On Representations of Non-semisimple Specialized Hecke Algebras

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Communicated by Nagayoshi Iwahori

Received June 1, 1990

INTRODUCTION

Let \((W, S)\) be a finite Coxeter system (see [4]), \(q\) an indeterminate, and \(H_q(W)\) a free \(\mathbb{C}[\sqrt{q}]\)-module with a basis \(\{T_w\}_{w \in W}\) parametrized by the elements of \(W\). Here \(\mathbb{C}\) denotes the field of complex numbers. Then \(H_q(W)\) has an associative \(\mathbb{C}[\sqrt{q}]\)-algebra structure characterized by the conditions

\[(T_s + 1)(T_s - q) = 0, \quad \text{if } s \in S\]

and

\[T_w T_{w'} = T_{w w'}, \quad \text{if } l(w) + l(w') = l(ww'),\]

where \(l\) is the length function. See [4, Chap. 4, Sect. 2, Ex. 23; 15]. Let \(\alpha\) be a complex number. Then we obtain a \(\mathbb{C}\)-algebra \(H_a(W)\) by specialization \(q \mapsto \alpha\). (See p. 637 of [7, II]. Note: We actually consider the specialization \(\sqrt{q} \mapsto \sqrt{\alpha}\) choosing \(\sqrt{\alpha}\). However, we indicate it by \(q \mapsto \alpha\) for convenience.) Suppose that \((W, S)\) is not of type \(A_n\). Let \(\alpha\) be a complex number. Then we obtain a \(\mathbb{C}\)-algebra \(H_{\alpha}(W)\) by specialization \(q \mapsto \alpha\). (See p. 637 of [7, II].) Note: We actually consider the specialization \(\sqrt{q} \mapsto \sqrt{\alpha}\) choosing \(\sqrt{\alpha}\). However, we indicate it by \(q \mapsto \alpha\) for convenience.) Suppose that \((W, S)\) is not of type \(A_n\). Then it is known that \(H_{\alpha}(W)\) is semisimple if and only if \(\alpha \neq 0\) and \(P_w(\alpha) \neq 0\), where \(P_w(\alpha)\) is the Poincaré polynomial of \((W, S)\) defined by \(P_w(\alpha) = \sum_{w \in W} q^{l(w)}\). (See [13, Theorem].) Moreover, if \(H_{\alpha}(W)\) is semisimple, then it is isomorphic to the group algebra \(\mathbb{C}(W)\). (See [7, Sect. 68].) Then one might be interested in what representations of non-semisimple specialized Hecke algebras look like. After we work with some examples, we recognize that in certain cases their representations quite resemble those of \(W\) over some finite field: Their indecomposable modules, the Loewy structure, the block decomposition, and so on. In particular, if \((W, S)\) is of type \(A_{p-1}\) and \(\alpha\) is a primitive \(p\)th root of unity for some prime \(p\), then, at least for small \(p\), we find by direct computation that representations of \(H_{\alpha}(W)\) look

* The author is partially supported by Grant-in-Aid for Scientific Research 01740041.
similar to those of the group algebra $F_p(W)$ of $W$ over the finite field $F_p$ of $p$ elements. (Note: In this case, $W$ is isomorphic to the symmetric group on $p$ letters.) In particular, $H_x(W)$ and $F_p(W)$ are both of finite representation type, namely, there are only finitely many isomorphism classes of indecomposable modules. In general, a modular group algebra $F_p(G)$ of $G$ is of finite representation type if and only if a $p$-Sylow subgroup of $G$ is cyclic [14]. For example, the symmetric group on $n$ letters has a (non-trivial) cyclic $p$-Sylow subgroup if and only if $n/2 < p \leq n$, which is equivalent to the fact that $p$ is the highest power of $p$ that divides the group order $n!$. Recall that a condition on the semisimplicity of specialized Hecke algebras can be stated in terms of the Poincaré polynomials while that of modular group algebras is related to the group order (Maschke's theorem). So, one might think that a condition that a specialized Hecke algebra is of finite representation type may also be obtained in terms of its Poincaré polynomial. Here we raise the following question.

**Question.** Let $\alpha$ be a non-zero complex number. Then is it true that the algebra $H_x(W)$ is of finite representation type if and only if $\alpha$ is a simple root of $P_x(q) = 0$?

In the case where $(W, S)$ is of type $A_l$, the Poincaré polynomial is $(1 + q)(1 + q + q^2) \cdots (1 + q + q^2 + \cdots + q^l)$. Thus $\alpha$ is a simple root of $P_x(q) = 0$ if and only if it is a primitive $r$th root of unity for some $r$ with $(l + 1)/2 < r \leq l + 1$. The above-mentioned case is precisely that of $l = p - 1$ and $r = p$. So it gives evidence of an affirmative answer to the question. The purpose of the present paper is to show that the question is affirmatively answered if $(W, S)$ is of type $A_l$ or of rank two. (See Theorems 3.8, 4.4, and 5.3.) In proving these, we determine the Loewy structure of all the indecomposable modules in the case where $(W, S)$ is of type $A_l$ and $\alpha$ is a primitive $(l + 1)$th root of unity. (See Theorem 3.6.) For those computations we use the result [8, 9, 18] on irreducible representations of $H_x(W)$. Also, techniques used in this paper come from Auslander–Reiten theory. (See Sect. 1.)

Notation is standard. For terminologies of representation theory, see [7, 10]. All the modules considered here are finitely generated right modules. Let $A$ be a (finite dimensional) algebra over some field, and let $\{\rho_i\}$ be a complete set of inequivalent irreducible representations of $A$. For an $A$-module $V$, we denote its Loewy series by

\[
\rho_{1,1}\rho_{1,2}\rho_{1,i_1},
\rho_{2,1}\rho_{2,2}\rho_{2,i_2},
\ldots,
\rho_{r,1}\rho_{r,2}\rho_{r,i_r},
\]
where \( \rho_{i,j} \) is some \( \rho_i \). This means that its maximal semisimple factor module \( V/V_1 \) gives a representation \( \rho_{1,1} \oplus \rho_{1,2} \oplus \cdots \oplus \rho_{1,i_1} \), and then, the maximal semisimple factor module \( V_1/V_2 \) of \( V_1 \) gives \( \rho_{2,1} \oplus \rho_{2,2} \oplus \cdots \oplus \rho_{2,i_2} \), and so on. Notice that the above \( V_1 \) is so called the radical of \( V \).

1. Representation Theory of Algebras

First, we mention Auslander–Reiten theory. For detail, see, for example, [7, II, Sect. 78; 16] and the references therein. Let \( A \) be a finite dimensional algebra over some field and let \( V \) be a non-projective indecomposable \( A \)-module. Then there is the unique (up to equivalence) AR-sequence (or almost split sequence)

\[
0 \rightarrow \delta V \rightarrow mV \rightarrow V \rightarrow 0 \quad \text{(exact)},
\]

where \( \delta V \) and \( mV \) are certain \( A \)-modules whose isomorphism classes are uniquely determined by \( V \). The module \( \delta V \) is indecomposable and if \( A \) is a symmetric algebra, then \( \delta V \) is known to be isomorphic to \( \Omega^2 V \), where \( \Omega^2 \) is the composite of two Heller operators [2]. Furthermore, \( \delta \) gives a bijection from the set of isomorphism classes of non-projective indecomposable \( A \)-modules into itself. The middle term \( mV \) is not indecomposable usually. Some AR-sequences can be obtained as follows. (See [3, Sect. 43.])

Let \( P \) be a projective indecomposable \( A \)-module which is not irreducible (i.e., not simple). Let \( \text{rad} P \) be the radical of \( P \) (i.e., the unique maximal submodule of \( P \)), and let \( \text{soc} P \) be the socle of \( P \) (i.e., the largest semisimple submodule of \( P \)). Then

\[
0 \rightarrow \text{rad} P \rightarrow P \oplus (\text{rad} P/\text{soc} P) \rightarrow P/\text{soc} P \rightarrow 0,
\]

is an AR-sequence. Here \( f(v) = (v, -\tilde{v}) \), where \( \tilde{v} \) means the element \( v + \text{soc} P \) in \( P/\text{soc} P \), and \( g(v, \tilde{u}) = \tilde{v} + \tilde{u} \). Moreover, it is known that the above is the only AR-sequence that involves the projective module \( P \). Furthermore, if we have an AR-sequence

\[
0 \rightarrow \delta V \rightarrow mV \rightarrow V \rightarrow 0,
\]

then we obtain another AR-sequence

\[
0 \rightarrow \Omega(\delta V) \rightarrow \Omega(mV) \oplus U \rightarrow \Omega(V) \rightarrow 0,
\]

where \( U \) is either zero or a projective module. In particular, if \( V \) is periodic, namely, \( \Omega^n(V) \simeq V \) for some \( n \), then so are all the indecomposable direct summands of \( mV \). We now explain which element in \( \text{Ext}_A^1(V, \Omega^2 V) \) corresponds to an AR-sequence. (See [7, II, Sect. 78] for detail.) Let \( V \) and
$U$ be $A$-modules. An $A$-homomorphism $f: V \to U$ is said to be projective if there is a projective $A$-module $P$ and $A$-homomorphisms $g: V \to P$ and $h: P \to U$ such that $f = hg$. The set $\text{Hom}_A(V, U)$ of all the projective homomorphisms from $V$ to $U$ is a subspace of $\text{Hom}_A(V, U)$. Moreover, if $f$ is projective, then, for all $v$ in $\text{End}_A(V)$ and $u$ in $\text{End}_A(U)$, the composite $ufv$ is also projective. Thus $\text{Hom}_A(V, U)$ can be regarded as an $\text{End}_A(U)$-$\text{End}_A(V)$-bimodule. Let $\text{Hom}_A(V, U)$ denote the factor space of $\text{Hom}_A(V, U)$ divided by $\text{Hom}_{\text{proj}}(V, U)$. Then $\text{Hom}_A(V, U)$ is an $\text{End}_A(U)$-$\text{End}_A(V)$-bimodule. The following is easy from the definition.

**Lemma 1.1.** If $V$ or $U$ is projective, then $\text{Hom}_A(V, U) = 0$. Also, the algebra $\text{End}_A(V, V)$ is (non-zero) local if and only if we have $V = V_0 \oplus P$, where $V_0$ is non-projective indecomposable and $P$ is zero or projective.

Now notice that if we take the dual space $\text{Hom}_A(V, U)^*$ of $\text{Hom}_A(V, U)$, then it is an $\text{End}_A(V)$-$\text{End}_A(U)$-bimodule. On the other hand, it is well known that $\text{Ext}^1_A(U, V)$ can be considered as an $\text{End}_A(V)$-$\text{End}_A(U)$-bimodule. However, by defining actions of endomorphisms suitably, these can be considered as $\text{End}_A(\Omega^n(V))$-$\text{End}_A(U)$-bimodules for all $n$. One of the key facts is the following. (See [7, II, (78.15), (78.19)].)

**Lemma 1.2.** Suppose that $A$ is a symmetric algebra. Then we have an isomorphism

$$\text{Hom}_A(V, U)^* \cong \text{Ext}^1_A(U, \Omega^2(V))$$

as $\text{End}_A(V)$-$\text{End}_A(U)$-bimodules.

If $V$ is non-projective indecomposable, then $\text{End}_A(V)$ is a local ring, and hence the socle of its dual is irreducible as an $\text{End}_A(V)$-module. Thus by Lemma 1.2, $\text{Ext}^1_A(V, \Omega^2(V))$ has an irreducible socle as well. Now we have:

**Theorem 1.3** [7, (78.25), (78.39)]. Suppose that $A$ is a symmetric algebra and $V$ is a non-projective indecomposable $A$-module. Then a generator of the socle of $\text{Ext}^1_A(V, \Omega^2(V))$ gives an AR-sequence.

Let $A$ be a symmetric algebra. The (stable) $AR$-quiver of $A$ is a directed and doubly weighted graph whose vertices are parameterized by the isomorphism classes of non-projective indecomposable $A$-modules. Let $V$ and $U$ be non-projective indecomposable $A$-modules. If $U$ appears in $mV$ as a direct summand, then it must follow that $V$ appears as a direct summand of the middle term of the $AR$-sequence

$$0 \to U \to m(\delta^{-1}U) \to \delta^{-1}U \to 0.$$

If this is the case, then we write an arrow from "$U$" to "$V$" whose weight is $(r', r)$, where $r'$ (resp. $r$) is the multiplicity of $V$ (resp. $U$) in $m(\delta^{-1}U)$.
(resp. \(mV\)). So, each AR-sequence can be regarded as a certain mesh of the AR-quiver, and conversely, the AR-quiver consists of meshes coming from AR-sequences. Notice also that if there is a periodic module \(V\) in the AR-quiver, then all the modules in the connected component containing \(V\) are also periodic. Now choose a maximal tree in a connected component such that, for every \(V_1 - V_2 - V_3\) in the tree, there holds \(\Omega^{\pm 2} V_3 \not\cong V_1\). Then such a maximal tree is uniquely determined up to isomorphisms of graphs and called the tree class of the component. If a maximal tree is isomorphic to some Dynkin diagram \(X\) (forgetting direction of the arrows), then we say that the tree class is of type \(X\). Concerning the finiteness of isomorphism classes of indecomposable modules, the following is a quite deep and important result.

**Theorem 1.4** [1, Theorem 6.5, 16, Theorem 4]. Suppose that \(1\) is the (unique) centrally primitive idempotent of \(A\). If the AR-quiver of \(A\) has a finite connected component (i.e., a component with finite vertices), then \(A\) is of finite representation type and its AR-quiver is connected.

**Remark.** The above theorem in particular says that, once we find a finite component of the AR-quiver, then we have already "obtained" all the (isomorphism classes of) non-projective indecomposable \(A\)-modules as its vertices.

The above can, of course, be used when we intend to conclude that some algebra is of finite representation type. For certain algebras, other conditions concerning the finiteness are known. Here we mention only one of them, which is due to P. Gabriel, et al. Let \(1 = \sum e_i\) be a primitive orthogonal idempotent decomposition of \(1\). Assume that \(e_i A \not\cong e_j A\) if \(i \neq j\). Then, the separated graph of \(A\) is defined to be a weighted graph whose vertices are parameterized by \(e_i\)'s and their copies \(e_i\)'s. Thus it has \(2n\) vertices, where \(n\) is the number of idempotents in the decomposition. Let \(\text{rad} A\) denote the Jacobson radical of \(A\). If the dimension of \(e_i (\text{rad} A) e_j / e_i (\text{rad} A)^2 e_j\) is equal to a non-zero \(a_{ij}\), then we write an edge between "\(e_i\)" and "\(e_j\)" with multiplicity \(a_{ij}\).

**Theorem 1.5** [11, 1.4 Satz]. Suppose that \((\text{rad} A)^2 = 0\), where \(\text{rad} A\) is the Jacobson radical of \(A\). Then \(A\) is of finite representation type if and only if its separated graph is a disjoint union of a finite number of Dynkin diagrams.

2. Representations of Hecke Algebras

Let \((\mathcal{W}, S)\) be a finite Coxeter system and let \(K = \mathbb{C}(\sqrt{q})\). It may be said that representations of \(K \otimes H_q(\mathcal{W})\) are quite well known. For example, all
the equivalence classes of irreducible representations are obtained by, for instance, using the method of $W$-graphs. (See [12], etc.) We first list some notions and results which will be needed in the sections thereafter. First of all, $K$ is a splitting field for $H_q(W)$. Also, since $C[\sqrt{q}]$ is a principal ideal domain, for any $K$-representation of $K \otimes H_q(W)$, there is an equivalent representation whose entries on $T_w (w \in W)$ lie in $C[\sqrt{q}]$. Hence one can choose such a representation $\rho$ and consider specialization $\rho|_{q \to x}$ to get a representation of the $C$-algebra $H_x(W)$ for any $x$ in $C$. Moreover, the values of the character $\chi_\rho$ of $\rho$ on $T_w$ lie in $C[\sqrt{q}]$ and $\chi_\rho|_{q \to x}$ is a character of the representation $\rho|_{q \to x}$. In the rest of this section, $H_{q,x}(W)$ denotes the algebra obtained by localizing $H_q(W)$ at the maximal ideal $(\sqrt{q} - \sqrt{x})$ of $C[\sqrt{q}]$. The above argument is also valid for $H_{q,x}(W)$. Also, if we write $H_{q}(W)$, then it means either $H_q(W)$ or $H_x(W)$, and in the former case, $\beta$ means $q$ while in the latter case, $\beta$ means $x$.

We now explain the Cartan-Brauer triangle. See p. 643 of [7, II] or [10, I, Sect. 17]. We first define the decomposition number $d_{\lambda,\gamma}$ for irreducible representations $\rho$ of $K \otimes H_q(W)$ and $\lambda$ of $H_x(W)$. It is the multiplicity of $\lambda$ in a composition series of $\rho|_{q \to x}$. On the other hand, the Cartan invariants $c_{\lambda,\gamma}$ of the algebra $H_x(W)$ are the multiplicity of $\lambda$ in a composition series of the projective cover $P_{\lambda'}$ of another irreducible representation $\lambda'$. Moreover, in our situation, for each irreducible representation $\lambda$ of $H_x(W)$, there is a representation $\bar{P}_{\lambda}$ of $H_{q,x}(W)$ such that $\bar{P}_{\lambda}|_{q \to x} = P_{\lambda}$. This is well known as the lifting idempotents theorem. (See also [7, I, Sect. 6, Ex. 16].) Let us regard $\bar{P}_{\lambda}$ as a representation of $K \otimes H_{q,x}(W)$. Then it can be written as a direct sum of irreducible representations of $K \otimes H_{q,x}(W)$. We define $e_{\lambda,\rho}$ as the multiplicity of $\rho$ in $\bar{P}_{\lambda}$. Notice that $c_{\lambda,\gamma} = \sum_{\rho} e_{\lambda,\rho} d_{\rho,\gamma}$. These numbers are known to be well defined. Namely, they do not depend on the choice of lifts or $H_{q,x}(W)$-lattices. Now the following holds.

**Theorem 2.1.** In the notations above, we have $d_{\rho,\gamma} - e_{\lambda,\rho}$ for all irreducible representations $\lambda$ of $H_x(W)$ and $\rho$ of $K \otimes H_q(W)$.

For each irreducible representation $\rho$ of $H_q(W)$, we define the generic degree $d_\rho = d_\rho(q)$ by

$$d_\rho = \frac{P_w(q) \deg(\rho)}{\sum_{\omega \in W} \chi_\rho(T_{\omega}) \chi_\rho(T_{W}^{-1}) q^{-i(w)},}$$

where $\chi_\rho$ is the character of $\rho$. The generic degree is an element of $K$. (See p. 649 of [7, II].)

**Example.** Each $(W, S)$ has two distinguished representations $IND$ and $SGN$ of degree one, which are given by

$IND(T_w) = q^{i(w)}$, \quad $SGN(T_w) = (-1)^{i(w)}$ \quad for all $w \in W$. 


Their generic degrees are 1 and \( q^{w_0} \), where \( w_0 \) is the unique longest element in \( W \).

Fix a non-zero complex number \( \alpha \) of \( \mathbb{C} \) and consider the specialization \( q \mapsto \alpha \). It is known that \( P_w(q)/d_\rho \) lies in \( \mathbb{C}[\sqrt[q]{\mathbb{Q}}, \sqrt[q^{-1}]{\mathbb{Q}}] \) for each irreducible representation \( \rho \), and thus we can also apply \( q \mapsto \alpha \) to it. The following is maybe the first general result concerning representations of non-semisimple Hecke algebras.

**Theorem 2.2 [19].** Using the above notation, the representation \( \rho\big|_{q \mapsto \alpha} \) is projective irreducible if and only if \( P_w(q)/d_\rho\big|_{q \mapsto \alpha} \neq 0 \).

Of course, the above can be considered as an analogue of a well-known result on modular representation theory: For an ordinary irreducible character \( \chi \) of a finite group \( G \), it gives a projective irreducible representation modulo \( p \) (for a prime \( p \)) if and only if \( |G|/\chi(1) \) is relatively prime to \( p \). An irreducible representation \( \rho \) with \( P_w(q)/d_\rho\big|_{q \mapsto \alpha} \neq 0 \) is called defect zero (under \( q \mapsto \alpha \)), and the irreducible representation \( \rho\big|_{q \mapsto \alpha} \) singly forms a block (i.e., an indecomposable two sided ideal) of \( H_\alpha(W) \). In particular, this block is semisimple and has the unique (up to equivalence) irreducible representation \( \rho\big|_{q \mapsto \alpha} \). The block is also called defect zero.

We now mention coset decompositions of \( W \). See [4, 5] or [7] for detail. Let \( J \) be a subset of \( S \). (\( J \) could be empty.) Then we have a Coxeter system \( (W_J, J) \), where \( W_J \) is the Coxeter group generated by \( J \) with the same relations as in \( W \). From each coset of \( W_J \) in \( W \) we can choose a representative \( t \) so that \( t \) is the shortest in \( W_Jt \). Let \( \mathcal{F} \) denote the set of these shortest elements. Then, \( W = \bigcup_{t \in \mathcal{F}} W_Jt \) is the left coset decomposition of \( W_J \) in \( W \) and any element \( w \) of \( W \) can be written as \( yt \) for some \( y \) in \( W_J \) and \( t \) in \( \mathcal{F} \) such that \( l(w) = l(y) + l(t) \). Moreover, if \( I \) and \( J \) are subsets of \( S \), then we have the double coset decomposition \( W = \bigcup_{t \in \mathcal{F}} W_I t W_J \), where \( t \) is the shortest element in \( W_I t W_J \) and every element \( w \) of \( W \) can be written as \( xty \) for some \( x \) in \( W_I \), \( t \) in \( \mathcal{F} \), and \( y \) in \( W_J \) such that \( l(w) = l(x) + l(t) + l(y) \). Thus, \( H_\beta(W) \) decomposes into a direct sum of \( H_\beta(W_I)-H_\beta(W_J) \)-bimodules \( H_\beta(W_I) T_I H_\beta(W_J) \) (\( t \in \mathcal{F} \)). In this paper whenever we have a (double) coset decomposition it is understood that each representative is taken so that it is shortest in the coset. Now we have the following. For the proof see [7, II, (64.40)] or [17].

**Lemma 2.3.** Let \( t \) be the shortest element in \( W_I t W_J \). Then we have \( t^{-1}W_I t \cap W_J = W_J \), where \( J_t = t^{-1} I_t \cap J \).

Fix \( t \) in \( \mathcal{F} \) and choose the shortest elements from each left coset of \( t^{-1}W_I t \cap W_J \) in \( W_J \). Let \( \mathcal{F}_t \) denote the set of these shortest elements, and let \( W_J \) be \( t^{-1}W_I t \cap W_J \). Then by Lemma 2.3 we have left coset
decomposition $W_J = \bigcup_{s \in \mathcal{F}} W_J s$. This gives a decomposition $W = \bigcup_{t \in \mathcal{F}} \bigcup_{s \in \mathcal{F}_t} W_I t W_J s$, and every element $w$ in $W$ can be written as $x t y s$ for some $x$ in $W_J$, $t$ in $\mathcal{F}$, $y$ in $W_J$, and $s$ in $\mathcal{F}_t$ such that $l(w) = l(x) + l(t) + l(y) + l(s)$. Hence we have a decomposition

$$H_\beta(W) = \bigoplus_{t \in \mathcal{F}} \bigoplus_{s \in \mathcal{F}_t} H_\beta(W_J) T_t H_\beta(W_J) T_s.$$  

This fact is used when we consider the Mackey decomposition theorem. (See Lemma 2.10.)

In the rest of this section, we fix a non-zero complex number $\alpha$ and a subset $J$ of $S$ and write $\mathcal{F}$ to mean the set of the representatives of left cosets of $W_J$ in $W$. Denote $H_\beta(W)$ and $H_\beta(W_J)$ by $H$ and $H_J$, respectively. Let $\tau$ denote the truncation map from $H$ to $H_J$ defined by $\tau(t_w) = t_w$ if $w$ lies in $W_J$, and $\tau(t_w) = 0$ otherwise. The Hecke algebra $H_J$ is a subalgebra of $H$. Also, the algebra extension $H/H_J$ is so-called Frobenius and most important results come from this fact. (See Lemma 2.7 below.) We now consider the relation between representations of $H$ and those of $H_J$ from this point of view. We first prove two easy lemmas.

**Lemma 2.4.** The above $\tau$ is an $H_J$-$H_J$-bimodule homomorphism.

**Proof.** It suffices to show that $\tau(t_w t_x) = \tau(t_{w x}) T_x$ and $\tau(t_w t_{x'}) = T_x \tau(t_{w})$ for all $w$ in $W$ and $x$ in $W_J$. Write $w = t^{-1} y t$ with $t$ in $\mathcal{F}$ and $y$ in $W_J$. Then $T_w T_x = T_{t^{-1}} t_x T_{t^{-1}} T_x$ and it is a linear combination of $T_{t^{-1}} T_x = T_{t^{-1}} x'$ $(x' \in W_J)$. Thus we have $\tau(t_w T_x) = T_w T_x$ if $t = 1$ and $\tau(t_w T_x) = 0$ otherwise. Since $\tau(t_w)$ is $T_w$ if $t = 1$ and is 0 otherwise, we have proved the first equality. The second one holds similarly.

**Lemma 2.5.** Let $t$ and $t'$ lie in $\mathcal{F}$. Then $\tau(t_{t'} T_{t^{-1}})$ is equal to $\beta^{l(t)} T_1$ if $t = t'$ and is 0 otherwise.

**Proof.** Use induction on $l(t)$. If $t = 1$, then the result is clear. So, assume that $l(t) \geq 1$. We can take $s$ in $S$ such that $t = x s$ for some $x$ in $W$ such that $l(t) = l(x) + 1$. Then one can easily show that $x$ is the shortest element in $W_J x$, namely, that $x$ lies in $\mathcal{F}$. Suppose first that $t = t'$. Then we have $$T_t T_{t'^{-1}} = T_x T_{t} T_{s} T_x T_{t'^{-1}} = (\beta - 1) T_x T_{t} T_{s} T_{t'^{-1}} + \beta T_x T_{t'^{-1}} - (\beta - 1) T_x T_{t'^{-1}} + \beta T_x T_{t'^{-1}}.$$ Since $l(x) < l(t)$, the inductive hypothesis yields that $\tau(T_t T_{t'^{-1}}) = \beta^{l(x)} T_1 = \beta^{l(t)} T_1$. Suppose then that $t \neq t'$. First notice that we can also see $t's \in \mathcal{F}$. If $l(t's) = l(t') - 1$, then $T_t T_{t'^{-1}}$ is equal to $$T_{t's} T_s T_{t's} T_{x^{-1}} = (\beta - 1) T_{t's} T_s T_{x^{-1}} + \beta T_{t's} T_{x^{-1}} = (\beta - 1) T_s T_{x^{-1}} + \beta T_{t's} T_{x^{-1}}.$$
Notice also that \( t' \neq x \) since \( l(t's) = l(t') - 1 \) while \( l(xs) = l(x) + 1 \). Since \( t' \neq t \) implies that \( t's \neq x \), we have \( \tau(T, T_1) = 0 \) by the inductive hypothesis. If \( l(t's) = l(t') + 1 \), then we have \( T_r T_{r-1} = T_r T_s T_{x-1} = T_{r'}, T_r T_{x-1} \). Hence, again by the inductive hypothesis, we similarly have \( \tau(T_r, T_{r-1}) = 0 \). Now the proof is completed.

If \( J \) is empty, then \( \tau \) induces a non-degenerate associative symmetric bilinear form. Thus we have the following. (See also [7], I, p. 1981.)

**Corollary 2.6.** \( H_p(W) \) is a symmetric algebra.

We now show that the extension \( H/H_J \) is Frobenius. The first equations of the following precisely say it, and the second one is crucial when proving Frobenius reciprocity.

**Lemma 2.7.** For any \( x \) in \( H \), we have

\[
x = \sum_{t \in \mathcal{F}} \beta^{-l(t)} T_{r-1} \tau(T, x) = \sum_{t \in \mathcal{F}} \tau(x \beta^{-l(t)} T_{r-1}) T_t
\]

and

\[
\sum_{t \in \mathcal{F}} x \beta^{-l(t)} T_{r-1} \otimes T_t = \sum_{t \in \mathcal{F}} \beta^{-l(t)} T_{r-1} \otimes T_t x.
\]

Here \( \otimes \) means the tensor product over \( H_J \).

**Proof.** Using Lemmas 2.4 and 2.5, it is routine to check the first equations. Let us see the second. Using the first, we can write \( T_r x = \sum_{t \in \mathcal{F}} \tau(T, x \beta^{-l(t)} T_{t-1}) T_t \). Then it follows that

\[
\left( \sum_{t \in \mathcal{F}} \beta^{-l(t)} T_{r-1} \otimes T_t \right) x = \sum_{t, t_1} \beta^{-l(t)} T_{r-1} \otimes \tau(T, x \beta^{-l(t_1)} T_{t-1}) T_{t_1}
\]

\[
= \sum_{t, t_1} \beta^{-l(t)} T_{r-1} \tau(T, x \beta^{-l(t_1)} T_{t-1}) \otimes T_{t_1}
\]

\[
= x \left( \sum_{t_1} \beta^{-l(t_1)} T_{r-1} \otimes T_{t_1} \right).
\]

Thus the result follows.

Let \( V \) be an \( H_J \)-module. Then we can form the induced \( H \)-module \( V^H = V \otimes_{H_J} H \). Also, if \( U \) is an \( H \)-module, then restricting the action of \( H \) to \( H_J \), we can consider it as an \( H_J \)-module. In this situation, Frobenius reciprocity laws hold. (See also [8, Sect. 2].)
THEOREM 2.8. In the above notation, we have isomorphisms

\[ i: \text{Hom}_H(V, U) \rightarrow \text{Hom}_H(V^H, U) \]

and

\[ i': \text{Hom}_H(U, V) \rightarrow \text{Hom}_H(U, V^H), \]

where \( i \) and \( i' \) are defined by \( i(f)(v \otimes x) = f(v)x \) and \( i'(g)(u) = \sum_{\tau \in \mathcal{F}} g(u \beta^{-1}(x)) \otimes T_\tau \) for all \( f \) in \( \text{Hom}_H(V, U) \), \( g \in \text{Hom}_H(U, V) \), \( u \in U \), \( v \in V \), and \( x \in H \).

Proof. We denote \( \otimes_H \) by \( \otimes \). The first part of the above is just a special case of the adjointness theorem [7, I, (2.19)] while the second holds only for Frobenius extensions. First claim that \( i' \) is well defined. For a fixed \( g \) in \( \text{Hom}_H(U, V) \), any \( x \otimes_H x' \), where \( x \) and \( x' \) lie in \( H \), gives a homomorphism \( \theta(x \otimes_H x') \) from \( U \) to \( V^H \) as additive groups defined by

\[ \theta(x \otimes_H x')(u) = g(ux) \otimes x' \]

for all \( u \) in \( U \). (Note: \( \theta \) is well defined since \( g \) is an \( H \)-homomorphism.) Then we have \( i'(g) = \theta(\sum \beta^{-1}(x) T_{\tau} \otimes T_\tau) \). Now Lemma 2.7 implies that \( \theta(\sum \beta^{-1}(x) T_{\tau} \otimes T_\tau) = \theta(\sum \beta^{-1}(x) T_{\tau} \otimes T_\tau \otimes x) \) for all \( x \) in \( H \). This just says that \( i'(g) \) is an \( H \)-homomorphism, and thus \( i' \) is well defined. Now define \( \kappa \) from \( \text{Hom}_H(U, V^H) \) to \( \text{Hom}_H(U, V) \) by \( \kappa(h) = \gamma_V h \) for all \( h \) in \( \text{Hom}_H(U, V^H) \), where \( \gamma_V \) is a map from \( V^H \) to \( V \) defined by \( \gamma_V(v \otimes x) = v\tau(x) \) for all \( v \in V \) and \( x \in H \). By Lemma 2.4, \( \gamma_V \) is a well-defined \( H \)-homomorphism. Then, to see that \( i' \kappa \) is the identity, fix \( h \) in \( \text{Hom}_H(U, V^H) \) and \( u \) in \( U \) and write \( h(u) = \sum_{\tau \in \mathcal{F}} v_{\tau} \otimes T_\tau \). Then we have

\[ i' \kappa(h)(u) = \sum_{\tau \in \mathcal{F}} \gamma_V h(u \beta^{-1}(x) T_{\tau} \otimes T_\tau) \]

\[ = \sum_{\tau \in \mathcal{F}} \gamma_V (h(u) \beta^{-1}(x) T_{\tau} \otimes T_\tau) \]

\[ = \sum_{\tau \in \mathcal{F}} \gamma_V (v_{\tau} \otimes T_\tau \beta^{-1}(x) T_{\tau} \otimes T_\tau) \]

\[ = \sum_{\tau \in \mathcal{F}} v_{\tau} \tau(T_\tau \beta^{-1}(x) T_{\tau} \otimes T_\tau) \]

\[ = \sum_{\tau \in \mathcal{F}} v_{\tau} \otimes (T_\tau \beta^{-1}(x) T_{\tau} \otimes T_\tau) \]

\[ = \sum_{\tau} v_{\tau} \otimes T_\tau \quad (\text{by Lemma 2.7}) \]

\[ = h(u). \]
Thus $\iota'\kappa$ is the identity. Now let us show that $\kappa\iota'$ is also the identity. Let $g$ lie in $\text{Hom}_{H_f} (U, V)$. Then $\kappa\iota'(g)$ sends any element $u$ of $U$ to

$$\gamma_U \left( \sum_i g(u\beta^{-i(t)} T_{i-1}) \otimes T_i \right)$$

$$= \sum_i g(u\beta^{-i(t)} T_{i-1}) \tau(T_i) = \sum_i g(u\beta^{-i(t)} T_{i-1} \tau(T_i)),$$

which is equal to $g(u)$ by Lemma 2.7. Therefore $\kappa\iota'$ is also identity, and this completes the proof.

**Remark.** Notice that the isomorphisms in the above theorem are actually those of modules over suitable endomorphism algebras. Moreover, if $X$ is an $H_f$-module and $g : P' \to X$ and $h : X \to U$ are $H_f$-homomorphisms, then it easily follows that $\iota(h g) = \iota(h)(g \otimes Id_H)$. Here, $\iota$ is roughly used for the isomorphisms in the Frobenius reciprocity between various spaces. An analogous equality holds for $\iota'$, and even for $\iota'^{-1}$ and $\iota'^{-1}$.

Now using the above remark, the following is easily shown.

**Proposition 2.9.** The above $\iota$ and $\iota'$ preserve projective homomorphisms. In particular, we still have isomorphisms when we replace $\text{Hom}$ by $\text{Hom}$ in Theorem 2.8.

In the rest of this section, we list several easy lemmas which are used in the next section. The first one is an analogue of the Mackey decomposition theorem. Let $I$ be another subset of $S$. Only in the first lemma (Lemma 2.10) do we use the notation in the paragraph following Lemma 2.3. Further, we denote $H_f(W_I)$, $H_f(W_I)$, and $H_f(W_J)$ by $H_f$, $H_f$, and $H_f$, respectively, for convenience.

**Lemma 2.10.** Let $V$ be an $H_f$-module. Then we have

$$V^H \cong \bigoplus_{\iota \in \mathcal{F}} V \otimes_{H_f} T_{\iota} H_f \otimes_{H_f} H_f$$

as $H_f$-modules.

**Proof.** By an argument in the paragraph following Lemma 2.3, we have

$$V^H = \bigoplus_{\iota \in \mathcal{F}} \bigoplus_{\delta \in \mathcal{F}} V \otimes_{H_f} T_{\iota} H_f T_{\delta}.$$  

Since $\bigoplus_{\iota \in \mathcal{F}} V \otimes_{H_f} T_{\iota} H_f T_{\delta} \cong (V \otimes_{H_f} T_{\iota}) \otimes_{H_f} H_f$, the result follows.

**Lemma 2.11.** Let $V$ be an $H_f$-module. Then, regarding $V^H$ as an $H_f$-module, $V$ is isomorphic to a direct summand of $V^H$. 


Proof. From the decomposition \( H = \bigoplus_{t \in \mathcal{F}'} H_t T H_t \), where \( \mathcal{F}' \) is the set of the representatives of double cosets \( W_J \setminus W / W_J \), this is clear.

Remark. The above is also true when \( \alpha = \beta = 0 \).

Lemma 2.12. Let \( V \) be an \( H_J \)-module. Then we have \( \Omega(V^H) \oplus P \cong (\Omega V)^H \), where \( P \) is zero or a projective \( H \)-module.

Proof. Clear from the definition since \( H \) is a free \( H_J \)-module.

Lemma 2.13. Let \( V \) and \( V' \) be \( H_J \)-modules, and \( f \) an \( H_J \)-homomorphism from \( V \) to \( V' \). Then, \( f \) is projective if and only if \( f \otimes_{H_J} \text{Id}_H \) is so.

Proof. It is clear that if \( f \) is projective then so is \( f \otimes_{H_J} \text{Id}_H \). Suppose then that \( f \otimes_{H_J} \text{Id}_H \) is projective. Then there are a projective \( H \)-module \( P \) and homomorphisms \( g : V^H \rightarrow P \) and \( g' : P \rightarrow V'^H \) such that \( f \otimes_{H_J} \text{Id}_H = g' g \).

By Frobenius reciprocity, \( g \) and \( g' \) induce \( H_J \)-homomorphisms from \( V \) to \( P \) and from \( P \) to \( V' \), respectively. Now it is routine to check that their composite gives \( f \). Thus \( f \) is projective, and this completes the proof.

3. Hecke Algebras of Type \( A_l \)

In this section, we assume that \( (W, S) \) is of type \( A_l \) \((l \geq 2)\). As is well known, all the equivalence classes of irreducible representations of the generic algebra \( H_q(W) \) are parameterized by partitions of \( l+1 \). See [8, 9]. For example, the representations \( IND \) and \( SGN \) correspond to the partitions \((l+1)\) and \((1, 1, \ldots, 1)\), respectively. Moreover, partitions of \( l+1 \) are in one-to-one correspondence naturally with Young diagrams with \( l+1 \) nodes. Namely, for any partition \((p_1, p_2, \ldots, p_k)\) of \( l+1 \), where \( p_1 \geq p_2 \geq \cdots \geq p_k > 0 \), the corresponding Young diagram consists of \( k \) rows and the \( i \)th row has \( p_i \) nodes. Henceforth we identify a partition of \( l+1 \) with the corresponding Young diagram. Let \( \mathcal{H}_{l+1} \) denote the set of Young diagrams with \( l+1 \) nodes, and let \( \mathcal{H}_{l+1}^\prime \) be the set of hooks in \( \mathcal{H}_{l+1} \). Here a hook means a Young diagram such that the first row is the only row which possibly has more than one node.

Let \( \alpha \) be a primitive \( r \)th root of unity. Then using the notion of \( r \)-cores, we can describe how the irreducible representations are distributed into the blocks of \( H_q(W) \), namely, an analogue of the Nakayama conjecture holds. For the proof see [9, (4.13)].

Theorem 3.1. Let \( \mu \) and \( \mu' \) be partitions of \( l+1 \), and let \( \rho \) and \( \rho' \) be the irreducible representations of \( H_q(W) \) corresponding to \( \mu \) and \( \mu' \), respectively. Then \( \rho|_{\varphi_{\mu, \rho}} \) and \( \rho'|_{\varphi_{\mu', \rho'}} \) lie in the same block if and only if \( \mu \) and \( \mu' \) have the same \( r \)-core.
Remark. Let $x$ be as above. We further assume that $l + 1 < 2r \leq 2(l + 1)$. Let $\mu$ be a partition of $l + 1$ and $\rho$ the corresponding irreducible representation of $H_q(W)$. Then there are only two possibilities: We can remove a single $r$-hook from $\mu$, or we cannot remove any $r$-hooks. In the former case, the number of remaining nodes is $l + 1 - r$; in the latter case, $\rho \mid q = x$ is projective irreducible since $r$-core of $\mu$ is $\mu$ itself. Hence besides defect zero blocks, $H_x(W)$ has at most $n$ blocks, where $n$ is the number of partitions of $l + 1 - r$ if $r < l + 1$ and $n = 1$ if $r = l + 1$.

For the time being, we assume that $x$ is a primitive $(l + 1)$th root of unity and that $W_f$ is a Weyl subgroup of $W$ of type $A_{l-1}$. All the equivalence classes of irreducible representations of $H_x(W)$ are obtained in [8, 9, 18], and here we mention what they are. (Note: (i) of the following is also derived from Theorem 2.2 and the explicit computation of the generic degrees [5, p. 446].)

**Theorem 3.2.** (i) The set \{ $\rho_Y \mid q = x$ $\mid Y \in \mathcal{H}_{l + 1} \}$ gives the set of equivalence classes of irreducible projective representations of $H_x(W)$.

(ii) For each $Y$ in $\mathcal{H}_l$, the irreducible representation $\lambda = \lambda_Y$ of $H_x(W_f)$ corresponding to $Y$ can be extended to a representation $\bar{\lambda}$ of $H_x(W)$.

(iii) The representations in (i) and all the $\bar{\lambda}$ in (ii) are, up to equivalence, all the irreducible representations of $H_x(W)$.

We now determine the Loewy structure of the projective covers of $\bar{\lambda}$ in (ii) above. Denote the element of $\mathcal{H}_{l + 1}$ having $l - i + 1$ nodes in the first row by $Y_i$ and the corresponding irreducible representation of $H_q(W)$ by $\rho_i$. Note that $Y_i$ has $i + 1$ rows and each row except the first one has only one node. Also, $i$ runs from 0 to $l$. Furthermore, $\lambda_i$ denotes the irreducible representation which corresponds to the hook in $\mathcal{H}_l$ having $l - i + 1$ nodes in the first row ($1 \leq i \leq l$). Notice that, since $H_x(W_f)$ is semisimple, each $\lambda_i$ can be regarded also as an irreducible representation of $H_x(W_f)$. (See also (ii) of the above theorem.)

**Lemma 3.3.** (i) For all $i$ with $1 \leq i \leq l - 1$ the representation $\rho_i \mid q = x$ has $\lambda_i$ and $\lambda_{i+1}$ as its irreducible constituents with multiplicity one.

(ii) $\rho_0 \mid q = x$ and $\rho_l \mid q = x$ are equivalent to $\lambda_1$ and $\lambda_l$, respectively.

**Proof.** (ii) is clear since these are IND and SGN. On the other hand, by the branching rule, the restriction of $\rho_i$ to $H_q(W_f)$ is equivalent to the direct sum of $\lambda_i$ and $\lambda_{i+1}$. Hence by Theorem 3.2, among irreducible representations of $H_x(W)$, the only possibility is the one in the statement. Therefore, the lemma is proved.
The Loewy structure of projective indecomposable modules are determined as follows. Let $P_i$ be the projective cover of $\mathcal{X}_i$ for all $i$ with $1 \leq i \leq l$.

**Theorem 3.4.** We have the following:

$$P_1 = \mathcal{X}_2, \quad P'_i = \mathcal{X}_{i+1}, \quad (2 \leq i \leq l-1), \quad P_l = \mathcal{X}_{l-1}$$

**Proof.** Using Theorem 2.1, Lemma 3.3 yields that the multiplicities of irreducible constituents of $P_i$ are precisely those in the statement. Recall that the head of $P_i$ is $\mathcal{X}_i$. On the other hand, by Corollary 2.6, $H_\mathcal{X}(W)$ is a symmetric algebra, and thus the socle of $P_i$ is also $\mathcal{X}_i$. Moreover, the remaining constituents must sit in the right places as in the statement. This proves the theorem.

**Remark.** H. Yamane has independently obtained the Loewy structures of $P_i$. In fact, he got matrix-representations of $P_i$'s concretely. The author is grateful to him for showing his computational result.

By Theorem 3.4, we obtain the following AR-sequences (see Sect. 1):

$$0 \to \mathcal{X}_1 \to P_1 \oplus \mathcal{X}_2 \to \mathcal{X}_1 \to 0,$$

$$0 \to \mathcal{X}_i \to P_i \oplus \mathcal{X}_{i+1} \oplus \mathcal{X}_{i-1} \to \mathcal{X}_i \to 0,$$

$2 \leq i \leq l-1$, and

$$0 \to \mathcal{X}_i \to P_i \oplus \mathcal{X}_{i-1} \to \mathcal{X}_i \to 0.$$
that these are just symbols for the present. However, if a certain symbol represents the Loewy series of some indecomposable \( H_\lambda(W) \)-module, then we consider it as the module. For example, \((i)\) is a “module” for all \(i\). Notice also that if they are actually “modules,” then they must be distinct.

It follows from the structure of projective indecomposable modules of \( H_\lambda(W) \) (Theorem 3.4) that \((1, 2), (1, 2)^*, (l-1, l), (l-1, l)^*\) and \((i, i+2), (i, i+2)^*\) \((1 \leq i \leq l-2)\) are modules. In fact, they represent \(\text{rad } P_i\) and \(P_i/\text{soc } P_i\) \((1 \leq i \leq l)\). Also, the AR-sequences given so far can be written as follows \((i\) runs from 2 to \(l-1\) in the following):

\[
0 \rightarrow (1, 2)^* \rightarrow P_i \oplus (2) \rightarrow (1, 2) \rightarrow 0
\]

\[
0 \rightarrow (i-1, i+1) \rightarrow P_i \oplus (i-1) \oplus (i+1) \rightarrow (i-1, i+1)^* \rightarrow 0
\]

\[
0 \rightarrow (l-1, l) \rightarrow P_i \oplus (l-1) \rightarrow (l-1, l)^* \rightarrow 0.
\]

We call the above “basic” AR-sequences. We now compute \(\Omega^k((i))\) by using the structure of projective covers of indecomposable modules. First, looking at projective modules, we have \(\Omega((1)) = (i-1, i+1)\) if \(2 \leq i \leq l-1\), \(\Omega((1)) = (1, 2)^*\), and \(\Omega((l)) = (l-1, l)\). Also, \(\Omega((1, 2)^*) = (2, 3)^*\) and, in general, we have \(\Omega(((i, i+1)^*)) = (i+1, i+2)^*\) \((1 \leq i \leq l-2)\) and \(\Omega((l-1, l)^*) = (l)\). Moreover, \(\Omega((i, i+1)) = (i-1, i)\) \((2 \leq i \leq l-1)\) and \(\Omega((1, 2)) = (1)\). Hence, we can conclude that the \(\Omega\)-orbit containing \((1)\) has 2\(l\) modules represented by the symbols and among them \((1)\) and \((l)\) are only irreducible modules. Now we go to general \(\Omega^k((i))\). Assume that \(2 \leq i \leq l/2\). Then we have \(\Omega^2((2)) = (1, 4)^*, \Omega^2((i)) = (i-2, i+2)\) if \(i \geq 3\), and going on computation, we generally have \(\Omega^k((i)) = (1, 2i)^*\). Then \(\Omega^{i+1}((i)) = (2, 2i+1)^*, \Omega^{-1}((i)) = (3, 2i+2)^*\) \((1 \leq i \leq l/2-2)\), and so on. Then we get \(\Omega^{i+1}((i)) = (l-2i+1, l)^*\). Also, \(\Omega^{l-i+1}((i)) = (l-2i+2, l)^*\), \(\Omega^{-l}((i)) = (l-2i+3, l-1)^*\), and so on. And we eventually obtain \(\Omega^{l-i}((i)) = (l-i, l-i+2)^*\). Thus \(\Omega^l((i)) = (l+1-i)\). Similarly, we can compute \(\Omega^k((i))\) when \(l/2 \leq i \leq l-1\) and see \(\Omega^l((i)) = (l+1-l)\). Hence it follows that the \(\Omega\)-orbit containing \((i)\) has 2\(l\) modules represented by the symbols and \((i)\) and \((l-i+1)\) are the only irreducible modules in the orbit. Now by the argument given so far, \(l/2 \times 2l = l^2\) symbols are actually modules if \(l\) is even and \((l-1)/2 \times 2l = (l-1)/l\) symbols are modules if \(l\) is odd. However, similar computations yield that, if \(l\) is odd, then the \(\Omega\)-orbit containing \(((l+1)/2)\) has \(l\) modules represented by the symbols and \(((l+1)/2)\) is the unique irreducible module in the orbit. Thus we have \(l^2\) modules in this case, too. Hence we have:

**Lemma 3.5.** All the symbols give distinct indecomposable \(H_\lambda(W)\)-modules. Furthermore, for all \(i\), every module in the \(\Omega\)-orbit of \(\lambda_i\) is represented by a certain symbol.
The stable AR-quiver of $H_\alpha(W)$ can be obtained from the basic AR-sequences and their translations by $Q$. Here we just give AR-quivers for small $l$ as examples. In the following, all arrows have weight $(1, 1)$, and the right and left most lines are identified:

The case $l = 2$

The case $l = 3$

The case $l = 4$

In general, we can determine the structure of all the indecomposable modules as follows.

**Theorem 3.6.** Suppose that $(W, S)$ is of type $A_1$. Let $\alpha$ be a primitive $(l + 1)$th root of unity. Then besides defect zero blocks $H_\alpha(W)$ has only one block which contains $l^2$ non-isomorphic non-projective indecomposable modules represented by the symbols and the tree class of its AR-quiver is $A_1$. In particular, $H_\alpha(W)$ is of finite representation type.
Proof. First recall that besides defect zero blocks, $H_\alpha(W)$ has only one block. (See remark following Theorem 3.1.) Let $\Gamma$ denote the connected component of the stable AR-quiver containing (1). Then, using the basic AR-sequences, we have a maximal tree in $\Gamma$ with required properties as follows:

If $l$ is odd and

$(1) \rightarrow (1,3)^* \leftarrow (3) \rightarrow (3,5)^* \leftarrow \cdots \leftarrow (l-2) \rightarrow (l-2, l)^* \leftarrow (l)$

if $l$ is even. In fact, since

\[0+(1)+(1,3)^*+(2,3)+0,\]
\[0-(1-2)+(1-2,1)+(1)+0,\]

are AR-sequences, $(1)$, $(l)$, and $(l-1, l)^*$ lie at the "end" of the AR-quiver. Also the basic AR-sequences tell us that we cannot extend the above tree from some vertex in the middle keeping the required properties. Hence, the above must be maximal and the tree class is of type $A_l$. Moreover, since $\Omega^2$ has period $l$ for the modules in the above tree, $\Gamma$ has at most $l^2$ modules. Hence $\Gamma$ is finite and, in view of Theorem 1.4, $\Gamma$ has all the non-projective modules. So, $\Gamma$ must have $l^2$ modules (see Lemma 3.5) and the proof is now completed.

For the direct product of two $H_\alpha(W)$'s the following holds.

**Proposition 3.7.** Suppose that $(W, S)$ is of type $A_l \times A_l$, where $l \geq 2$, and that $\alpha$ is a primitive $(l+1)$th root of unity. Then $H_\alpha(W)$ is of infinite representation type.

**Proof.** Use the notation in the discussions given so far. Write by $A$ the Hecke algebra of type $A_l$ specialized by $q \mapsto \alpha$. Then we have $H_\alpha(W) \cong A \otimes \mathbb{C} A$. Let $R$ be $H_\alpha(W)/(\text{rad } H_\alpha(W))^2$. Then it suffices to show that $R$ is of infinite representation type. Let $e_1$ and $e_2$ be primitive idempotents of $A$ corresponding to the irreducible representations $\chi_1$ and $\chi_2$, respectively. Then it follows from Theorem 3.4 that $e_1 \text{ rad } A e_2/e_1 (\text{ rad } A)^2 e_2 \neq 0$ and $e_2 \text{ rad } A e_1/e_2 (\text{ rad } A)^2 e_1 \neq 0$. Let $e_{ij}$ be the image of $e_i \otimes e_j$ in $R$ under the natural epimorphism $(1 \leq i, j \leq 2)$. Then, these four elements are non-equivalent primitive idempotents of $R$. Moreover, it is easy to see that $e_{ij} \text{ rad } Re_{ij} \neq 0$, $e_{ii} \text{ rad } Re_{ii} \neq 0$, $e_{ij} \text{ rad } Re_{ii} \neq 0$, and $e_{ii} \text{ rad } Re_{ii} \neq 0$. (Here different letters represent different numbers.) For example, $e_{ii} \text{ Rad } Re_{ii}$ is the
image of $e_iAe_i \otimes e_iAe_i + e_iAe_i \otimes e_iAe_i$. So, $e_i \otimes x$, where $x$ is an element of $e_iAe_i \setminus e_i(\text{rad } A)^2e_i$, gives a non-zero element. This observation yields that the separated diagram of $R$ contains the following:

$$
\begin{array}{cccc}
  e_{11} & e_{12} & e_{21} & e_{22} \\
  e'_{11} & e'_{12} & e'_{21} & e'_{22}
\end{array}
$$

Therefore, by Theorem 1.5, $R$ must be of infinite representation type. This completes the proof.

Now we can prove one of our main results. Let $\alpha$ be a primitive $r$th root of unity and assume that $(l+1)/2 < r \leq l+1$. Also, let $n$ be the number of partitions of $l+1-r$ if $r < l+1$ and let $n=1$ if $r = l+1$.

**Theorem 3.8.** Suppose that $(W, S)$ is of type $A_l$. Let $\alpha$ be a primitive $r$th root of unity with $(l+1)/2 < r \leq l+1$. Then besides defect zero blocks $H_\alpha(W)$ has $n$ blocks and each of those blocks contains $(r-1)^2$ non-isomorphic non-projective indecomposable modules and the tree class of its $AR$-quiver is $A_{r-1}$. In particular, $H_\alpha(W)$ is of finite representation type.

Before going to the proof, we fix notations as follows. First we choose disjoint subsets $I$ and $J$ of $S$ such that $(W, I)$ and $(W, J)$ are of type $A_{r-1}$ and $A_{l-r}$, respectively. Note that $(W_{I \cup J}, I \cup J)$ is a subsystem of $(W, S)$ of type $A_{r-1} \times A_{l-r}$. Let $W = \bigcup_{t \in \mathcal{T}} W_{t}tW_{t}$ be the double coset decomposition such that $t$ is the shortest in $W_{t}tW_{t}$. For each $t$ in $\mathcal{T}$, we have the coset decomposition $W_{t} = \bigcup_{s \in \mathcal{J}} W_{t}s$, where $W_{t}s = t^{-1}W_{t}t \cap W_{t}$. (See the paragraph following Lemma 2.3.) Note that $W_{t}$ has nothing to do with this decomposition and the notations. We write $H, H_{I}, H_{J}$, and $H_{I \cup J}$ to mean $H_{\alpha}(W)$, $H_{\alpha}(W_{I})$, $H_{\alpha}(W_{J})$, and $H_{\alpha}(W_{I \cup J})$, respectively. Since $P_{W_{t}}(q) = (1 + q) \cdots (1 + q + \cdots + q^{l-r})$ and since $l-r+1 < r$, $H_{I}$ is semisimple. Also, $H_{I}$ is of finite representation type by Theorem 3.6. Let $\{ V_{i} \}$ and $\{ U_{j} \}$ be the complete sets of non-isomorphic non-projective indecomposable modules over $H_{I}$ and $H_{J}$, respectively (1 $\leq i, j \leq 1$, 1 $\leq j \leq n$). (Recall that every $U_{j}$ is irreducible.) Put $V_{i,j} = V_{i} \otimes_{C} U_{j}$, which is a module over $H_{I \cup J} \simeq H_{I} \otimes_{C} H_{J}$. Then $\{ V_{i,j} \}$ form the complete set of non-isomorphic indecomposable modules over $H_{I \cup J}$. To prove the theorem, we need some lemmas. The first two are easy to show and we omit their proofs.

**Lemma 3.9.** If $w$ in $W$ satisfies $wW_{t}w^{-1} = W_{t}$, then $w$ lies in $W_{t} \times W_{J}$.

**Lemma 3.10.** The space $\text{Hom}_{H_{I \cup J}}(V_{i,j}, V_{i,j})$ is zero if $j \neq j'$, and is naturally isomorphic to $\text{Hom}_{H_{I}}(V_{i}, V_{j})$ otherwise. Moreover, we have similar conclusions when we replace $\text{Hom}$ by $\text{Hom}$ in the above.
Lemma 3.11. For any \( f \) in \( \text{Hom}_{\mathcal{H}_{i,j}}(V_{i,j}, V_{i,j}') \), define \( \iota(f) \) by \( \iota(f) = f \otimes_{\mathcal{H}_{i,j}} \text{Id}_{\mathcal{H}} \). Then \( \iota(f) \) lies in \( \text{Hom}_{\mathcal{H}}(V^H_{i,j}, V^H_{i,j}') \) and \( \iota \) gives an isomorphism

\[
\text{Hom}_{\mathcal{H}_{i,j}}(V_{i,j}, V_{i,j}') \cong \text{Hom}_{\mathcal{H}}(V^H_{i,j}, V^H_{i,j}')
\]
of \( \text{End}_{\mathcal{H}_{i,j}}(V_{i,j}) \)-\( \text{End}_{\mathcal{H}_{i,j}}(V_{i,j}') \)-bimodules.

Proof. It follows from Proposition 2.9 and Lemma 2.10 that

\[
\text{Hom}_{\mathcal{H}}(V^H_i, V^H_{i'}) \cong \bigoplus_{i \in \mathcal{I}} \text{Hom}_{\mathcal{H}_i}(V_i, V^H_{i'})
\]

If \( t \) is not in \( W \), then Lemma 3.9 yields that \( W \) is a proper subgroup of \( W \), which implies that \( H_z(W) \) is semisimple. In particular, \( V_i \otimes_{H_t} T_i, H_t \otimes_{H_t} H, \) is a projective \( H \)-module. On the other hand, if \( t \) is in \( W \), then we have

\[
V_i \otimes_{H_t} T_i, H_t \otimes_{H_t} H, \cong V_i \otimes_{H_t} T_i,
\]
which is isomorphic to \( V_i \) as \( H \)-modules. Note that \( W \) is contained in \( \mathcal{I} \). Thus we obtain

\[
\dim \text{Hom}_{\mathcal{H}}(V^H_i, V^H_{i'}) = |W| \dim \text{Hom}_{\mathcal{H}_i}(V_i, V_{i'}).
\]

On the other hand, it follows that

\[
V^H_i \cong (V^H_{i'-j})^H \cong \left( \bigoplus_j (\dim U_j) V_{i,j} \right)^H \cong \bigoplus_j (\dim U_j) (V^H_{i,j}).
\]

This together with Lemmas 2.13 and 3.10 implies that

\[
\dim \text{Hom}_{\mathcal{H}}(V^H_i, V^H_{i'}) = \sum_{i \neq j} (\dim U_j)(\dim U_j') \dim \text{Hom}_{\mathcal{H}}(V^H_{i,j}, V^H_{i,j'})
\]

\[
\geq \sum_{j \neq j'} (\dim U_j)(\dim U_{j'}) \dim \text{Hom}_{\mathcal{H}_{i,j}}(V_{i,j}, V_{i,j'})
\]

\[
= \sum_j (\dim U_j)^2 \dim \text{Hom}_{\mathcal{H}_i}(V_i, V_{i})
\]

\[
= |W| \dim \text{Hom}_{\mathcal{H}_i}(V_i, V_{i}).
\]

Therefore, comparing this to the previous result, we can conclude that

\[
\dim \text{Hom}_{\mathcal{H}}(V^H_{i,j}, V^H_{i,j'}) = \dim \text{Hom}_{\mathcal{H}_{i,j}}(V_{i,j}, V_{i,j'})
\]
for all $i$, $i'$, $j$, and $j'$. Moreover looking at isomorphisms used in the above computation, we can see that these spaces are isomorphic as $\text{End}_{H_{I+J}}(V_{i,j}) - \text{End}_{H_{I+J}}(V_{i,j})$-bimodules. This completes the proof.

**Corollary 3.12.** For any $i$ and $j$, the module $V_{i,j}$ is non-projective and isomorphic to a direct sum of an indecomposable module $U_{i,j}$ with a zero or a projective module. Moreover, $U_{i,j} \cong U_{i',j'}$ if and only if $i = i'$ and $j = j'$.

**Proof.** Since $V_{i,j}$ is non-projective, and $\text{End}_{H}(V^H_{i,j})$ is local by the above lemma, the first half follows from Lemma 1.1. To show the second half, assume that $U_{i,j} \cong U_{i',j'}$. Then we first have $j = j'$ by Lemma 3.10. Also, by an argument in the proof of Lemma 3.11, $V_i$ is the only non-projective direct summand of the restriction of $U_{i,j}$ to $H_i$. Hence we get $i = i'$. This completes the proof.

In the above notation let

$$\mathcal{S}_{i,j}: 0 \to \Omega^2 U_{i,j} \to mU_{i,j} \to U_{i,j} \to 0$$

and

$$\mathcal{S}'_{i,j}: 0 \to \Omega^2 V_{i,j} \to mV_{i,j} \to V_{i,j} \to 0$$

be $AR$-sequences.

**Corollary 3.13.** The induced sequence $\mathcal{S}'_{i,j} \otimes_{H_{I+J}} \text{Id}_H$ is equivalent to a direct sum of $\mathcal{S}_{i,j}$ with a split short exact sequence. In particular, $mU_{i,j}$ is isomorphic to a direct summand of $(mV_{i,j})^H_i$.

Moreover, every indecomposable module in the component of the $AR$-quiver containing $U_{i,j}$ can be written as $U_{i',j'}$ for some $i'$.

**Proof.** Recall that the sequence $\mathcal{S}'_{i,j}$ corresponds to a generator of the socle of $\text{End}_{H_{I+J}}(V_{i,j})^*$. It follows from Lemma 3.11 that $\mathcal{S}'_{i,j} \otimes \text{Id}_H$ corresponds to a generator of the socle of $\text{End}_{H}(V^H_{i,j})^*$. Therefore, it is equivalent to a direct sum of $\mathcal{S}_{i,j}$ with a split short exact sequence, which implies that $mU_{i,j}$ is isomorphic to a direct summand of $(mV_{i,j})^H_i$. (See also Lemma 2.12.) This proves the first part. The second part is clear from the first and Lemma 3.10 since every module in the $AR$-quiver of $H_{I+J}$ can be written as $V_{r,j}$, and since $V_{i,j}$ and $V_{i,j}$ lie in the same component if and only if $j = j'$.

**Proof of Theorem 3.8.** First note that the stable $AR$-quiver $\Gamma_{I+J}$ of $H_{I+J}$ consists of $n$ components $\Gamma_j$ and that each component has $\{V_{i,j}\}_{i}$ as its vertices. Moreover, for all $j$ and $j'$, a map from $\Gamma_j$ to $\Gamma_{j'}$ sending $V_{i,j}$ into $V_{i,j}$ gives an isomorphism of quivers. Let $\Gamma$ be the stable $AR$-quiver of $H$. Define a map $\phi$ from the vertices of $\Gamma_{I+J}$ to those of $\Gamma$ by $\phi(V_{i,j}) = U_{i,j}$. It
follows from Corollaries 3.12 and 3.13 that $\phi$ gives an injective quiver homomorphism. Furthermore, each $\phi(\Gamma_j)$ is a component of $\Gamma$. (See Lemma 3.10 and Corollary 3.13.) Thus by Theorem 1.4, these components must correspond to blocks. Hence by Theorem 3.1, $\{\phi(\Gamma_j)\}_{1 \leq j \leq n}$ and defect zero blocks give all the blocks of $H$. (See also the remark following Theorem 3.1.) Therefore, $H$ is of finite representation type. The other statements in the theorem are also clear. This completes the proof.

4. RANK TWO CASES

In this section we assume that $(W, S)$ is rank two, that is, $S = \{s_1, s_2\}$. Thus, the types which we consider are $A_1 \times A_1, A_2, B_2, G_2,$ and $I(p)$ ($p = 5, 7, 8, 9, 10...$). Irreducible representations of the generic Hecke algebras are constructed in [6, Sect. 8], which we describe now. Write $|W| = 2n$. Let $\varepsilon$ be a primitive $n$th root of unity. For any integer $j$, define

$$R_{j,1} = \begin{pmatrix} -1 & a_j \\ 0 & q \end{pmatrix} \quad \text{and} \quad R_{j,2} = \begin{pmatrix} q & 0 \\ b_j & -1 \end{pmatrix},$$

where $a_j$ and $b_j$ are any complex numbers with $a_jb_j = q(2 + \varepsilon^j + \varepsilon^{-j})$. Then we have:

THEOREM 4.1. [6, Theorem 8.1]. If $n = 2r$ is even, then $H_q(W)$ has four representations of degree one, $\dot{\lambda}_1, \dot{\lambda}_2, \dot{\lambda}_3, \dot{\lambda}_4$ given by

$$\begin{align*}
\dot{\lambda}_1(T_{s_1}) &= q, & \dot{\lambda}_1(T_{s_2}) &= q, \\
\dot{\lambda}_2(T_{s_1}) &= -1, & \dot{\lambda}_2(T_{s_2}) &= -1, \\
\dot{\lambda}_3(T_{s_1}) &= q, & \dot{\lambda}_3(T_{s_2}) &= -1, \\
\dot{\lambda}_4(T_{s_1}) &= -1, & \dot{\lambda}_4(T_{s_2}) &= q,
\end{align*}$$

and $r - 1$ inequivalent irreducible representations $\pi_1, ..., \pi_{r-1}$ of degree two given by

$$\pi_j(T_{s_1}) = R_{j,1} \quad \text{and} \quad \pi_j(T_{s_2}) = R_{j,2}.$$ 

If $n = 2r + 1$ is odd, then it has two representations of degree one, $\dot{\lambda}_1, \dot{\lambda}_2$, and $r$ inequivalent irreducible representations $\pi_1, ..., \pi_r$ of degree two given similarly as the above. In each case, they are, up to equivalence, all the irreducible representations of $H_q(W)$.

The Poincaré polynomial $P_w(q)$ of $H_q(W)$ is easily computed as $(q + 1)(1 + q + q^2 + \cdots + q^{n-1})$. So, the roots of $P_w(q) = 0$ are $-1$ and $\varepsilon^j$. 
for some $j$ with $e^j \neq 1$. Also, $\alpha$ is a multiple root of $P_w(q) = 0$ if and only if $n$ is even and $\alpha = -1$. From the irreducible representations given explicitly above, we can compute their generic degrees as follows.

**Proposition 4.2.** We have

$$d_{i_1} = 1, \quad d_{i_2} = q^n, \quad P_w(q)/d_{i_3} = P_w(q)/d_{i_4} = n(q + 1)^2/2q.$$

$$P_w(q)/d_{n_j} = nq(q - e^j)(q - e^{-j})/(1 - e^j)(1 - e^{-j}) \quad (1 \leq j < n/2).$$

Take a root $\alpha$ of $P_w(q) = 0$ and consider the specialization $q \mapsto \alpha$. The following is a direct consequence of Theorem 2.2 [19].

**Proposition 4.3.** The following lists, up to equivalence, all the irreducible projective representations of $H_2(W)$.

- $\pi_1, ..., \pi_{r-1}$ if $n = 2r$ is even and if $\alpha = -1$;
- $\lambda_3, \lambda_4, \pi_j (j \neq k \text{ or } n - k)$ if $n = 2r$ is even and $\alpha = e^k \neq \pm 1$;
- $\pi_1, ..., \pi$, if $n = 2r + 1$ is odd and if $\alpha = -1$;
- $\pi_j (j \neq k \text{ or } n - k)$ if $n = 2r + 1$ is odd and $\alpha = e^k \neq 1$.

Our conjecture has an affirmative answer in this case.

**Theorem 4.4.** Let $\alpha$ be a root of $P_w(q) = 0$. Then:

(i) If $n = 2r$ is even and if $\alpha = -1$, then $H_2(W)$ has infinitely many inequivalent indecomposable representations.

(ii) If $n = 2r$ and if $\alpha = e^k (\neq \pm 1)$, then $H_2(W)$ has four inequivalent non-projective indecomposable representations and $r + 2$ projective indecomposable representations.

(iii) If $n = 2r + 1$ and if $\alpha = -1$, then $H_2(W)$ has one non-projective indecomposable representation and $r + 1$ inequivalent projective indecomposable representations.

(iv) If $n = 2r + 1$ and if $\alpha = e^k (\neq 1)$, then $H_2(W)$ has four inequivalent non-projective indecomposable representations and $r + 1$ inequivalent projective indecomposable representations.

Moreover, the tree class of the stable AR-quiver is of type $A_2$ in cases (ii) and (iv) above and is of type $A_1$ in case (iii).

**Proof:** (i) For any non-zero element $x$ in $C$, let $M_x = (\begin{smallmatrix} -1 & \cdots \cr 0 & \ddots \end{smallmatrix})$. Then for any pair $(x, y) (x, y \in C \setminus \{0\})$, an algebra homomorphism $\rho_{x,y} : H^{-1}(W) \to M_y(C)$ defined by $\rho_{x,y}(T_x) = M_x$ and $\rho_{x,y}(T_{y_2}) = M_y$ gives an indecomposable representation. We now claim that $\rho_{x,y}$ and $\rho_{x,y'}$ are equivalent if
and only if \((x, y) = (x', y')\) in \(\mathbb{P}^1(\mathbb{C})\), the projective line. Suppose that a matrix \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) satisfies \(M^*M = MM^*\) and \(M'_*M' = MM'_*\). Then, it follows that \(e = 0\), \(ax' = dx\), and \(ay' = dy\). Thus, the claim clearly holds, and this proves (i).

(ii) and (iv). Suppose that \(x = \varepsilon^k\). Recall the explicit definition of the representations in Theorem 4.1. Let \(k'\) be \(k\) if \(k < n/2\) and \(n - k\) if \(n/2 < k\). Then the irreducible representations which are not projective irreducible when specializing by \(q \mapsto \alpha\) are precisely \(\lambda_1, \lambda_2\), and \(\pi_{k'}\). Since \(a_{k'}b_{k'}|_{q \mapsto \alpha} = \varepsilon^k(2 + \varepsilon^{k'} + \varepsilon^{-k'})\) is equal to \(\varepsilon^k(2 + \varepsilon^k + \varepsilon^{-k})\), we have

\[
\begin{pmatrix}
-1 & a_{k'} \\
0 & \varepsilon^{k'}
\end{pmatrix}
= \begin{pmatrix}
(b_{k'}, -a_{k'}b_{k'} + \varepsilon^{2k} + \varepsilon^k) = (-b_{k'}, \varepsilon^k + 1)
\end{pmatrix}

\[
\begin{pmatrix}
\varepsilon^{k'} & 0 \\
b_{k'} & -1
\end{pmatrix}
= \begin{pmatrix}
(-\varepsilon^{k}b_{k'} + \varepsilon^{k}b_{k'} + b_{k'} - \varepsilon^{k} - 1)
\end{pmatrix}

= (-b_{k'}, \varepsilon^k + 1).

Thus \(\pi_{k'}\) has an irreducible constituent \(\lambda_2\). Moreover, by looking at the values of the traces, it follows that \(\pi_{k'}\) has \(\lambda_1\) as its irreducible constituent. On the other hand, \(\lambda_1\) and \(\lambda_2\) are irreducible when specializing by \(q \mapsto \alpha\). So, \(\lambda_i\) \((i = 1, 2)\) are the only irreducible representations of \(H_x(W')\) up to equivalence. Hence, by the Cartan–Brauer theorem, the projective cover \(P_i\) of \(\lambda_i\) \((i = 1, 2)\) has irreducible constituents \(\lambda_i\) with multiplicity two and the other irreducible representation with multiplicity one. Since \(H_x(W)\) is a symmetric algebra (Corollary 2.6), the Loewy series of \(P_i\)'s must be

\[P_1 = \lambda_2 \quad \text{and} \quad P_2 = \lambda_1\]

Thus, we have AR-sequences

\[0 \to \lambda_1 \to P_2 \oplus \lambda_1 \to \lambda_2 \to 0,\]

\[0 \to \lambda_2 \to P_1 \oplus \lambda_2 \to \lambda_1 \to 0,\]

\[0 \to \lambda_2 \to \lambda_1 \to \lambda_1 \to 0,\]

\[0 \to \lambda_1 \to \lambda_2 \to \lambda_2 \to 0.\]
Therefore, the tree class is $A_2$ and non-projective indecomposable representations are given by $\lambda_1$, $\lambda_2$, $\tilde{\lambda}_1$, and $\tilde{\lambda}_2$. Projective indecomposable representations are $P_1$, $P_2$, and those in Proposition 4.3. Thus (ii) and (iv) are proved.

(iii) Since the dimension of the non-simple summand of $H_1(W)$ is two, there is the unique irreducible representation $\lambda_1 = \lambda_2$. Also, its projective cover has dimension two. Thus $\lambda_1$ is the unique non-projective indecomposable representation. Projective indecomposable representations are those in Proposition 4.3 and the projective cover of $\lambda_1$. This completes the proof.

5. HECKE ALGEBRAS WHICH ARE OF INFINITE REPRESENTATION TYPE

Let $(W, S)$ be a Coxeter system. Here we prove that in certain cases the specialized algebra $H_\alpha(W)$ is of infinite representation type. First we remark the following.

**Lemma 5.1.** Let $J$ be a subset of $S$. If $H_\alpha(W_J)$ is of infinite representation type for a complex number $\alpha$, then so is $H_\alpha(W)$.

**Proof.** Suppose that there are only finitely many isomorphism classes of indecomposable $H_\alpha(W_J)$-modules, $V_1, \ldots, V_n$, say. Let $U$ be an indecomposable $H_\alpha(W_J)$-module. Then, by Lemma 2.11, $U$ is isomorphic to a direct summand of $U \otimes H_\alpha(W)$ as $H_\alpha(W_J)$-modules. Thus $U$ must be isomorphic to a direct summand of $V_1 \oplus \cdots \oplus V_n$. Hence we can conclude that $H_\alpha(W_J)$ is of finite representation type. This completes the proof.

By the above lemma, in order to show that some $H_\alpha(W)$ is of infinite representation type, it suffices to prove that there is some $J$ such that $H_\alpha(W_J)$ is of infinite representation type. One of the devices is Proposition 3.7.

**Theorem 5.2.** Let $(W, S)$ be an irreducible Coxeter system of rank $l$, and let $\alpha$ be a primitive $r$th root of unity. If $4 \leq 2r \leq l$, then $H_\alpha(W)$ is of infinite representation type.

**Proof.** One can show that $(W, S)$ has a subsystem $(W_J, J)$ of type $A_{r-1} \times A_{r-1}$ by a case by case argument. Then $H_\alpha(W_J)$ is of infinite representation type by Proposition 3.7 ($r > 3$) and Theorem 4.4 ($r = 2$). Therefore, the result holds from Lemma 5.1.

Of course, in certain cases, we can say more. The proof of the following should be almost the same as that of the above and we omit it. Note that
(ii) of the following theorem says that one implication of the question in the introduction holds in the case where \((W, S)\) is of type \(A_r\).

**Theorem 5.3.**  
(i) If \((W, S)\) has rank greater than two, then \(H_-(W)\) is of infinite representation type.

(ii) If \((W, S)\) is of type \(A_r\), and if \(\alpha\) is a primitive \(r\)th root of unity with \(4 \leq 2r \leq 1 + l\), then \(H_+(W)\) is of infinite representation type.

**Acknowledgments**

The author owes a great debt to Professors A. Gyoja and T. Okuyama for their many valuable suggestions. Indeed, in response to a question of the author, Prof. Gyoja showed many computational results on representations of Hecke algebras. The question in the introduction arose in discussion with Prof. Okuyama on the computational result of low rank cases. Also, thanks are extended to H. Yamane and H. Asashiba, who gave the author much information on Hecke algebras and artinian algebras, respectively.

**References**


