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On the time decay of solutions in one-dimensional theories of porous materials

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Abstract

In this paper we investigate the temporal asymptotic behavior of the solutions of the one-dimensional porous-elasticity problem when several damping effects are present. We show that viscoelasticity and temperature produce slow decay in time, and the same result is obtained when the porous viscosity is combined with microtemperatures. However, when the viscoelasticity is coupled with porous damping or with microtemperatures the decay is controlled by a negative exponential.
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1. Introduction

Elasticity problems have attracted the attention of researchers from different fields interested in the temporal decay behavior of the solutions. This interest has given many results that can be found in the literature. In the one-dimensional case, for instance, it is known that combining the equations of elasticity with thermal effects provokes that a negative exponential controls the solutions decay (Jiang and Racke, 2000; Quintanilla and Racke, 2003; Slemrod, 1981).

If elastic solids with voids are considered, as in this paper, one should look into the theory of porous elastic materials. Here we deal with the theory established by Nunziato and Cowin (1979), Cowin and Nunziato (1983), and Cowin (1985). In their setting, the bulk density is the product of two scalar fields, the matrix material density and the volume fraction field. It is deeply discussed in the book of Ieşan (2004).

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Grot (1969) developed a theory of thermodynamics for elastic materials with microstructure whose microelements, in addition to microdeformations, possess microtemperatures. It is based on the continuum theories with microstructure, where the microelements undergo homogeneous deformations called microdeformations. It was extensively developed by Eringen (1964, 1967, 1999) and Eringen and Kafadar (1976), but we also want to recall the contributions of Ieşan (2001) and Ieşan and Quintanilla (2000). In this paper, we will see that the microthermal effects are mechanisms that produce a fast decay of the porous variables.

The analysis of the temporal decay in one-dimensional porous-elastic materials was first studied by Quintanilla (2003). The author showed that porous viscosity was not powerful enough to obtain exponential stability in the solutions. For the sake of completeness we recall that the solutions generated by a semigroup $U(t)$ are said to be *exponentially stable* if there exist two constants (independent of the initial conditions) $C > 0$ and $\varpi > 0$ such that $\|U(t)\| \leqslant C\exp(-\varpi t)\|U(0)\|$.

Casas and Quintanilla (2005) proved that the addition of temperature neither gives exponential stability. However, Casas and Quintanilla (2005) found that the combination of porous viscosity with thermal effects does produce it. In the same way, mixing temperature and microtemperature gives rise to exponential stability (see Casas and Quintanilla, 2005, again). Some of these results have been recently extended to generalized thermoelasticity by Magaña and Quintanilla (2005).

From the above comments, it seems natural to think that the porous-elastic coupling is not very strong. In this work we want to deepen that matter and clear it up. In order to do so, we will consider dissipation mechanisms that have not been studied before in this context. To be exact, we will introduce the elastic viscosity of rate type and combine it with temperature, microtemperature or porous viscosity. In each case we will study the temporal asymptotic behavior of the solutions of the corresponding problem.

Through the paper, to simplify our expressions, we speak several times about *slow decay* or *exponential decay* of the solutions. We will say that the decay of the solutions is exponential if they are exponentially stable and, if they are not, we will say that the decay of the solutions is slow. Perhaps it is worth recalling the main difference between these two concepts in a thermomechanical context. If the decay is exponential, then after a short period of time, the thermomechanical displacements are very small and can be neglected. However, if the decay is slow, then the solutions weaken in a way that thermomechanical displacements could be appreciated in the system after some time. Therefore, the nature of the solutions highly determines the temporal behavior of the system and, from a thermomechanical point of view, it is relevant to be able to classify them.

The paper is structured as follows. In Section 2, we recall the general three-dimensional theory and we state the equations for the one-dimensional case. In Section 3, we study first the effect of viscoelasticity in the porous-elasticity problem, and we show the slow decay of its solutions. We examine as well this problem adding the corresponding thermal effects and we find the same result. In Section 4, we consider again the porous-elasticity problem with elastic and porous dissipation but without heat conduction. In this case we get exponential decay of the solutions. In Section 5, we remove the porous dissipation from the system but we introduce microtemperatures, and we also get exponential stability. Section 6 is devoted to study the system with porous dissipation, microtemperatures but without elastic damping, and we prove the slow decay of the solutions. Finally, in Section 7, we summarize all these results and some conclusions are listed. It is worth mentioning that in Sections 4 and 5 we adapt to our situations the semigroup arguments, following the guidelines used in the book of Liu and Zheng (1999), that have shown to be a powerful tool to study this kind of problems.

2. Preliminaries

The theory of elastic solids with voids was introduced by Nunziato and Cowin (1979). Ieşan (1986, 2001, 2004) added temperature and also microtemperatures to this theory. Let us make a short presentation of the general three-dimensional theory. The evolution equations are

$$\begin{cases} \rho\ddot{u}_i = t_{ji,j}, \\ J\ddot{\phi} = h_{i,i} + g, \\ \rho T_0\dot{\eta} = q_{i,i}, \\ \rho\dot{E}_i = P_{ji,j} + q_i - Q_i, \end{cases}$$

where t_{ji} is the stress tensor, h_i is the equilibrated stress vector, g is the equilibrated body force, q_i is the heat flux vector, η is the entropy, P_{ji} is the first heat flux moment tensor, Q_i is the mean heat flux vector, E_i is the first moment of energy vector and T_0 is the absolute temperature in the reference configuration. The variables u_i and ϕ are, respectively, the displacement of the solid elastic material and the volume fraction. We assume that ρ and J are positive constants whose physical meaning is well known.

To state the field equations, we need first the constitutive equations. In the general case of solids with viscoelasticity, porous-viscosity, temperature and microtemperature we assume the following (see Ieşan, 2001, 2004):

$$\begin{cases} t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\phi\delta_{ij} + \lambda^*\dot{e}_{rr}\delta_{ij} + 2\mu^*\dot{e}_{ij} - \beta\theta\delta_{ij}, \\ h_i = \delta\phi_{,i} - dw_i, \\ g = -be_{rr} + m\theta - \xi\phi - \tau\dot{\phi}, \\ \rho\eta = \beta e_{rr} + c\theta + m\phi, \\ q_i = k\theta_{,i} + \kappa_1 w_i, \\ P_{ij} = -\kappa_4 w_{r,r}\delta_{i,j} - \kappa_5 w_{i,j} - \kappa_6 w_{j,i}, \\ Q_i = (\kappa_1 - \kappa_2)w_i + (k - \kappa_3)\theta_{,i}, \\ \rho E_i = -\alpha w_i - d\phi_{,i}. \end{cases}$$

Here λ, μ, b, λ^*, μ^*, β, δ, d, m, ξ, τ, c, k, κ_i ($i = 1, \ldots, 6$) and α are the constitutive coefficients, and θ and w_i are the temperature and microtemperatures, respectively.

From Cowin and Nunziato (1983), the constitutive coefficients for isotropic bodies satisfy the following inequalities:

$$\mu > 0, \quad \xi > 0, \quad \delta > 0, \quad 2\mu + 3\lambda > 0, \quad (2\mu + 3\lambda)\xi > 3b^2. \tag{2.1}$$

The other coefficients satisfy the Clausius–Duhem conditions (Ieşan, 2001).

As we are considering here the one-dimensional theory, the evolution equations become easier and are given by

$$\begin{cases} \rho\ddot{u} = t_x, \\ J\ddot{\phi} = h_x + g, \\ \rho\dot{\eta} = q^*_x, \\ \rho\dot{E} = P_x + q^* - Q, \end{cases}$$

where q^* stands for $T_0^{-1}q$.

And the constitutive equations:

$$\begin{cases} t = \mu u_x + b\phi + \gamma \dot{u}_x - \beta\theta, \\ h = \delta\phi_x - dw, \\ g = -bu_x - \xi\phi + m\theta - \tau\dot{\phi}, \\ \rho\eta = \beta u_x + c\theta + m\phi, \\ q^* = k^*\theta_x + \kappa_1^* w, \\ P = -\kappa_4 w_x, \\ Q = (\kappa_1^* - \kappa_2)w + (k^* - \kappa_3)\theta_x, \\ \rho E = -\alpha w - d\phi_x, \end{cases}$$

where, abusing a little bit the notation, we write μ instead of $\lambda + 2\mu$ and γ instead of $\lambda^* + 2\mu^*$. We also use κ_1^* and k^* to denote $T_0^{-1}\kappa_1$ and $T_0^{-1}k$, respectively, but in the sequel, we will omit the stars.

Thus, the constitutive coefficients, in the one-dimensional case and with the new notation, satisfy the following inequalities:

$$\xi > 0, \quad \delta > 0, \quad \mu\xi > b^2. \tag{2.2}$$

It is assumed that the internal mechanical energy density is a positive definite form.

As coupling is considered, b must be different from 0, but its sign does not matter in the analysis.

When thermal effects are considered, we assume that the thermal capacity c and the thermal conductivity k are strictly positive. Analogously, if microtemperatures are present, parameters α, κ_4 and κ_2 are positive.

Note that γ and τ are nonnegative. If $\gamma > 0$ viscoelastic dissipation is assumed in the system, and if $\tau > 0$ porous dissipation is present.

We would like to say that we do not know any reference with an explicit formulation of the equations for porous-elastic materials with microtemperatures. Nevertheless, it is known that in the one-dimensional linear theory, the equations that describe porosity and microstretch coincide. Therefore, we think it is appropriate to use the equations proposed by Ieşan (2001) to describe this theory.

In this paper we analyze five problems. All of them are particular cases of the above system. However, it is worth noting that we do not consider in any place the complete system. In fact, our aim is to know if the solutions decay can be controlled by an exponential when only one or two of the damping effects are present. In other words, we want to know when the solutions are exponentially stable.

The boundary conditions used in this work pretend to ease the analysis carried out. To prove the slow decay we use spectral arguments, and to show the exponential decay we make use of functional analysis. It is important to notice that the spectral arguments strongly depend on the boundary conditions. Alternative boundary conditions do not allow to use the same arguments. However, the method we use to prove exponential decay can be extended without difficulties to other boundary conditions.

3. Viscoelasticity, porosity and temperature

Our analysis begins with the most elementary question: which is the asymptotic behavior of the solutions of the porous-elasticity problem if elastic dissipation is taken into account? We will show that viscoelasticity is not strong enough to make the solutions decay in an exponential way. If we introduce the constitutive equations into the evolution equations, but neglecting the thermal and microthermal effects, we obtain the system

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + b\phi_x + \gamma\dot{u}_{xx}, \\ J\ddot{\phi} = \delta\phi_{xx} - bu_x - \xi\phi. \end{cases} \tag{3.1}$$

Conditions (2.2) are assumed for the system coefficients. As we are now considering viscoelasticity, the constant γ is assumed to be strictly positive.

To have the problem determined, we impose boundary and initial conditions. Thus, we assume that the solutions satisfy the boundary conditions

$$u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = 0, \tag{3.2}$$

and the initial conditions

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), \quad \phi(x,0) = \phi_0(x), \quad \dot{\phi}(x,0) = \varphi_0(x). \tag{3.3}$$

There are solutions (uniform in the variable x) that do not decay. To avoid these cases, we will also assume that

$$\int_0^{\pi} \phi_0(x)\,\mathrm{d}x = \int_0^{\pi} \varphi_0(x)\,\mathrm{d}x = 0.$$

Theorem 3.1. *Let (u,ϕ) be a solution of the problem determined by (3.1)–(3.3). Then (u,ϕ) decays in a slow way.*

Proof. We will prove that there exists a solution of system (3.1) of the form

$$u = K_1 \mathrm{e}^{\omega t}\sin(nx), \qquad \phi = K_2 \mathrm{e}^{\omega t}\cos(nx),$$

such that $\mathrm{Re}(\omega) > -\epsilon$ for all positive ϵ. Hence, a solution ω as near as desired to the imaginary axis can be found. This fact shows that it is impossible to have uniform exponential decay on the solutions of the problem determined by (3.1)–(3.3).

Imposing that u and ϕ are as above and replacing them in (3.1) the following homogeneous system in the unknowns K_1 and K_2 is obtained:

$$\begin{pmatrix} \rho\omega^2 + n^2\mu + n^2\gamma\omega & bn \\ bn & J\omega^2 + n^2\delta + \xi \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system will have nontrivial solutions if, and only if, the determinant of the coefficients matrix is equal to zero. We denote by $p(x)$ the determinant of the coefficient matrix once ω is replaced by x:

$$p(x) = J\rho x^4 + Jn^2\gamma x^3 + \left(Jn^2\mu + n^2\delta\rho + \xi\rho\right)x^2 + \left(n^4\delta\gamma + n^2\gamma\xi\right)x - b^2n^2 + n^4\delta\mu + n^2\mu\xi.$$

Notice that $p(x)$ is a fourth degree polynomial and, hence, its roots can be computed by formula. Nevertheless, the expressions of the roots we have obtained using Mathematica do not let us decide whether their real parts are negative or not. To prove that $p(x)$ has roots as near as we want to the complex axis, we will show that for any $\epsilon > 0$ there are roots of $p(x)$ located on the right side of the vertical line $\mathrm{Re}(z) = -\epsilon$. This fact will be shown if the polynomial $p(x-\epsilon)$ has a root with positive real part. To prove that, we use the Routh–Hurwitz theorem. It assesses that, if $a_0 > 0$, then all the roots of polynomial

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4,$$

have negative real part if, and only if, all the leading minors of matrix

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{pmatrix},$$

are positive. We denote by L_i, for $i = 1,2,3,4$, the leading minors of this matrix.

Direct computations show that the third leading minor, L_3, is an eight degree polynomial on n:

$$L_3 = -2J^2\delta\epsilon\gamma^3 n^8 + R(n),$$

where $R(n)$ is a polynomial on n of degree 6. Thus, for n large enough, L_3 is negative and $p(x-\epsilon)$ has a root with positive real part. (We have used Mathematica to compute L_3.)

This argument shows that the solutions of system (3.1) decay in a slowly way, or, in other words, that a uniform rate of decay of exponential type for all the solutions cannot be obtained. □

One may think now that perhaps the addition of thermal effects would produce a qualitative change in the behavior of the solutions of system (3.1). Now, we study the system obtained for the variables u, ϕ and θ when we assume viscoelasticity ($\gamma > 0$), but we do not assume porousviscosity ($\tau = 0$). This consideration leads to the following system:

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + b\phi_x - \beta\theta_x + \gamma\dot{u}_{xx}, \\ J\ddot{\phi} = \delta\phi_{xx} - bu_x - \xi\phi + m\theta, \\ c\dot{\theta} = k\theta_{xx} - \beta\dot{u}_x - m\dot{\phi}. \end{cases} \tag{3.4}$$

Now we impose that the boundary conditions are given by (3.2) and

$$\theta_x(0,t) = \theta_x(\pi,t) = 0, \tag{3.5}$$

and that the initial conditions are given by (3.3) and

$$\theta(x,0) = \theta_0(x). \tag{3.6}$$

As in the isothermal case, there are uniform solutions in the variable x such that do not tend to zero as time goes to infinity. To avoid this possibility we again impose that the average of the initial condition for ϕ_0, φ_0 and θ_0 vanishes. In the remaining of this manuscript we suppose that the initial conditions satisfy these assumptions.

Theorem 3.2. *Let (u,ϕ,θ) be a solution of the problem determined by the system (3.4), the boundary conditions (3.2), (3.5) and the initial conditions (3.3), (3.6). Then (u,ϕ,θ) decays in a slow way.*

Proof. The proof follows the same guidelines of those of Theorem 3.1. Let us point out the main facts.

We will prove the existence of a solution of system (3.4) of the form

$$u = K_1 e^{\omega t}\sin(nx), \qquad \phi = K_2 e^{\omega t}\cos(nx), \qquad \theta = K_3 e^{\omega t}\cos(nx),$$

with $\mathrm{Re}(\omega) > -\epsilon$ for all positive ϵ. Imposing that u, ϕ and θ are as above and replacing them in (3.4) the following homogeneous system on the unknowns K_1, K_2 and K_3 is obtained:

$$\begin{pmatrix} n^2\mu + n^2\gamma\omega + \rho\omega^2 & bn & -\beta n \\ bn & n^2\delta + \xi + J\omega^2 & -m \\ n\beta\omega & m\omega & c\omega + kn^2 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In the present case, ω will be a root of the polynomial

$$\begin{aligned} q(x) = {} & cJ\rho x^5 + \left(cJn^2\gamma + Jkn^2\rho\right)x^4 + \left(Jn^2\beta^2 + Jkn^4\gamma + cJn^2\mu + m^2\rho + cn^2\delta\rho + c\xi\rho\right)x^3 \\ & + \left(m^2n^2\gamma + cn^4\delta\gamma + Jkn^4\mu + cn^2\gamma\xi + kn^4\delta\rho + kn^2\xi\rho\right)x^2 \\ & + \left(n^4\beta^2\delta + kn^6\delta\gamma + m^2n^2\mu + cn^4\delta\mu + n^2\beta^2\xi + kn^4\gamma\xi + cn^2\mu\xi - b^2cn^2 - 2bmn^2\beta\right)x \\ & + kn^6\delta\mu + kn^4\mu\xi - b^2kn^4. \end{aligned}$$

We consider $q(x-\epsilon)$ and use the Routh–Hurwitz theorem again (coefficient a_0 in $q(x-\epsilon)$ is $c\,J\rho > 0$). We denote by M_i, for $i = 1,2,3,4,5$, the leading minors of the corresponding matrix. M_4 is a polynomial on n of degree sixteen with negative main coefficient. In fact,

$$M_4 = -2J^3k^3\delta\epsilon\gamma^3(c\gamma + k\rho)n^{16} + S(n),$$

where $S(n)$ is a polynomial on n of fourteenth degree. Thus, $M_4 < 0$ for all ϵ and for n large enough.

This shows that it is impossible to obtain a uniform rate of decay of exponential type for all the solutions of system (3.4). □

To sum up this section in physical terms: we have shown that nor the elastic dissipation nor the elastic dissipation plus the addition of heat are strong enough to make the solutions decay in an exponential way.

4. Viscoelasticity and viscoporosity

In this section we consider the case where viscoelasticity and porous dissipation are both present, and we prove that this is enough to have exponential stability. We use the semigroup arguments due to Liu and Zheng (1999).

The system we want to study is

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + b\phi_x + \gamma\dot{u}_{xx}, \\ J\ddot{\phi} = \delta\phi_{xx} - bu_x - \xi\phi - \tau\dot{\phi}, \end{cases} \tag{4.1}$$

where $\gamma, \tau > 0$.

The boundary and initial conditions are given by (3.2) and (3.3), respectively.

Again, the existence of solutions that do not decay is clear. But if the average of the initial condition for ϕ_0 and φ_0 vanishes, then we avoid this possibility.

We consider the Hilbert space

$$\mathscr{H} = \left\{(u, v, \phi, \varphi) \in H_0^1 \times L^2 \times H^1 \times L^2, \int_0^\pi \phi(x)\,\mathrm{d}x = \int_0^\pi \varphi(x)\,\mathrm{d}x = 0\right\}.$$

Taking into account that $\dot{u} = v$ and $\dot{\phi} = \varphi$, and writing $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x}$, we can restate system (4.1) in the following way:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = \frac{1}{\rho}[\mu \mathrm{D}^2 u + b\mathrm{D}\phi + \gamma \mathrm{D}^2 v], \\ \dot{\phi} = \varphi, \\ \dot{\varphi} = \frac{1}{J}[\delta \mathrm{D}^2\phi - b\mathrm{D}u - \xi\phi - \tau\varphi]. \end{cases} \tag{4.2}$$

And, if $U = (u, v, \phi, \varphi)$, then our initial-boundary value problem can be written as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathscr{A}U, \quad U_0 = (u_0, v_0, \phi_0, \varphi_0),$$

where $\mathscr{A}$ is the following 4×4 matrix:

$$\mathscr{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\mu}{\rho}\mathrm{D}^2 & \frac{\gamma}{\rho}\mathrm{D}^2 & \frac{b}{\rho}\mathrm{D} & 0 \\ 0 & 0 & 0 & I \\ -\frac{b}{J}\mathrm{D} & 0 & \frac{\delta \mathrm{D}^2 - \xi}{J} & -\frac{\tau}{J} \end{pmatrix},$$

and I is the identity operator.

Now, we define an inner product in $\mathscr{H}$. If $U^* = (u^*, v^*, \phi^*, \varphi^*)$, then

$$\langle U, U^* \rangle = \int_0^\pi \left(\rho v \bar{v}^* + J \varphi \bar{\varphi}^* + \mu u_x \bar{u}_x^* + \delta \phi_x \bar{\phi}_x^{\,*} + \xi \phi \bar{\phi}^* + b\left(u_x \bar{\phi}^* + \bar{u}_x^* \phi \right) \right) \mathrm{d}x. \tag{4.3}$$

Here a superposed bar denotes the conjugate complex number. It is worth recalling that this product is equivalent to the usual product in the Hilbert space $\mathscr{H}$. It can be proved that the general solutions of system (4.1) are given by the semigroup of contractions generated by the operator $\mathscr{A}$.

Direct calculation gives

$$\mathrm{Re}\langle \mathscr{A} U, U \rangle = -\int_0^\pi \left(\gamma \|v_x\|^2 + \tau \|\varphi\|^2 \right) \mathrm{d}x \leqslant 0. \tag{4.4}$$

To show the exponential stability we use a result due to Gearhart (Liu and Zheng, 1999) which states that a semigroup of contractions on a Hilbert space is exponentially stable if and only if

$$\{\mathrm{i}\lambda, \lambda \text{ is real}\} \text{ is contained in the resolvent of } \mathscr{A}, \tag{4.5}$$

and

$$\overline{\lim_{|\lambda| \to \infty}} \|(\mathrm{i}\lambda \mathscr{I} - \mathscr{A})^{-1}\| < \infty, \tag{4.6}$$

where $\mathscr{I}$ denotes the identity matrix.

To prove these conditions we need first the following result.

Lemma 4.1. *Let $\mathscr{A}$ be the above defined matrix. Then,* 0 *is in the resolvent of $\mathscr{A}$.*

Proof. For any $\mathscr{F} = (f_1, f_2, f_3, f_4) \in \mathscr{H}$ we will find $U \in \mathscr{H}$ such that $\mathscr{A} U = \mathscr{F}$, or equivalently:

$$\left.\begin{array}{l} v = f_1, \\ \frac{1}{\rho}[\mu \mathrm{D}^2 u + b \mathrm{D}\phi + \gamma \mathrm{D}^2 v] = f_2, \\ \varphi = f_3, \\ \frac{1}{J}[\delta \mathrm{D}^2 \phi - b \mathrm{D} u - \xi \phi - \tau \varphi] = f_4. \end{array}\right\}$$

Therefore, the second and fourth equations can be written in terms of f_1 and f_3 as follows:

$$\left.\begin{array}{l} \mu \mathrm{D}^2 u + b \mathrm{D}\phi = \rho f_2 - \gamma \mathrm{D}^2 f_1, \\ -b \mathrm{D} u + \delta \mathrm{D}^2 \phi - \xi \phi = J f_4 + \tau f_3. \end{array}\right\}$$

The unique solvability of this system is guaranteed using the usual elliptic arguments. It is also clear from the regularity theory of linear elliptic systems that

$$\|U\|_{\mathscr{H}} \leqslant K \|\mathscr{F}\|_{\mathscr{H}},$$

where K is a constant independent of U. □

Lemma 4.2. *Let $\mathscr{A}$ be the same matrix as in Lemma* 4.1. *Then condition (4.5) is satisfied.*

Proof. We split the proof in three steps.

(i) Since 0 is in the resolvent of $\mathscr{A}$, by the contraction mapping theorem, for any real λ such that $|\lambda| < \|\mathscr{A}^{-1}\|^{-1}$, the operator $\mathrm{i}\lambda \mathscr{I} - \mathscr{A} = \mathscr{A}(\mathrm{i}\lambda \mathscr{A}^{-1} - \mathscr{I})$ is invertible. Moreover, $\|(\mathrm{i}\lambda \mathscr{I} - \mathscr{A})^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathscr{A}^{-1}\|^{-1}, \|\mathscr{A}^{-1}\|^{-1})$.

(ii) If $\sup\{\|(i\lambda\mathscr{I}-\mathscr{A})^{-1}\|,|\lambda|<\|\mathscr{A}^{-1}\|^{-1}\}=M<\infty$, then, using the contraction theorem again, the operator

$$i\lambda\mathscr{I}-\mathscr{A}=(i\lambda_0\mathscr{I}-\mathscr{A})\Big(\mathscr{I}+i(\lambda-\lambda_0)(i\lambda_0\mathscr{I}-\mathscr{A})^{-1}\Big),$$

is invertible for $|\lambda-\lambda_0|<M^{-1}$. Hence, choosing λ_0 close enough to $\|\mathscr{A}^{-1}\|^{-1}$, the set $\{\lambda,|\lambda|<\|\mathscr{A}^{-1}\|^{-1}+M^{-1}\}$ is contained in the resolvent of $\mathscr{A}$ and $\|(i\lambda\mathscr{I}-\mathscr{A})^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathscr{A}^{-1}\|^{-1}-M^{-1},\|\mathscr{A}^{-1}\|^{-1}+M^{-1})$.

(iii) Suppose that the statement of this lemma is not true. Then, there exists a real number $\sigma\neq 0$ with $\|\mathscr{A}^{-1}\|^{-1}\leqslant|\sigma|<\infty$ satisfying that the set $\{i\lambda,|\lambda|<|\sigma|\}$ is in the resolvent of $\mathscr{A}$ and $\sup\{\|(i\lambda\mathscr{I}-\mathscr{A})^{-1}\|,|\lambda|<|\sigma|\}=\infty$. In this case, we can find a sequence of real numbers, λ_n, with $\lambda_n\to\sigma$, $|\lambda_n|<|\sigma|$, and a sequence of unit norm vectors in the domain of $\mathscr{A}$, $U_n=(u_n,v_n,\phi_n,\varphi_n)$, such that

$$\|(i\lambda_n\mathscr{I}-\mathscr{A})U_n\|\to 0.$$

Writing this condition term by term we get

$$i\lambda_n u_n-v_n\to 0\quad\text{in } H^1,\tag{4.7}$$

$$i\lambda_n v_n-\frac{\mu}{\rho}D^2u_n-\frac{\gamma}{\rho}D^2v_n-\frac{b}{\rho}D\phi_n\to 0\quad\text{in } L^2,\tag{4.8}$$

$$i\lambda_n\phi_n-\varphi_n\to 0\quad\text{in } H^1,\tag{4.9}$$

$$i\lambda_n\varphi_n+\frac{b}{J}Du_n-\frac{\delta}{J}D^2\phi_n+\frac{\xi}{J}\phi_n+\frac{\tau}{J}\varphi_n\to 0\quad\text{in } L^2.\tag{4.10}$$

Taking the inner product of $(i\lambda_n\mathscr{I}-\mathscr{A})U_n$ times U_n in $\mathscr{H}$ using (4.4) and selecting its real part we obtain $\|Dv_n\|^2\to 0$ and $\|\varphi_n\|^2\to 0$ in L^2. Thus, we have also $v_n\to 0$, and, from (4.7) and (4.9), $u_n\to 0$ and $\phi_n\to 0$.Therefore, conditions (4.8) and (4.10) become, respectively,

$$-\frac{\mu}{\rho}D^2u_n-\frac{\gamma}{\rho}D^2v_n-\frac{b}{\rho}D\phi_n\to 0\quad\text{in } L^2\tag{4.11}$$

and

$$\frac{b}{J}Du_n-\frac{\delta}{J}D^2\phi_n\to 0\quad\text{in } L^2.\tag{4.12}$$

Taking the inner product of (4.12) times ϕ_n we obtain $D\phi_n\to 0$. Taking then the product of (4.11) times u_n we get $Du_n\to 0$.This argument shows that U_n cannot be of unit norm, which finishes the proof of this lemma. □

Lemma 4.3. *Let $\mathscr{A}$ be the above defined matrix. Then condition (4.6) holds.*

Proof. Suppose that the statement of the lemma is not true. Then, there is a sequence λ_n with $|\lambda_n|\to\infty$ and a sequence of unit norm vectors in the domain of $\mathscr{A}$, $U_n=(u_n,v_n,\phi_n,\varphi_n)$, such that conditions (4.7)–(4.10) hold. Again Dv_n and φ_n tend to zero and then $u_n\to 0$ and $\phi_n\to 0$ (in fact, $\lambda_n u_n\to 0$ and $\lambda_n\phi_n\to 0$).

Taking the inner product of (4.8) times u_n we obtain

$$i\lambda_n\langle v_n,u_n\rangle-\frac{\mu}{\rho}\langle D^2u_n,u_n\rangle-\frac{\gamma}{\rho}\langle D^2v_n,u_n\rangle-\frac{b}{\rho}\langle D\phi_n,u_n\rangle\to 0\quad\text{in } L^2.$$

Integrating by parts and using (4.7), this expression becomes

$$-\|v_n\|^2 + \frac{\mu}{\rho}\|\mathrm{D}u_n\|^2 \to 0 \quad \text{in } L^2,$$

which implies $\mathrm{D}u_n \to 0$.

Finally, taking the inner product of (4.10) times ϕ_n and using (4.9) we get

$$-\|\varphi_n\|^2 + \frac{\delta}{J}\|\mathrm{D}\phi_n\|^2 \to 0 \quad \text{in } L^2.$$

Once again, this proves that U_n cannot be of unit norm. □

Theorem 4.1. *Let* (u, ϕ) *be a solution of the problem determined by* (4.1), (3.2) *and* (3.3). *Then* (u, ϕ) *decays exponentially.*

Proof. The proof is a direct consequence of Lemmas 4.2 and 4.3. □

5. Viscoelasticity, porosity and microtemperatures

We study now if the introduction of microtemperatures in the problem gives rise to exponential stability of the solutions. In this case the system we should analyze is

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + b\phi_x + \gamma\dot{u}_{xx}, \\ J\ddot{\phi} = \delta\phi_{xx} - bu_x - \xi\phi - dw_x, \\ \alpha\dot{w} = \kappa_4 w_{xx} - d\dot{\phi}_x - \kappa_2 w. \end{cases} \tag{5.1}$$

To have a well-posed problem we impose the boundary conditions

$$u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = w(0,t) = w(\pi,t) = 0, \tag{5.2}$$

and the initial conditions

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), \quad \phi(x,0) = \phi_0(x), \quad \dot{\phi}(x,0) = \varphi_0(x), \quad w(x,0) = w_0(x). \tag{5.3}$$

As usual, we denote $v = \dot{u}$ and $\varphi = \dot{\phi}$. For $U = (u, v, \phi, \varphi, w)$ we consider the Hilbert space

$$\mathscr{H} = \left\{ U \in H_0^1 \times L^2 \times H^1 \times L^2 \times L^2, \int_0^\pi \phi(x)\,\mathrm{d}x = \int_0^\pi \varphi(x)\,\mathrm{d}x = 0 \right\}.$$

We use again the semigroup method of Liu and Zheng, but now we omit part of the proofs.

System (5.1) can be rewritten as

$$\begin{cases} \dot{u} = v, \\ \dot{v} = \frac{1}{\rho}(\mu\mathrm{D}^2u + b\mathrm{D}\phi + \gamma\mathrm{D}^2v), \\ \dot{\phi} = \varphi, \\ \dot{\varphi} = \frac{1}{J}(\delta\mathrm{D}^2\phi - b\mathrm{D}u - \xi\phi - d\mathrm{D}w), \\ \dot{w} = \frac{1}{\alpha}(\kappa_4\mathrm{D}^2w - d\mathrm{D}\varphi - \kappa_2 w), \end{cases}$$

or, equivalently, $\dot{U} = \mathscr{B}U$, where

$$\mathscr{B} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \frac{\mu}{\rho}\mathrm{D}^2 & \frac{\gamma}{\rho}\mathrm{D}^2 & \frac{b}{\rho}\mathrm{D} & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ -\frac{b}{J}\mathrm{D} & 0 & \frac{\delta\mathrm{D}^2-\xi}{J} & 0 & -\frac{d}{J}\mathrm{D} \\ 0 & 0 & 0 & -\frac{d}{\alpha}\mathrm{D} & \frac{\kappa_4\mathrm{D}^2-\kappa_2}{\alpha} \end{pmatrix}.$$

Let $U^* = (u^*, v^*, \phi^*, \varphi^*, w^*)$. We define an inner product in $\mathscr{H}$:

$$\langle U, U^*\rangle = \int_0^\pi \left(\rho v\bar{v}^* + J\varphi\bar{\varphi}^* + \alpha w\bar{w}^* + \mu u_x\bar{u}_x^* + \delta\phi_x\bar{\phi}_x^{\,*} + \xi\phi\bar{\phi}^* + b\left(u_x\bar{\phi}^* + \bar{u}_x^*\phi\right)\right)\mathrm{d}x.$$

Direct computation gives

$$\mathrm{Re}\langle\mathscr{B}U, U\rangle = -\int_0^\pi \left(\gamma\|v_x\|^2 + \kappa_4\|w_x\|^2 + \kappa_2\|w\|^2\right)\mathrm{d}x \leqslant 0. \tag{5.4}$$

Lemma 5.1. *Let $\mathscr{B}$ be the above defined matrix. Then,* 0 *is in the resolvent of $\mathscr{B}$.*

Proof. Analogous to that of Lemma 4.1. □

Lemma 5.2. *Let $\mathscr{B}$ be the same matrix as in Lemma* 5.1. *Then condition* (4.5) *is satisfied.*

Proof. The proof is *mutatis mutandi* analogous to that of Lemma 4.2. We write only the proof of the third step. In this case, we can find a sequence of real numbers, λ_n, with $\lambda_n \to \sigma$, $|\lambda_n| < |\sigma|$, and a sequence of unit norm vectors in the domain of $\mathscr{B}$, $U_n = (u_n, v_n, \phi_n, \varphi_n, w_n)$, such that

$$\|(\mathrm{i}\lambda_n\mathscr{I} - \mathscr{B})U_n\| \to 0.$$

Writing this condition term by term we get

$$\mathrm{i}\lambda_n u_n - v_n \to 0 \quad \text{in } H^1, \tag{5.5}$$

$$\mathrm{i}\lambda_n v_n - \frac{\mu}{\rho}\mathrm{D}^2u_n - \frac{b}{\rho}\mathrm{D}\phi_n - \frac{\gamma}{\rho}\mathrm{D}^2v_n \to 0 \quad \text{in } L^2, \tag{5.6}$$

$$\mathrm{i}\lambda_n\phi_n - \varphi_n \to 0 \quad \text{in } H^1, \tag{5.7}$$

$$\mathrm{i}\lambda_n\varphi_n + \frac{b}{J}\mathrm{D}u_n - \frac{\delta}{J}\mathrm{D}^2\phi_n + \frac{\xi}{J}\phi_n + \frac{d}{J}\mathrm{D}w_n \to 0 \quad \text{in } L^2, \tag{5.8}$$

$$\mathrm{i}\lambda_n w_n - \frac{\kappa_4}{\alpha}\mathrm{D}^2w_n + \frac{d}{\alpha}\mathrm{D}\varphi_n + \frac{\kappa_2}{\alpha}w_n \to 0 \quad \text{in } L^2. \tag{5.9}$$

From (5.4), it is easy to see that $\mathrm{D}v_n$, $\mathrm{D}w_n$ and w_n tend to 0. This implies $u_n \to 0$ in H^1. Therefore, $\mathrm{D}u_n \to 0$.

After simplifying, condition (5.9) becomes

$$-\frac{\kappa_4}{\alpha}\mathrm{D}^2w_n + \frac{d}{\alpha}\mathrm{D}\varphi_n \to 0,$$

which, integrating with respect to x, gives $\varphi_n \to 0$ in L^2. Notice that here the boundary conditions play an important role.

Taking the inner product of (5.8) times ϕ_n and simplifying we get

$$-\|\varphi_n\|^2 + \frac{\delta}{J}\|D\phi_n\|^2 + \frac{\xi}{J}\|\phi_n\|^2 \to 0.$$

In consequence, $D\phi_n \to 0$. This shows that U_n cannot be of unit norm. □

Lemma 5.3. *Let $\mathscr{B}$ be the matrix defined before. Then condition (4.6) holds.*

Proof. We argue as in Lemma 4.3. Suppose now that $|\lambda_n| \to \infty$. Dividing (5.8) by λ_n it is clear that $\frac{D^2\phi_n}{\lambda_n}$ is bounded. It is also clear from (5.7) that $\frac{\varphi_n}{\lambda_n} \approx i\phi_n$. Thus, dividing (5.9) by λ_n and simplifying we obtain

$$-\frac{\kappa_4}{\alpha\lambda_n}D^2 w_n + \frac{d}{\alpha}iD\phi_n \to 0.$$

Multiplying this expression by $D\phi_n$ and integrating by parts it yields

$$-\frac{\kappa_4}{\alpha\lambda_n}\langle Dw_n, D^2\phi_n\rangle + \frac{d}{\alpha}i\|D\phi_n\|^2 \to 0.$$

And then, $\|D\phi_n\|^2 \to 0$.

Taking the inner product of (5.8) times ϕ_n and simplifying we get

$$-\|\varphi_n\|^2 + \frac{\delta}{J}\|D\phi_n\|^2 + \frac{\xi}{J}\|\phi_n\|^2 \to 0.$$

In consequence, $\varphi_n \to 0$. Again this contradicts the assumption that U_n has unit norm. □

Theorem 5.1. *Let (u,ϕ,w) be a solution of the problem determined by (5.1)–(5.3). Then (u,ϕ,w) decays exponentially.*

Proof. The proof is a direct consequence of Lemmas 5.2 and 5.3. □

6. Elasticity, viscoporosity and microtemperatures

Finally, we consider the presence of porous viscosity but not elastic dissipation, and we combine it with microtemperatures. The system we obtain is

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + b\phi_x, \\ J\ddot{\phi} = \delta\phi_{xx} - bu_x - \xi\phi - dw_x - \tau\dot{\phi}, \\ \alpha\dot{w} = \kappa_4 w_{xx} - d\dot{\phi}_x - \kappa_2 w. \end{cases} \tag{6.1}$$

In this case the boundary and initial conditions are given by

$$u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = w(0,t) = w(\pi,t) = 0, \tag{6.2}$$

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), \quad \phi(x,0) = \phi_0(x), \quad \dot{\phi}(x,0) = \varphi_0(x), \quad w(x,0) = w_0(x). \tag{6.3}$$

Theorem 6.1. *Let (u,ϕ,w) be a solution of the problem determined by (6.1)–(6.3) and assume that $J\mu \neq \delta\rho$. Then (u,ϕ,w) decays in a slow way.*

Proof. We proceed as in Section 3. We will prove the existence of a solution of system (6.1) of the form

$$u = K_1 e^{\omega t} \sin(nx), \qquad \phi = K_2 e^{\omega t} \cos(nx), \qquad w = K_3 e^{\omega t} \sin(nx).$$

Replacing them in system (6.1) and simplifying we obtain the following homogeneous linear system in the unknowns K_1, K_2 and K_3:

$$\begin{pmatrix} n^2\mu + \rho\omega^2 & bn & 0 \\ bn & n^2\delta + \tau\omega + J\omega^2 + \xi & dn \\ 0 & -dn\omega & \alpha\omega + \kappa_2 + \kappa_4 n^2 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We denote by $r(x)$ the determinant of the above matrix after replacing ω by x. Direct calculation shows that $r(x)$ is a fifth degree polynomial. We consider $r(x - \epsilon)$ and study the sign of its roots. Applying the Routh–Hurwitz theorem we see that the fourth leading minor of the corresponding matrix is a twelfth degree polynomial on n with negative main coefficient for ϵ small enough, provided that $J\mu \neq \delta\rho$. Let N_4 be this minor, then

$$N_4 = -2J\epsilon\kappa_4^3\rho^2(J\mu - \delta\rho)^2\big(d^2 + \kappa_4(\tau - 2J\epsilon)\big)n^{12} + T(n),$$

where $T(n)$ is a 10th degree polynomial on n. □

7. Conclusions

In this paper we have analyzed the porous-elastic problem. First of all, we have proved that elastic dissipation and porosity do not produce exponential decay in the solutions. Neither does the addition of temperature. Nevertheless, if the viscoelasticity and the porous dissipation are both present, then the solutions are exponentially stable.

As temperature is insufficient to cause exponential decay, we have also studied the effects of including microtemperature. In this case, when there is elastic dissipation, the solutions decay exponentially, but if only porous damping is taken into account then, generically, microtemperature is not strong enough to produce exponential stability.

It is worth noting that these results, added to previous ones (temperature combined with porous viscosity and temperature combined with microtemperature), analyze and solve the porous-elastic problem in many possible situations. We summarize the main conclusions with the help of a scheme:

thermal effect viscoelastic effect	→	elasticity ↕ porosity	←	microthermal effect viscoporous effect

If we take simultaneously one effect from the right square and other one from the left square, then we get exponential stability. However, if we consider two simultaneous effects from one square only, then we get slow decay. Of course, if more than two of these mechanisms are taken, we have exponential stability.

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