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Solution of The Differential Equation $\left(\frac{\partial^2}{\partial x \partial y} + ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cxy + \frac{\partial}{\partial t}\right) P(x, y, t) = 0$ And The Bogoliubov Transformation*,†

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An eigen function expansion for the solution of the Lambropolous partial differential equation is obtained by the use of a transformation similar to the Bugolubov transformation familiar in Bose gas theory. Also the technique of normal ordering of operators is employed. The orthogonality properties of such solutions are also analysed.

1. INTRODUCTION

Various closed-form solutions to Lambropoulos' [1] partial differential equation

$$\left(\frac{\partial^2}{\partial x\, \partial y} + ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cxy + \frac{\partial}{\partial t}\right) P(x, y, t) = 0 \tag{1}$$

subject to the initial condition

$$P(x, y, 0) = \Phi(x, y) \tag{2}$$

have recently been given by Neuringer [2], Goldstein [3], Kolsrud [4],

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Wilcox [5], and Multhei [6]. Although these forms have various nice features, for some purposes it is desirable to separate out the t dependence at the outset by setting

$$P(x, y, t) = \Psi(x, y)\, e^{-\lambda t} \tag{3}$$

and thus work with the eigenvalue equation

$$H\Psi = \lambda\Psi \tag{4}$$

with

$$H = \left(\frac{\partial^2}{\partial x\, \partial y} + ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cxy\right). \tag{4'}$$

This will be the case, for instance, if one wishes to solve a boundary-value problem rather than an initial-value problem.

In the present paper we obtain a solution $\psi(x, y)$ of the eigenvalue equation (4) and hence of Eq. (1) under the condition (3) by using a transformation of the operators x, $\partial/\partial x$, y, and $\partial/\partial y$. This transformation is similar to the well known Bogoliubov transformation which is well suited to diagonalize the Hamiltonian of the Bose gas below its λ transition. It is to be noted that the operator H is a linear combination of operators which form a closed Lie algebra under commutation. The transformation reduces H to a sum of two diagonal operators which commute with each other, and whose eigensolutions are readily found. This is carried out in Section 2, and the eigensolutions with respect to the original H are obtained by means of a normal-ordering technique such as we have used in Refs. [5] and [7]. In Section 3, the orthogonality properties of such solutions are analysed by employing Bargmann's approach (Ref. [8]).

2. Diagonalization of H

In this section we transform the operator H to $\bar{H}$ given by

$$\bar{H} = e^{S} H e^{-S} \tag{5}$$

where S is the operator defined by

$$S = -AXY + Bxy. \tag{6}$$

Here X and Y denote the operators $X \equiv \partial/\partial x$ and $Y \equiv \partial/\partial y$ satisfying the commutation relations

$$[X, x] = I = [Y, y], \tag{7}$$

where I is the identity operator while A and B are constants to be determined. To find $\bar{H}$, we first find the effect of the transformation on x, y, X, and Y separately. Using a familiar parameter differentiation technique (Ref. [7]), we find

$$e^S x e^{-S} = x \cosh \theta - Y(A/B)^{1/2} \sinh \theta \tag{8a}$$

$$e^S y e^{-S} = y \cosh \theta - X(A/B)^{1/2} \sinh \theta \tag{8b}$$

$$e^S X e^{-S} = X \cosh \theta - y(B/A)^{1/2} \sinh \theta \tag{8c}$$

$$e^S Y e^{-S} = Y \cosh \theta - x(B/A)^{1/2} \sinh \theta \tag{8d}$$

where θ is defined by

$$\theta \equiv (AB)^{1/2}. \tag{9}$$

Substituting Eq. (4′) into Eq. (5) and using Eqs. (8), we find

$$\begin{aligned}\bar{H} = {} & XY[\cosh^2 \theta + (a + b)(A/B)^{1/2} \sinh \theta \cosh \theta + (cA/B) \sinh^2 \theta] \\ & + xy[(B/A) \sinh^2 \theta - (a + b)(B/A)^{1/2} \sinh \theta \cosh \theta + c \cosh^2 \theta] \\ & + xX[a \cosh^2 \theta + b \sinh^2 \theta - ((B/A)^{1/2} + c(A/B)^{1/2}) \sinh \theta \cosh \theta] \\ & + yY[a \sinh^2 \theta + b \cosh^2 \theta - ((B/A)^{1/2} + c(A/B)^{1/2}) \sinh \theta \cosh \theta] \\ & + I[a \sinh^2 \theta + b \sinh^2 \theta - ((B/A)^{1/2} + c(A/B)^{1/2}) \sinh \theta \cosh \theta].\end{aligned} \tag{10}$$

To diagonalize the above expression, we set the coefficients of XY and xy equal to zero. After some manipulations we obtain the important results

$$B/A = c, \tag{11}$$

and

$$\cosh 2\theta = (a + b)/g \tag{12}$$

or

$$\sinh 2\theta = 2c^{1/2}/g, \tag{13}$$

where

$$g = [(a + b)^2 - 4c]^{1/2}. \tag{14}$$

(For convenience, throughout this paper we will assume that a, b, and c are positive numbers. Other cases can similarly be worked out.)

The diagonalized operator $\bar{H}$ is then given by

$$\bar{H} = \tfrac{1}{2}xX[g + a - b] + \tfrac{1}{2}yY[g + b - a] + \tfrac{1}{2}[g - a - b]. \tag{15}$$

Clearly $\bar{H}$ satisfies the eigenvalue equation

$$\bar{H}\Psi_{mn} = \lambda_{mn}\Psi_{mn} \tag{16}$$

where

$$\bar{\psi}_{mn} = x^m y^n \tag{17}$$

and

$$\lambda_{mn} = \tfrac{1}{2}g[m + n + 1] + \tfrac{1}{2}a(m - n - 1) + \tfrac{1}{2}b(n - m - 1). \tag{18}$$

The requirement that $\bar{\psi}_{mn}$ be an analytic function of x and y at $x = 0$ and $y = 0$ (which is the condition also that ψ_{mn} be analytic) restricts m and n to be non-negative integers.

If the condition

$$ab \geqslant c \tag{19}$$

is satisfied, the eigenvalues are bounded from below, and

$$\lambda_{00} = \tfrac{1}{2}[g - a - b] \tag{20}$$

is the lowest eigenvalue.

Once the $\bar{\psi}_{mn}$ are known, the eigenfunctions of the operator H itself can be found from

$$\psi_{mn} = e^{-S}\bar{\psi}_{mn}, \tag{21}$$

as follows from Eqs. (4), (5), and (6).

To obtain $\psi(x, y)$ we express e^{-rS} in a normal-ordered form

$$e^{-rS} = N \exp[-\beta XY - \mu xX - \nu yY - \omega xy - \delta], \tag{22}$$

where β, μ, ν, ω, and δ are functions of r, and the superoperator N orders the derivative operators X and Y to the right of the coordinate operators. The differential equations in r which β, μ, ν, ω, and δ satisfy can be determined as explained in Refs. [5] and [6]. The inclusion of operators xX, yY, and δ in Eq. (22) become necessary when one remembers the fact that XY, xX, yY, xy, and I constitute the members of a closed Lie algebra.

Hence one obtains

$$e^{-S} = \operatorname{sech}\theta \exp[-c^{1/2}\tanh\theta xy] \cdot N\exp[-(1 - \operatorname{sech}\theta)(xX + yY)] \cdot \exp[c^{-1/2}\tanh\theta XY], \tag{23}$$

where θ denotes $(AB)^{1/2}$ as in Eq. (9), and we have used Eq. (11) to replace B/A by c.

From Eqs. (17), (21), and (23), the eigenfunctions Ψ_{mn} are given by

$$\Psi_{mn} = \operatorname{sech}\theta \exp[-c^{1/2} \tanh \theta xy]\, Q_{mn}(x \operatorname{sech}\theta, y \operatorname{sech}\theta), \tag{24}$$

where

$$\begin{aligned} Q_{mn}(x, y) &= \exp[c^{-1/2} \tanh \theta XY]\, x^m y^n \\ &= \sum_{j=0}^{[m,n]} \frac{m!n![c^{-1/2} \tanh \theta]^j\, x^m y^n}{j!(m-j)!\,(n-j)!}. \end{aligned} \tag{25}$$

The quantity $Q_{mn}(x \operatorname{sech}\theta, y \operatorname{sech}\theta)$ is, of course, obtained from Eq. (25) by replacing x and y by $x \operatorname{sech}\theta$ and $y \operatorname{sech}\theta$, respectively.

The quantities $\operatorname{sech}\theta$ and $\tanh\theta$ may, if desired, be expressed in terms of a, b, and c by

$$\operatorname{sech}\theta = [(a + b + g)/2g]^{1/2} \tag{26}$$

and

$$\tanh\theta = [(a + b - g)/(a + b + g)]^{1/2} \tag{27}$$

as follows from Eqs. (12), (13), and (14).

That the eigensolutions defined by Eqs. (24), (25), (26), (27), (14), and (18) satisfy the eigenvalue equation (4) with H defined by Eq. (4′) can be verified by direct substitution. The solution $P(x, y, t)$ of Eq. (3) in now known for every eigenstate designated by λ_{mn}. Before taking the most general solution, which is a linear combination of the eigenfunctions, we have to analyze their orthogonality. This is done in the following section.

3. Orthogonality of the Ψ_{mn}

In Ref. [8], Bargmann has analyzed in detail the function space in which Fock's solutions are realized, and its analogy with the conventional Hilbert space of square-integrable functions. In this space in which the non-commutativity of z and $\partial/\partial z$ is built in, he defines complex variables z_1, z_2,... etc., and deduces the scalar product of two analytic functions $f(z)$ and $g(z)$ to be

$$(f, g) = \pi^{-1} \int f^*(z)\, g(z)\, e^{-zz^*}(d^2z), \tag{28}$$

where the integration is over the entire z plane. The following property follows:

$$(zf, g) = (f, \partial g/\partial z). \tag{29}$$

This means that

$$z^{+} = \partial/\partial z, \tag{30}$$

that is the operators z and $\partial/\partial z$ are adjoint.

For our present pursuit we define the scalar product (f, g) by

$$(f, g) = \pi^{-2} \int d^2x \int d^2y f^*(x, y)\, g(x, y)\, e^{-xx^* - yy^*}, \tag{31}$$

where both x and y are two-dimensional complex quantities. We define χ_{mn} dual to ψ_{mn} such that

$$(\chi_{m,n}, \psi_{mn}) = \delta_{mm'}\delta_{nn'} . \tag{32}$$

Just as ψ_{mn} is defined from Eqs. (21) and (17), we define χ_{mn} by

$$\chi_{mn} = e^{S^+}\bar{x}_{mn} , \tag{32'}$$

where $\bar{x}_{mn}$ is defined by

$$\bar{\chi}_{mn} = x^m y^n / m!n! \tag{33}$$

Here S^+ is the hermitian adjoint of S. With this choice of normalization, one can easily verify from (31) (by performing the integrations using polar coordinates) that

$$(\bar{\chi}_{m,n}, \bar{\psi}_{mn}) = \delta_{mm'}\delta_{nn'} . \tag{34}$$

The ortho-normality condition (32) then follows from Eqs. (34), (32), and (21) since

$$(e^{S^+}\bar{\chi}_{m'n'} , e^{-S}\bar{\psi}_{mn}) = (\bar{\chi}_{m'n'} , e^{S}e^{-S}\bar{\psi}_{mn}) = \delta_{mm'}\delta_{nn'} . \tag{35}$$

In order to explicitly determine the dual eigenfunctions x_{mn} by means of Eq. (32), we need the adjoint of S defined in Eq. (6)

$$S^+ = - A^*xy + B^*XY. \tag{36}$$

Note that if $A^* = B$, then $S^+ = - S$, so that

$$\chi_{m,n} = \psi_{mn}/(m!n!). \tag{37}$$

This will be the case if the original H-operator defined in Eq. (4') is hermitian, i.e., if $c = 1$, $a = a^*$, and $b = b^*$.

For the general case by comparing Eqs. (32) and (21), Eqs. (36) and (6), and Eqs. (33) and (17), we see that χ_{mn} may be obtained from Eq. (24) for ψ_{mn} by interchanging A and B^* and dividing through by $m!n!$. Thus

$$\chi_{mn} = \text{sech}\, \theta^* \exp[-(c^*)^{-1/2} \tanh \theta^* xy]\, R_{mn}(x\, \text{sech}\, \theta^*, y\, \text{sech}\, \theta^*) \qquad (38)$$

where

$$R_{mn}(x, y) = \sum_{j=0}^{[m,n]} \frac{[(c^*)^{1/2} \tanh \theta^*]^j\, x^{m-j} y^{n-j}}{j!(m-j)!\,(n-j)!}\,. \qquad (39)$$

The quantities sech θ^* and tanh θ^* are obtained by taking the complex conjugates of Eqs. (26) and (27), respectively.

Note that the $\chi_{m,n}$ are eigenfunctions of the adjoint eigenvalue equation

$$H^+ \chi_{m,n} = \lambda_{mn}^* \chi_{m,n}\,, \qquad (40)$$

where H^+ is the adjoint of the operator H defined in Eq. (4′),

$$H^+ = \left(c^* \frac{\partial^2}{\partial x\, \partial y} + a^* x \frac{\partial}{\partial x} + b^* y \frac{\partial}{\partial x} + xy\right), \qquad (41)$$

while λ_{mn}^* is the complex conjugate of the eigenvalue λ_{mn} defined by Eq. (18).

Since the $\bar{\psi}_{mn}$ defined by Eq. (17) constitute a complete set of orthogonal analytic functions in the variables x and y, the same is true of the ψ_{mn} also. From the orthonormality relation (32) it follows that any reasonable function $\Phi(x, y)$ can be expanded as a linear combination of these functions as follows:

$$\Phi(x, y) = \sum_{m,n=0}^{\infty} (\chi_{m,n}\,, \Phi)\, \psi_{mn}(x, y). \qquad (42)$$

Thus the solution to the original differential equation (1) subject to the initial condition (2) is given by

$$P(x, y, t) = \sum_{m,n=0}^{\infty} (\chi_{m,n}\,, \Phi)\, \psi_{mn}(x, y)\, e^{-\lambda_{mn} t}. \qquad (43)$$

4. Discussion

The novel method of solving the partial differential equation (1) using the Bogoliubov-type transformation (5) can readily be extended to solve partial differential equations containing any number of pairs of variables x_i, y_i, $i = 1, \ldots, N$, since the transformation reduces the operator H to its diagonal form. This procedure itself originated from the fact that the H operator is similar to the simplest form of the Bose-gas Hamiltonian in its condensed phase after replacing the zero-momentum operators by c-numbers. (See Ref. 9.) The similarity arises from the fact that the creation and annihilation operators of the Fock space can be mapped, respectively, onto the x and $\partial/\partial x$ operators of coordinate space. Setting the coefficients of XY and xy equal to zero is equivalent to the "cancellation of dangerous diagrams" in Bose-gas theory. The extension of the method to differential equations with $2n$ variables like

$$\left(\frac{\partial}{\partial t} + \sum_i \frac{\partial}{\partial x_i\, \partial y_i} + a_i x_i \frac{\partial}{\partial x_i} + b_i y_i \frac{\partial}{\partial y_i} + c_i x_i y_i\right) \cdot P(x_1, x_2 \cdots x_n; y_1, y_2 \cdots y_n, t) = 0 \tag{44}$$

is quite easily accomplished.

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