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Comparing models of higher type computation

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1 Introduction

Models of higher type computation appear basically in two flavours: as realizability models/partial equivalence relations over some appropriate weak algebraic model of function application, and as some kind of "limit-preserving" functionals for some, suitably weakened, notion of topology.

The aim of this presentation is to show that the two approaches are no different, in the sense that, provided a certain theorem can be proved about the "limit structures", the higher order structures defined via the limit-preserving functionals coincides with one defined in terms of PER's.

Glimpses of this fact can be found in various guises in the literature, see *e.g.* [1,12,14,16,18]. We shall take advantage of the presentation of a category of partial equivalence relations on a category $\boldsymbol{C}$ with pullbacks as a full subcategory of the exact completion $\boldsymbol{C}_{\mathrm{ex}}$, and of the corollary from the Yoneda Lemma that there is a full embedding $\boldsymbol{C}_{\mathrm{ex}} \xhookrightarrow[\text{full}]{} [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ into the (possibly large) category of presheaves on $\boldsymbol{C}$ which preserves all existing constructions involving limits and exponentiation, see [13,19].

Beyond the obvious interest in unifying two seemingly different approaches, we believe that the study raises a specific attention into the *sites* of definition for the various hierarchies.

2 Extensional presheaves

Consider a locally small category $\boldsymbol{C}$ with a terminal object 1. Say that a contravariant set-valued functor $A\colon \boldsymbol{C}^{\mathrm{op}} \longrightarrow \boldsymbol{Set}$ on $\boldsymbol{C}$ is *extensional* if equality on A is determined by the points: for every $a, a' \in A(C)$, we have that $a = a'$ if and only if

$$\forall c\colon 1 \longrightarrow C.\ a_{|c} = a'_{|c}$$

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where we wrote the action of $A(c)$ on a as $a_{|c}$.

In other words, the extensional presheaves are precisely those contravariant set-valued functors A on $\boldsymbol{C}$ such that the composites

$$\eta_C : A(C) \xrightarrow{\simeq} \mathrm{Nat}(\boldsymbol{C}(-,C),A) \xrightarrow{\Gamma} \Gamma(A)^{\boldsymbol{C}(1,C)}, \qquad \text{for } C \text{ in } \boldsymbol{C}$$

are monic. As remarked in [10,20], it follows that the extensional presheaves are also the separated presheaves for the topology on $[\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ determined by the adjunction $\Gamma \dashv \nabla : \boldsymbol{Set} \longrightarrow [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ where Γ denotes the global section functor $\hom(1,-) = \mathrm{Nat}(1,-)$ on $[\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ and

$$\nabla(S) = S^{\boldsymbol{C}(1,-)}.$$

The usefulness of the notion of extensional presheaf rests in the reduction on the size of the homsets: the full subcategory $\mathrm{Ext}(\boldsymbol{C})$ of $[\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ on the extensional presheaves is locally small, since a natural transformation between two such functors is totally determined by its value on the global elements.

One can also improve the logical properties of embedding by choosing a suitable subcanonical Grothendieck topology J on $\boldsymbol{C}$ and describing the notion of extensional (J-)sheaves, and their category thereof.

It is well-known that extensional sheaves form a quasitopos, see [10,20] and also [5]. Examples of categories of extensional sheaves are filter spaces and limit spaces [9], directed-complete posets, (a modification of) hypercoherences, (reflexive) logical relations,

Summing up the properties in this section we have the following.

Theorem 2.1 *Suppose $\boldsymbol{C}$ is a category with a terminal object. Then the full inclusion $\mathrm{Ext}(\boldsymbol{C}) \hookrightarrow [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ has a reflector which commutes with pullbacks along maps in $\mathrm{Ext}(\boldsymbol{C})$. Hence $\mathrm{Ext}(\boldsymbol{C})$ is a regular, locally cartesian closed category. Moreover, it has a regular subobject classifier, hence $\mathrm{Ext}(\boldsymbol{C})$ is a quasitopos.*

Proof. The reflector is defined by forcing "point-determination": Given any presheaf A, let $A'(C)$ be the image of $\eta_C : A(C) \longrightarrow A(1)^{\boldsymbol{C}(1,C)}$. □

Corollary 2.2 *The Yoneda embedding $y : \boldsymbol{C} \longrightarrow [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ factors via the inclusion $\mathrm{Ext}(\boldsymbol{C}) \hookrightarrow [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ if and only if the global section functor $\boldsymbol{C}(1,-) : \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ on $\boldsymbol{C}$ is faithful. When this is the case, the embedding preserves any locally cartesian closed structure existing in $\boldsymbol{C}$.*

As we mentioned above, one may take extensional sheaves with respect to the canonical topology on $\boldsymbol{C}$: with this, one would obtain an embedding which also preserves stable colimits in $\boldsymbol{C}$ such as a strict initial object, or a stable disjoint coproduct. Most of the results in the paper can be extended to the case of extensional sheaves for the canonical topology, but we shall not pursue that further generality in this note.

3 Partial equivalence relations

Partial equivalence relations over a category $\boldsymbol{C}$ with a terminal object can be seen as a quotient of a reflective subcategory of a certain comma category. First, considered the squared global section functor

$$P = \boldsymbol{C}(1,-) \times \boldsymbol{C}(1,-)\colon \boldsymbol{C} \longrightarrow \boldsymbol{Set},$$

take the full subcategory of the comma $(\boldsymbol{Set} \downarrow P)$ on the equivalence relations. Then quotient the homsets by the equivalence relation induced by (that given on) the codomain to obtain the category $\mathrm{Mod}(\boldsymbol{C})$ of so-called *modest sets* on $\boldsymbol{C}$.

The process intends to achieve, at least, two goals: usually the category $\boldsymbol{C}$ is defined in terms of "codes" or "notations" for an intended function, and PER's are introduced just to identify "extensionally" equal maps. Also, the category of modest sets extends definability in $\boldsymbol{C}$ by allowing quotienting of objects, see *e.g.* [1]. Examples are many, see [2] for a reasonably exhaustive list of references.

By taking an object C in $\boldsymbol{C}$ to the diagonal relation on $\boldsymbol{C}(1, C)$, one obtains the following property, cf. [15].

Proposition 3.1 *There is a full functor* $d\colon \boldsymbol{C} \longrightarrow \mathrm{Mod}(\boldsymbol{C})$*. Moreover, the induced functor* $\mathrm{Mod}(\boldsymbol{C}) \longrightarrow [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ *is an embedding which preserves any locally cartesian closed structure existing in* $\mathrm{Mod}(\boldsymbol{C})$*.*

Proof. (Sketch) The functor d takes an object C to the diagonal relation on the global elements of A, the set of maps $\boldsymbol{C}(1, A)$. This induces a functor $\mathrm{Mod}(\boldsymbol{C}) \longrightarrow [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ as follows: a modest set (R, A), where R is an equivalence relation on $\boldsymbol{C}(1, A)$, is mapped to the functor

$$\hom(d(-), (R, A))\colon \boldsymbol{C}^{\mathrm{op}} \longrightarrow \boldsymbol{Set}.$$

Explicitly, at an object C, it is the quotient of the set of all maps $\boldsymbol{C}(C, A)$ with respect to the equivalence relation induced by R by global elements: $f \sim g$ exactly when

$$\forall c\colon 1 \longrightarrow C.\, fc\; R\; gc.$$

It is easy to see that the functor is faithful as the presheaf represented by A covers $\hom(d(-), (R, A))$. Fullness follows from the projectivity of the representables. □

Note that it follows that hierarchies of types defined in modest sets or in the extensional presheaves based on an interpretation of types in $\boldsymbol{C}$ coincide, since these could as well be defined in $[\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$.

A general explanation of this can be obtained using the construction of the regular completion and of the exact completion, see [4,8]. (From now on, we shall assume that the category $\boldsymbol{C}$ has weak finite limits, as is the case in all the examples.) We refer the reader to some of the references above or [3,6,17,18] for the definitions of the regular completion and of the exact completion. Intuitively, the first freely adds stable quotients of kernel pairs (possibly strengthening the weak products, hence all finite limits), the second does the same for general quotients of equivalence relations in $\boldsymbol{C}$. The following theorem is well-known.

Theorem 3.2 *Suppose* $\boldsymbol{C}$ *is a category with weak limits. Then the regular completion and the exact completion are full subcategories of the category of presheaves* $[\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ *by embeddings which preserve any existing locally cartesian closed structure.*

Theorem 3.3 *In the same hypothesis of the previous theorem, the regular completion* $\boldsymbol{C}_{\mathrm{reg}}$ *is a full subcategory of the category of the extensional presheaves on* $\boldsymbol{C}$ *if and only if the global section functor* $\boldsymbol{C}(1,-)\colon \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ *is faithful.*

Proof. (Sketch) Any object in the regular completion is canonically a subobject of an object in $\boldsymbol{C}$, see Theorem 5.1(ii) in [3]. Hence, by 2.2, it is a subobject of an extensional presheaf. □

The condition that $\boldsymbol{C}_{\mathrm{ex}}$ is cartesian closed can be translated purely in terms of $\boldsymbol{C}$ as stating that the category $\boldsymbol{C}$ has weak simple products, see [7]. In case the regular completion $\boldsymbol{C}_{\mathrm{reg}}$ is a reflective subcategory of $\boldsymbol{C}_{\mathrm{ex}}$, the property just stated ensures that that is also cartesian closed. Again, by the embedding into $[\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$, one has that hierarchies of types defined in either of these, based on objects in $\boldsymbol{C}$, would result in the same as defined in extensional presheaves or modest sets.

It is interesting to note at this point the following property for modest sets on $\boldsymbol{C}$.

Theorem 3.4 *If* $\boldsymbol{C}$ *has weak simple products, then* $\mathrm{Mod}(\boldsymbol{C})$ *is cartesian closed.*

4 Connecting the constructions

As we already mentioned, the category of modest sets is related to the completions: a modest set can be turned into an equivalence relation in $\boldsymbol{C}$ under suitable assumptions on $\boldsymbol{C}$. Again, it is the Yoneda embedding which comes to help: by 3.1, the category $\mathrm{Mod}(\boldsymbol{C})$ is a full subcategory of $[\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$. A sufficient condition for inclusion into the subcategory of extensional presheaves is given below, strengthening 3.1.

Theorem 4.1 *Suppose* $\boldsymbol{C}$ *has a terminal object. Then* $\mathrm{Mod}(\boldsymbol{C})$ *is a full subcategory of* $\mathrm{Ext}(\boldsymbol{C})$.

While the descriptions of $\mathrm{Mod}(\boldsymbol{C})$ and $\mathrm{Ext}(\boldsymbol{C})$ depend on the semantic realization $\boldsymbol{C}(1,-)\colon \boldsymbol{C} \longrightarrow \boldsymbol{Set}$, those of the completions $\boldsymbol{C}_{\mathrm{reg}}$ and $\boldsymbol{C}_{\mathrm{ex}}$ depend only on the category $\boldsymbol{C}$, and its inner, syntactic, structure. Thus, one should expect that a connection between the two kinds of constructions requires some definability properties of $\boldsymbol{C}$.

Proposition 4.2 *Suppose $\boldsymbol{C}$ has a terminal object and weak limits. Suppose the global section functor $\boldsymbol{C}(1,-)\colon \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ is faithful.*

(i) *If pullbacks of the form*

$$\begin{array}{ccc} P & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ yA & \xrightarrow{\eta} & \nabla(\Gamma(yA)) \end{array}$$

are always representable in $\boldsymbol{C}$, then Mod($\boldsymbol{C}$) *is a reflective subcategory of the exact completion $\boldsymbol{C}_{\mathrm{ex}}$ with a reflector which preserves reindexing along maps in* Mod($\boldsymbol{C}$)*. Hence* Mod($\boldsymbol{C}$) *is closed under any locally cartesian closed structure which exists in $\boldsymbol{C}_{\mathrm{ex}}$.*

(ii) *If $\boldsymbol{C}$ has weak coequalizers, then* Mod($\boldsymbol{C}$) *is a subcategory of the regular completion $\boldsymbol{C}_{\mathrm{reg}}$.*

Remark 4.3 The hypothesis on pullbacks in the previous statement can be expressed in logical terms: it states that, if we let J be the topology determined by the adjunction $\Gamma \dashv \nabla\colon \boldsymbol{Set} \longrightarrow [\boldsymbol{C}^{\mathrm{op}}, \boldsymbol{Set}]$ introduced in section 2, the J-closed subobjects of a representable are representable. It is satisfied when $\boldsymbol{C}(1,-)\colon \boldsymbol{C} \longrightarrow \boldsymbol{Set}$ has a right adjoint D. In that case, one can in fact define a reflector from the regular completion $\boldsymbol{C}_{\mathrm{reg}}$ into Mod($\boldsymbol{C}$) which preserves reindexing along maps in Mod($\boldsymbol{C}$). To see this, considered an equivalence relation

$$R \hookrightarrow P(C) = \boldsymbol{C}(1,C) \times \boldsymbol{C}(1,C),$$

take first its quotient $\boldsymbol{C}(1,C) \longrightarrow (\boldsymbol{C}(1,C)/_R$ in $\boldsymbol{Set}$, then the object in $\boldsymbol{C}_{\mathrm{reg}}$ defined by its transpose $C \longrightarrow D((\boldsymbol{C}(1,C)/_R)$.

Example 4.4 In case $\boldsymbol{C}$ is the category of T_0-topological spaces, then Mod($\boldsymbol{C}$) is the category of equilogical spaces, and Ext($\boldsymbol{C}$) is an extension of the category of filter spaces.

In case $\boldsymbol{C}$ is the category of topological spaces, then Mod C coincide with the regular completion of $\boldsymbol{C}$, and Ext($\boldsymbol{C}$) is an extension of the category of ultralimit spaces (which is the category of extensional sheaves for the canonical topology).

In case $\boldsymbol{C}$ is the category of subsets of a partial combinatory algebra A, then Mod($\boldsymbol{C}$) is the usual category of partial equivalence relations on A, and Ext($\boldsymbol{C}$) is a category of partial A-path preserving functionals. It seems possible to work out a connection between some categories of Kripke logical relations (as a category of extensional sheaves) and categories of modest sets on some appropriate weakly cartesian closed category.

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