498
			(0.3324)	(0.0265)	(0.1469)	(0.0132)

Table 3 The mean of estimates ($\bar{\hat{\alpha}}_{ML}$, $\bar{\hat{\alpha}}_B$) of the parameter α and the mean squared errors ($MSE(\alpha_{ML}), MSE(\alpha_B)$) when $\beta = 1,\ a = 0.2,\ b = 0.3,\ \alpha = 0.473285$, based on lower record values.

n	r	s	$\bar{\hat{\alpha}}_{ML}$	$MSE(\alpha_{ML})$	$\bar{\hat{\alpha}}_B$	$MSE(\alpha_B)$
6	6	1	0.5661	0.0923	0.5807	0.0923
		2	0.5639	0.0851	0.5787	0.0862
	5	1	0.6035	0.1608	0.6192	0.1546
		2	0.6057	0.1362	0.6223	0.1352
9	9	1	0.5496	0.0472	0.5601	0.0489
		2	0.5424	0.0491	0.5528	0.0505
	7	1	0.5496	0.0598	0.5678	0.0618
		2	0.5384	0.0551	0.5515	0.0565
12	12	1	0.5581	0.0267	0.5665	0.0283
		2	0.5628	0.0328	0.5711	0.0344
	8	1	0.5932	0.0632	0.6052	0.0658
		2	0.5947	0.0657	0.6085	0.0683

Table 4 the mean of estimates (ML and Bayes) of the parameters α and β and the mean squared error when $a = 0.5, b = 1.5, c = 3, d = 2, \beta = 1.0868, \alpha = 1.0126$.

n	r	s	$\bar{\hat{\alpha}}_{ML}$ (MSE)	$\bar{\hat{\beta}}_{ML}$ (MSE)	$\bar{\hat{\alpha}}_{B}$ (MSE)	$\bar{\hat{\beta}}_{B}$ (MSE)
6	5	1	1.5824 (0.3243)	2.5259 (2.0712)	1.0544 (0.0017)	1.2034 (0.0138)
		2	3.2701 (5.0963)	0.9189 (0.0281)	1.0808 (0.0046)	1.2175 (0.0171)
10	7	1	1.5346 (0.2724)	1.5448 (0.2098)	1.2029 (0.0387)	1.0754 (0.0001)
		2	1.4536 (0.1945)	1.2039 (0.0137)	1.2215 (0.0436)	1.0129 (0.0055)
12	8	1	0.7881 (0.0504)	0.1881 (0.8075)	0.9245 (0.0078)	0.5733 (0.2636)
		2	0.7501 (0.0689)	0.1125 (0.9492)	0.9215 (0.0083)	0.6450 (0.1951)

4.2.1. One unknown parameter case

In this case the algorithm is similar to Algorithm 1, except for, $m = -1$ and steps 4 and 5 are replaced by the following step:

- generate the first n lower record values from Burr type X distribution with parameters α and β.

The results of this case are summarized in Table 3.

4.2.2. Two unknown parameters case

For this case the algorithm is similar to Algorithm 2, except for, $m = -1$, steps 4 and 5 are replaced by the following step:

- generate the first n lower record values from Burr type X distribution with parameters α and β,

and the ML estimates of the parameter α and β by solving the nonlinear equations (2.5) with $m = -1$ and $k = 1$. The results are presented in Table 4.

5. Concluding remarks

1. The ML and Bayes methods of estimation are used to estimate the parameters of the Burr type X distribution based on doubly Type
2. II censored samples from dgos. Estimation based on reversed ordered order statistics and lower records are presented as special cases.
3. From the simulation study, it is observed that, in most cases Bayes estimates are better than their corresponding ML estimates.
4. In general, it is noted that the accuracy of the estimates increases as n increase.
5. The results in the case of one unknown parameter in lower records are satisfactory compared with the results of Ali-Mousa [3].

Acknowledgements

The authors would like to thank the anonymous referees for constructive suggestions and comments that improve the representation substantially.

References

[1] Z.F. Jaheen, Empirical Bayes estimation of the reliability and failure rate functions of the Burr type X failure model, Appl. Stat. Sci. 3 (1996) 281–288.
[2] E.K. Ahmad, M. Fakhry, Z.F. Jaheen, Impirical Bayes estimation of $p(y < x)$ and characterizations of the Burr type x model, J. Stat. Plan. Infer. 64 (1997) 297–308.
[3] M. Ali Mousa, Inference and prediction for the Burr type X model based on records, Statistics 35 (2001) 415–425.
[4] Z.F. Jaheen, B.N. Al-Matrafi, Bayesian prediction bounds from the scaled Burr type X model, Comput. Math. Appl. 44 (2002) 587–594.
[5] M.Z. Raqab, Order statistics from the Burr type X model, Comput. Math. Appl. 36 (1998) 111–120.
[6] Z.F. Jaheen, Estimation based on generalized order statistics from the Burr model, Commun. Stat. Theor. Meth. 34 (4) (2005) 785–794.
[7] D. Kundu, M.Z. Raqab, Generalized Rayleigh distribution: different methods of estimation, Comput. Stat. Data Anal. 49 (2005) 187–200.
[8] M.Z. Raqab, D. Kundu, Burr type X distribution: revisited, J. Prob. Stat. Sci. 4 (2) (2006) 179–193.
[9] K.M. Aludaat, M.T. Alodat, T.T. Alodat, Parameter estimation of Burr type X distribution for grouped Data, Appl. Math. Sci. 2 (9) (2008) 415–423.
[10] H. Al-Nachawati, S.E. Abu-Youssef, A Bayesian analysis of order statistics from the generalized Rayleigh distribution, Appl. Math. Sci. 3 (27) (2009) 1315–1325.
[11] U. Kamps, A Concept of Generalized Order Statistics, Teubner, Stuttgart, 1995.
[12] M. Burkschat, E. Cramer, U. Kamps, Dual generalized order statistics, Metron LXI (1) (2003) 13–26.
[13] H.M. Barakat, M.E. El-Adll, A.E. Amany, Exact prediction intervals for future exponential lifetime based on random generalized order statistics, Comput. Math. Appl. 61 (5) (2011) 1366–1378.
[14] M.E. El-Adll, S.F. Ateya, M.M. Rizk, Prediction intervals for future lifetime of three parameters Weibull observations based on generalized order statistics, Arab. J. Math. 1 (2012) 295–304.
[15] M.A.W. Mahmoud, M.G.M. Ghazal, Characterizations of mixture of two-component exponentiated family of distributions based on generalized order statistics, J. Egypt. Math. Soc. 20 (2012) 205–210.
[16] M. Ahsanullah, A characterization of the uniform distribution by dual generalized order statistics, Commun. Stat. Theor. Meth. 33 (2004) 2921–2928.
[17] H.M. Barakat, M.E. El-Adll, Asymptotic theory of extreme dual generalized order statistics, Statist Probab. Lett. 79 (9) (2009) 1252–1259.
[18] A.A. Ahmad, Joint moment generating functions of nonadjacent dual generalized order statistics from reflected generalized Pareto distributions, Commun. Stat. Theor. Meth. 41 (15) (2012) 2762–2772.
[19] A.H. Khan, I.A. Shah, M. Ahsanullah, Characterization through distributional properties of dual generalized order statistics, J. Egypt. Math. Soc. 20 (2012) 211–214.