MATHEMATICS

PERIODIC SOLUTIONS OF THE VAN DER POL EQUATION

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While testing some of the ALGOL 60 procedures for solving ordinary differential equations given in [6], we solved, using $RK4n$, the van der Pol equation

$$\frac{d^2x}{dt^2} - \nu(1-x^2)\frac{dx}{dt} + x = 0 \tag{1}$$

and computed the amplitude a and the period T of the periodic solution for $\nu = 1\ (1)\ 20\ (5)\ 50\ (10)\ 100$.

We tried to check our results against tables given by KROGDAHL [2] and URABE [5] as well as against asymptotic formulas given by DORODNICYN [1] and URABE. We found that the tables contain errors. The discrepancies we found suggested that both Dorodnicyn's formula for the period T and his formula as "corrected" by Urabe, were wrong. Checking Dorodnicyn's paper we found instead of his formula (8.8):

$$\left\{\begin{aligned} T_3 \sim &\left(\frac{3}{2}-\log 2\right)\nu - \nu^{1/3} + 1/3 + \left(\frac{a}{2}-\frac{1}{4}\right)\nu^{-1/3} + \frac{7}{10}\nu^{-2/3} - \frac{3}{2}\frac{\log \nu}{\nu} + \\ &\left(\frac{1}{2}b_0 - \frac{7}{12} + \frac{11}{6}\log\frac{2}{3}\right)\nu^{-1} + O(\nu^{-4/3}), \end{aligned}\right. \tag{2}$$

and for the full period

$$\left\{\begin{aligned} T \sim &(3-2\log 2)\,\nu + 3a\,\nu^{-1/3} - \frac{2}{3}\frac{\log \nu}{\nu} + \\ &\left(-\log 3 + 3\log 2 - \frac{3}{2} + b_0 - 2d\right)\nu^{-1} + O(\nu^{-4/3}). \end{aligned}\right. \tag{3}$$

Formula (7.2) for the amplitude proved to be correct:

$$a \sim 2 + \frac{a}{3}\nu^{-4/3} - \frac{16}{27}\frac{\log \nu}{\nu^2} + \frac{1}{9}(3b_0 - 1 + 2\log 2 - 8\log 3)\,\nu^{-2} + O(\nu^{-8/3}), \tag{4}$$

where $a = 2.338107$, $b_0 = 0.1723$, $d = 0.4889$.

Recently PONZO and WAX [3], using a different method, found the first three terms of T and a in agreement with (3) and (4).

In an attempt to compute the next term in the expansion we found that the expansions for $p = \mathrm{d}x/\mathrm{d}t$ at the boundary of Dorodnicyn's regions II and III differed:

$$\text{II: } p \sim \frac{2}{3}\nu^{-1} - \left(\frac{2}{3} + \frac{5a}{27}\right)\nu^{-7/3} + O\,(\log \nu/\nu^2)$$

$$\text{III: } p \sim \frac{2}{3}\nu^{-1} - \frac{5a}{27}\nu^{-7/3} + O\,(\log \nu/\nu^2).$$

Therefore, in order to find more terms one has to use a different method.

To conclude we give numerical results. Here, T and a are found by numerical integration, whereas T_{asympt} and a_{asympt} are the results of the first four terms of (3) and (4) respectively.

ν	T	T_{asympt}	a	a_{asympt}
1	6.66329		2.00862	
2	7.62987		2.01989	
3	8.85909		2.02330	
4	10.20352		2.02296	
5	11.61223		2.02151	
6	13.06187		2.01983	
7	14.53975		2.01822	
8	16.03818		2.01675	
9	17.55218		2.01544	
10	19.07837	19.10684	2.01429	2.02253
11	20.61431	20.63896	2.01326	2.02011
12	22.15822	22.17981	2.01236	2.01814
13	23.70876	23.72786	2.01156	2.01650
14	25.26487	25.28193	2.01084	2.01512
15	26.82575	26.84108	2.01020	2.01394
16	28.39074	28.40461	2.00962	2.01291
17	29.95929	29.97192	2.00909	2.01202
18	31.53097	21.54252	2.00862	2.01123
19	33.10542	33.11603	2.00819	2.01054
20	34.68232	34.69212	2.00779	2.00992
25	42.59579	42.60268	2.00624	2.00761
30	50.54369	50.54885	2.00516	2.00612
35	58.51444	58.51848	2.00438	2.00509
40	66.50137	66.50463	2.00379	2.00433
45	74.50026	74.50296	2.00333	2.00376
50	82.50833	82.51061	2.00296	2.00330
60	98.54479	98.54648	2.00240	2.00264
70	114.60067	114.60198	2.00201	2.00219
80	130.67020	130.67126	2.00172	2.00186
90	146.74979	146.75066	2.00150	2.00160
100	162.83707	162.83781	2.00132	2.00141

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