Global Existence for Coupled Reaction–Diffusion Systems

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Submitted by Chia Ven Pao

Received March 2, 1999

We prove here global existence in time of classical solutions for reaction–diffusion systems with strong coupling in the diffusion and with natural structure conditions on the nonlinear reactive terms. This extends some similar results in the case of a diagonal diffusion-operator associated with nonlinearities preserving the positivity and the total mass of the solutions or for which the total mass is a priori bounded. Here, however, no positivity assumption is made since nondiagonal systems do not preserve it.

Key Words: reaction–diffusion systems; global existence; semilinear parabolic systems.

INTRODUCTION

The goal of this paper is the study of global existence in time of solutions to some reaction–diffusion systems of the type

\[
\begin{align*}
  u_t - a \Delta u &= f(x, t, u, v) & \text{on } \Omega \times (0, \infty), \\
  v_t - c \Delta u - d \Delta v &= g(x, t, u, v) & \text{on } \Omega \times (0, \infty), \\
  u &= v = 0 & \text{on } \partial \Omega \times (0, \infty), \\
  u(x, 0) &= u_0(x); & v(x, 0) = v_0(x) & \text{on } \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded open set in \( \mathbb{R}^N \) with a regular boundary, \( a > 0, \) \( d > 0, \) \( c \in \mathbb{R} \), and \( f, g \) are regular functions satisfying structure assumptions that are described below. It is classical that (1) has local solutions in time. We look here at sufficient “natural” conditions on the nonlinearities \( f, g \) to have global solutions.
Let us first comment on the diagonal case $c = 0$. Then the system is only coupled by the nonlinearities $f$ and $g$. Much work has been done in the literature to understand global existence for this system when the two “natural” conditions hold on $f$ and $g$,

$$\forall u, v \geq 0, \quad f(0, v) \geq 0, \quad g(u, 0) \geq 0,$$

which implies that the nonnegativity of the solutions $u, v$ is preserved in time

$$f + g \leq 0,$$

which, with nonnegativity, implies that the total mass $\int u(t) + v(t)$ is uniformly bounded in time. We know that a uniform $L^\infty$-bound rather than this uniform $L^1$-bound would provide global existence in time. Here, conditions (A), (B) imply at least that the associated system of ordinary differential equations has global solutions, and so has the complete system (1) when the diffusion coefficients are moreover equal ($a = d, c = 0$), as one easily deduces from the maximum principle applied to the sum of the two equations in $u$ and $v$. But the situation turns out to be quite more complicated when the diffusion coefficients are not equal. Two of the main results for these systems state that

(C) If, moreover, one of the components, say $u$, is a priori known to be bounded, and if the nonlinearities are at most polynomial, then the other component $v$ is also uniformly bounded and global existence holds [5, 8, 17–19]. This is for instance the case when, besides (A) and (B), we also have

$$f \leq 0.$$

A uniform bound on $u$ then follows from the maximum principle applied to the first equation of (1). Next

(D) In general, under the only two conditions (A), (B), it might happen that blow up of the $L^\infty$-norm occurs in finite time (see [22]).

We now look at the coupled system (i.e., (1) with $c \neq 0$). Our goal is to understand how the above results extend to the nondiagonal situation, at least triangular. A first main difference has to be taken into account: indeed, it is well known that, as soon as the system is not diagonal ($c \neq 0$), then the positive cone is no longer invariant for the linear P.D.E. system (that is, (1) with $f = g = 0$), see [20]. Therefore, one cannot deal any more with nonnegative solutions and we may, in particular, drop the assumption (A) (see, however, the remarks in Section 3). On the other hand, we must work here with “bilateral” conditions on the data rather than “unilateral”
ones as those recalled above. For instance, the condition \((C_1)\) will be naturally replaced by
\[
(C'_1) \quad \text{sign}(u)f \leq 0,
\]
and the condition \((B)\) by
\[
(B') \quad \text{sign}(u)f + \text{sign}(v)g \leq 0.
\]
We mention in Section 4 some explicit examples satisfying these conditions. More comments will also be made on these assumptions. Note that \((C'_1)\) implies, at least formally, an easy a priori \(L^\infty\)-bound on \(u\). In the diagonal case, the condition \((B')\) would imply an \(L^1\)-bound on \(u, v\) without any nonnegativity assumption, but even the existence of \(L^1\)-bounds is not so obvious when \(c \neq 0\) because of crossed terms of the form \(\text{sign}(v) \Delta u\) coming from the equation in \(v\). This will be a consequence of our analysis to overcome this difficulty and to prove even an \(L^1\)-bound on \(v\) over all finite intervals \((0, T)\). The proof is actually based on extensive use of \(L^p\)-estimates on \(v\) and on some other auxiliary functions, for \(p\) finite but large, coming in particular from the \(L^p\)-regularity theory for parabolic operators. The main novelty is to treat the “bad” term \(\text{sign}(v) \Delta u\) appearing in the natural combinations of the two equations. This is essentially done in Step 3 of the proof where some precise analysis is needed to reach the \(L^p\)-estimate. This leads to \(L^\infty\)-estimates when the nonlinearities are assumed to be polynomial which we will do here as in most previous work dealing with the diagonal situation. Note that these techniques can be extended to exponential growth, but not farther.

Let us mention that some results have also been recently obtained for the nondiagonal case in [10, 11, 14]. It is there mainly assumed that \(g = -f\) and \(u\) is nonnegative. Extensions are made of the techniques in [5, 7] handling exponential growth of the nonlinearity in \(v\) for some range of the diffusion coefficients, or of the techniques in Martin and Pierre [17] for the “better case” \(a > d\) for which no growth assumption may be required on the nonlinearities. We refer to Section 3 for more comments on the particular structure \(g = -f\) and \(u \geq 0\). We indicate for this case a very direct and simple proof which is also valid for any diffusion coefficients.

Note that the choice of Dirichlet boundary conditions is not essential and everything would work similarly with Neumann boundary conditions.

We finally remark that the case of strongly coupled systems which are not triangular in the diffusion part is quite more difficult. As a consequence of the blow-up examples found in [22], we can indeed prove that there is in general blow-up of the \(L^\infty\)-norm of the solutions in finite time for such nontriangular systems even if one assumes \(f \leq 0, g \leq 0\). This is
explained in detail in [3]. This is why we first treat here the triangular case. It would be interesting to understand for instance the case of a small perturbation $\varepsilon \Delta u$ in the first equation of (1). The extension of our approach to this case does not seem straightforward. See [12] for partial results in this direction.

Note that strongly coupled systems often appear in applications; see for instance [4, 9, 12, 13, 15].

The paper is organized as follows: Section 1 deals with formulation of the result, Section 2 with proof of the theorem, Section 3 with some remarks about a different approach, and Section 4 with examples and comments.

We will use the following notations: $Q_T = \Omega \times (0, T)$, $\|u\|_{p,n} = \left(\int_0^T |u(x)|^p \, dx\right)^{1/p}$, $\|u\|_{p,q,T} = \left(\int_0^T \int_{\Omega} |u(x,t)|^q \, dx \, dt\right)^{1/p}$, $\|\cdot\|_n$ will denote the $L^n$-norm in $\Omega$, and $W^{2,1}_p(Q_T) = \{ \phi \in L^p(Q_T); \, \phi_n, \phi_{n,t}, \phi_{n,x} \in L^p(Q_T)\}$, equipped with its natural norm.

1. FORMULATION OF THE RESULT

Let us consider the problem (1) where we assume

\begin{align}
  a, d > 0, \quad c \in R \\
  u_0, v_0 \in L^r(\Omega) \tag{2}
\end{align}

\begin{align}
  f, g: \Omega \times [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ are measurable} \\
  \text{and locally Lipschitz continuous in } u, v, \text{ namely} \tag{3}
\end{align}

\begin{align}
  \text{a.e. } x, t, \forall 0 \leq |u|, |\hat{u}|, |v|, |\hat{v}| \leq r \\
  |f(x,t,u,v) - f(x,t,\hat{u},\hat{v})| + |g(x,t,u,v) - g(x,t,\hat{u},\hat{v})| \\
  \leq K(r)(|u - \hat{u}| + |v - \hat{v}|) \tag{4}
\end{align}

We now come to the main structure assumptions on the nonlinearities $f, g$

\begin{align}
  \text{a.e. } x, t, \forall u, v \in \mathbb{R}, \quad (\text{sign } u)f(x,t,u,v) \leq 0. \tag{5}
\end{align}

\begin{align}
  \exists \alpha > 0, \text{ such that } a.e. x, t, \forall u, v \in \mathbb{R} \\
  (\text{sign } u)f(x,t,u,v) + \alpha(\text{sign } v)g(x,t,u,v) \leq 0. \tag{6}
\end{align}

Here the function sign is defined as usual by

\[
  \text{sign } t = \begin{cases}
    -1 & \text{if } t < 0, \\
    0 & \text{if } t = 0, \\
    1 & \text{if } t > 0.
  \end{cases}
\]
We finally assume a polynomial growth condition on \( f, g \):

There exists \( L(r), M(r) > 0, \sigma \geq 1 \) such that \( \forall u, v \in \mathbb{R}, \)
\[
|u| \leq r \\
|f(x, t, u, v)| + |g(x, t, u, v)| \leq L(r)|v|^\sigma + M(r).
\]

(7)

**Remark 1.1.** For the main result of this paper, conditions (5) and (6) could be weakened, for instance, by replacing the right-hand sides by adequate linear combinations of \( u \) and \( v \) (see [3]).

The diffusions could also be replaced by more general ones. We also refer to [3] for more details and discussion.

It is standard (see [1, 2]) that the system (1) has a unique local bounded classical solution on the interval \((0, T)\). Moreover, the maximal time of existence \( T_{\text{max}} \) is characterized by
\[
(\forall t \in (0, T_{\text{max}}), \|u(t)\|_\infty + \|v(t)\|_\infty \leq C) \Rightarrow (T_{\text{max}} = +\infty).
\]
Consequently, to show the global existence of classical solutions, it suffices to prove that they remain bounded on \((0, T_{\text{max}})\) (besides [1, 2], see also, e.g., the books dealing with reaction–diffusion systems for this type of arguments [6, 21, 23]). By classical solution, we mean a solution \( u, v \) belonging to \( W^{1,p}_\sigma(\Omega) \cap C([0, T_{\text{max}}]; L^p(\Omega)), \forall p \in [1, +\infty), \forall \eta \in (0, T_{\text{max}}) \) and satisfying (1) in the usual sense (see [1, 2, 16]).

We now state the main result of this paper.

**Theorem 1.1.** Under the hypotheses (2)–(7), the system (1) has a unique global classical solution.

2. PROOF OF THE THEOREM

**Step 1.** We start with some estimates on \( u \). We introduce the following regularizing functions:
\[
h_\epsilon(t) = \sqrt{t^2 + \epsilon^2} - \epsilon.
\]

Note that \( h_\epsilon \geq 0, h_\epsilon \) is convex, \( 0 \leq h_\epsilon(t) \leq |t|, \) and \( \forall t \in \mathbb{R}, \)
\[
h_\epsilon(t) \to |t| \quad \text{when} \ \epsilon \to 0
\]
\[
h_\epsilon'(t) \to \text{sign}(t) \quad \text{when} \ \epsilon \to 0.
\]

Let \( u, v \) be the local solutions of the system (1). Then
\[
h_\epsilon(u)_t - a \Delta h_\epsilon(u) = h_\epsilon(u)f(x, t, u, v) - ah_\epsilon''(u)|\nabla u|^2.
\]
Since $h_\varepsilon$ is convex, regular
\[ h_\varepsilon(u)_t - a \Delta h_\varepsilon(u) \leq h_\varepsilon'(u)f(x, t, u, v). \]  \hspace{1cm} (8)

But,
\[ h_\varepsilon'(u)f(x, t, u, v) = \frac{u}{\sqrt{u^2 + \varepsilon^2}} f(x, t, u, v). \]

According to (5), it follows that
\[ h_\varepsilon(u)_t - a \Delta h_\varepsilon(u) \leq 0. \]

Since, moreover, $h_\varepsilon(u) = 0$ at the boundary, by the maximum principle, we have
\[ \|h_\varepsilon(u)(t)\|_{\infty} = \|h_\varepsilon(u)(0)\|_{\infty} \leq \|u_0\|_{\infty} \quad \text{for } 0 \leq t < T_{\max}. \]

Since $h_\varepsilon(u)$ converges uniformly to $|u|$ when $\varepsilon \to 0$
\[ \|u(t)\|_{\infty} \leq \|u_0\|_{\infty} \quad \text{for } 0 \leq t < T_{\max}. \]  \hspace{1cm} (9)

**Step 2.** We will now estimate $f(x, t, u, v)$ by duality. For this, we introduce $\theta \in C_0^\infty(Q_T)$, $\theta \geq 0$, $T = T_{\max}$, and $\phi$ a nonnegative solution of (see, e.g., [16])
\[ -\phi_t - d \Delta \phi = \theta \quad \text{on } Q_T \]
\[ \phi = 0 \quad \text{on } \partial\Omega \times (0, T) \]
\[ \phi(., T) = 0 \quad \text{on } \Omega. \]  \hspace{1cm} (10)

By $L^p$-regularity properties of the heat operator (see [16]), $\forall 1 < q < \infty$, there exists a constant $C$ independent of $\theta$ such that
\[ \|\phi\|_{H^2_\theta(Q_T)} + \|\phi(., 0)\|_{\partial\Omega} \leq C\|\theta\|_{\theta, Q_T}. \]  \hspace{1cm} (11)

Thanks to (8) and (5),
\[ \left| h_\varepsilon'(u)f(x, t, u, v) \right| \leq -h_\varepsilon(u)_t + a \Delta h_\varepsilon(u). \]  \hspace{1cm} (12)

Multiplying this by the solution $\phi$ of (10) and integrating by parts leads to
\[ \int_{Q_T} |h_\varepsilon'(u)f(x, t, u, v)| \phi \leq \int_{Q_T} h_\varepsilon(u)(\phi_t + a \Delta \phi) + \int_{\Omega} h_\varepsilon(u_0)\phi(., 0). \]

Now, by letting $\varepsilon \to 0$,
\[ \int_{Q_T} |f(x, t, u, v)| \phi \leq \int_{Q_T} |u|(\phi_t + a \Delta \phi) + \int_{\Omega} |u_0|\phi(., 0). \]
By using the regularity of $\phi$ in (11) with $q = p/(p - 1)$, $p \in (1, \infty)$,
\[
\int_{Q_T} |f(x, t, u, v)| \phi \leq (1 + a)\|u\|_{p, Q_T}\|\phi\|_{W^{1,p}(Q_T)} + \|u_0\|_{p, \Omega} \|\phi(., 0)\|_{q, \Omega}.
\]

Thanks to (9) and (11), we deduce
\[
\int_{Q_T} |f(x, t, u, v)| \phi \leq C\|u_0\|_{\infty, \Omega} \|\theta\|_{q, Q_T}. \tag{13}
\]

Next, in order to estimate $v$, we use the following lemma.

**Lemma 2.1.** Let $z$ be the solution of the equation $T = T_{\text{max}}$
\[
\begin{align*}
  z_t - d \Delta z &= c \Delta u & \text{on } Q_T \\
  z &= 0 & \text{on } \partial \Omega \times (0, T) \\
  z(x, 0) &= v_0(x) & \text{on } \Omega.
\end{align*} \tag{14}
\]

Then for all $p \in (1, \infty)$, there exists $C > 0$ such that
\[
\|z\|_{L^p(Q_T)} \leq C(\|u_0\|_{\infty, \Omega} + \|v_0\|_{\infty, \Omega}). \tag{15}
\]

**Proof of the lemma.** First note that the classical solution $z$ of (14) exists at least when the initial data $u_0$ are regular, since then $\Delta u$ is in $L^p_{\text{loc}}([0, T_{\text{max}}), L^p(\Omega))$. By regularizing $u_0$ if necessary, it is sufficient to prove (15) for regular $u_0$.

Let $\theta \in C^0_0(Q_T)$, $\theta \geq 0$, and let $\phi$ be the nonnegative solution of (10). Multiply Eq. (14) by $\phi$ to obtain
\[
-\int_{Q_T} \phi_t z - \int_{Q_T} \phi(., 0)v_0 - d\int_{Q_T} z \Delta \phi = c\int_{Q_T} u \Delta \phi.
\]

Using the definition of $\phi$, it follows that
\[
\int_{Q_T} \theta z = \int_{Q_T} \phi(., 0)v_0 + c\int_{Q_T} u \Delta \phi. \tag{16}
\]

Let $p \in (1, \infty)$ and $q = p/(p - 1)$; using (11), we deduce from (16)
\[
\left| \int_{Q_T} \theta z \right| \leq C(\|v_0\|_{p, \Omega} + \|u\|_{p, Q_T})\|\theta\|_{q, Q_T}.
\]

The inequality (15) follows by duality from this estimate together with (9).
Step 3. We now come back to the proof of the main theorem. By subtracting Eq. (14) from the equation in \( v \) in (1), it follows that
\[
(v-z)_t - d\Delta (v-z) = g(x,t,u,v) \quad \text{on } Q_T
\]
\[
(v-z) = 0 \quad \text{on } \partial \Omega \times (0,T)
\]
\[
(v-z)(x,0) = 0 \quad \text{on } \Omega.
\]
Thus,
\[
h_x(v-z)_t - d\Delta h_x(v-z) \leq h'_x(v-z)g(x,t,u,v), \quad (17)
\]
where
\[
h'_x(v-z) = \frac{v - z}{\sqrt{(v-z)^2 + \varepsilon^2}} = \frac{|v - z| \text{sign}(v-z)}{\sqrt{(v-z)^2 + \varepsilon^2}}.
\]
But
\[
\text{sign}(v-z) = \begin{cases} 
\text{sign } v & \text{if } |v| \geq |z| \\
-\text{sign } z & \text{if } |v| \leq |z|,
\end{cases}
\]
so that, thanks to (5), (6), (17), we have on the set where \(|v| \geq |z|\),
\[
h_x(v-z)_t - d\Delta h_x(v-z) \leq \frac{1}{\alpha} |f(x,t,u,v)|. \quad (18)
\]
Now, thanks to (7), (17), on the set where \(|v| \leq |z|\), we have
\[
h_x(v-z)_t - d\Delta h_x(v-z) \leq L|z|^\sigma + M. \quad (19)
\]
Adding up (18) and (19), we obtain
\[
h_x(v-z)_t - d\Delta h_x(v-z) \leq \frac{1}{\alpha} |f(x,t,u,v)| + L|z|^\sigma + M. \quad (20)
\]
Using again a duality argument as above, we multiply (20) by the \( \phi \) solution of (10) to obtain
\[
\int_{Q_T} h_x(v-z)_t \theta \leq \int_{Q_T} |f(x,t,u,v)| \phi + L \int_{Q_T} |z|^\sigma \phi + M \int_{Q_T} \phi.
\]
Now, by letting \( \varepsilon \to 0 \),
\[
\int_{Q_T} |v-z| \theta \leq \int_{Q_T} |f(x,t,u,v)| \phi + L \int_{Q_T} |z|^\sigma \phi + M \int_{Q_T} \phi.
\]
Again, by using (11) with \( q = p/(p - \sigma) \), \( p \in (\sigma, \infty) \), and successively the estimates (13) and (15)

\[
\int_{Q_T} |v - z| \theta \leq C \|u_0\|_{\infty, \Omega} \|\theta\|_{p/(p - \sigma), Q_T} \\
+ L \|z\|^\sigma_{p/\sigma, Q_T} \|\phi\|_{p/(p - \sigma), Q_T} + M' \|\phi\|_{p/(p - \sigma), Q_T} \\
\leq C (\|u_0\|_{\infty, \Omega} + \|v_0\|_{\infty, \Omega})^\sigma + \|u_0\|_{\infty, \Omega} + 1) \|\theta\|_{p/(p - \sigma), Q_T}.
\]

Consequently, by duality

\[
v - z \in L^{p/\sigma}(Q_T), \quad \forall p \in (\sigma, +\infty)
\]

and

\[
\|v - z\|_{p/\sigma, Q_T} \leq C (\|u_0\|_{\infty, \Omega} + \|v_0\|_{\infty, \Omega})^\sigma + \|u_0\|_{\infty, \Omega} + 1). \quad (21)
\]

By combining (15) and (21),

\[
\|v\|_{p/\sigma, Q_T} \leq C (\|u_0\|_{\infty, \Omega} + \|v_0\|_{\infty, \Omega})^\sigma + \|u_0\|_{\infty, \Omega} + \|v_0\|_{\infty, \Omega} + 1). \quad (22)
\]

Thanks to (22) and (7),

\[
f(x, t, u, v), g(x, t, u, v) \in L^{p/\sigma^2}(Q_T), \quad \forall p \in (\sigma^2, +\infty), \quad (23)
\]

with an estimate of the corresponding norm depending only on the data. Recall that \( T = T_{\text{max}} \) here.

**Step 4.** Next, we go back to Eq. (1). We know that for some positive small enough \( \delta, u(\delta), v(\delta) \) are regular (at least \( C^1 \)) with a regularity depending only on the data. Then, thanks to the \( L^p \)-regularity theory for the heat equation in \( u \) (see [16]), we deduce from the estimate of \( f \) in (23)

\[
c \Delta u \in L^{p/\sigma^2}(\delta, T_{\text{max}}) \times \Omega), \quad \forall \delta \in (0, T_{\text{max}}), \quad \forall p \in (\sigma^2, +\infty),
\]

with a bound depending only on the data. Now, it can be deduced from the equation in \( v \) in (1) that, for \( p > \sigma^2(N + 2)/2 \),

\[
v \in L^p(\delta, T) \times \Omega), \quad \forall \delta \in (0, T)
\]

and also \( v \in L^p(Q_{T_{\text{max}}}) \) (with an estimate depending only on the data), since by local existence, there exists \( \delta = \delta(\|u_0\|_{\infty, \Omega}, \|v_0\|_{\infty, \Omega}) < T \) such that \( v \in L^p(Q_\delta) \). This implies \( T_{\text{max}} = +\infty \).
3. SOME REMARKS ABOUT A DIFFERENT APPROACH

When \( a \neq d \), we can obtain a different proof of the same result as follows.

The matrix \( \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \) can then be diagonalized and the system (1) can be transformed into a diagonal system by a corresponding change of functions. We then check that the right-hand side \( (F, G) \) of the new system satisfies the same conditions (5) and (6) with \( a \) replaced by

\[
\hat{a} = a |a - d|/(|a - d| + \alpha |c|).
\]

Then, we are reduced to checking that the result of the main theorem is valid for a diagonal system. This is well known if an extra assumption ensuring the nonnegativity of \( u, v \) is made (for this one could for instance assume conditions of type (A) on \( f, g \) and that \( u_0, v_0 \) are so that the initial data of the diagonalized system are nonnegative; without sign conditions, we just have to prove that the bilaterality of the conditions (5), (6) is sufficient to handle solutions without sign. Details of this approach may be found in [3].

This approach cannot, however, treat the case \( a = d \), since then \( \hat{a} = 0 \). This is surprising: indeed, at least, when \( c = 0 \), global existence is immediate in (1) since \( u + v \) satisfies the heat equation. If \( c \neq 0 \) (and \( a = d \)), we have

\[
(|u| + |v|)_t - d \Delta (|u| + |v|) - c \operatorname{sign} v \Delta u \leq 0.
\]  

If \( v \geq 0 \), this implies that \( |u| + |v| \) is bounded in \( L^p(Q_T) \), for all \( p \in (1, +\infty) \), since \( u \in L^r(Q_T) \) (use again the \( L^p \)-duality technique). But, since \( v \) does not have a sign in general, this approach fails.

A particular case. We come back to general \( a, d > 0, c \in \mathbb{R} \). Assume we work with nonnegative \( u \) (this is the case if \( u_0 \geq 0, f(0, v) = 0, \forall v \in \mathbb{R} \)) and that (5) is satisfied so that \( u \) is uniformly bounded. Assume now that we precisely have

\[
\forall u \geq 0, \quad \forall v \in \mathbb{R}, \quad f + g = 0,
\]  

(see below for comparison with (6)). Then \( v \) can easily be estimated in \( L^p(Q_T) \), as follows: we write

\[
v_t - d \Delta v = -u_t + a \Delta u + c \Delta u.
\]  

Dual \( L^p \)-regularity theory then implies that the \( L^p \)-norm of \( v \) on \( Q_{T_{\max}} \) is bounded by the same norm of \( u \) which we know is bounded on \( L^r \). Global existence follows.
In general, we only have inequality in (26) so that only \( v^+ \) can be bounded in \( L^p \). As shown by examples below, the negative part of \( v \) needs also to be controlled.

4. SOME EXAMPLES AND COMMENTS

As an example to illustrate what has been previously presented, we can consider the system, where \( T \) is arbitrary positive and \( \eta \geq 0 \),

\[
\begin{align*}
  u_t - a \Delta u &= -u|v|^p & \text{on } Q_T \\
  v_t - c \Delta u - d \Delta v &= u(|v|^p - \eta uv|v|^p + q) & \text{on } Q_T \\
  u &= v = 0 & \text{on } \partial \Omega \times (0, T) \\
  u(x, 0) &= u_0(x) ; \quad v(x, 0) &= v_0(x) & \text{on } \Omega,
\end{align*}
\]

with \( p > 1, q > 0 \). It can easily be checked that the hypotheses of the theorem are satisfied. Thus the system admits a unique classical global solution.

More generally, it works for systems of the form

\[
\begin{align*}
  u_t - a \Delta u &= -uF(u, v) & \text{on } Q_T \\
  v_t - c \Delta u - d \Delta v &= u(F(u, v) - \eta uvG(u, v)) & \text{on } Q_T \\
  u &= v = 0 & \text{on } \partial \Omega \times (0, T) \\
  u(x, 0) &= u_0(x) ; \quad v(x, 0) &= v_0(x) & \text{on } \Omega,
\end{align*}
\]

where \( F, G \) are regular nonnegative with polynomial growth and \( \eta \geq 0 \). Note that if \( \eta = 0 \), then these systems satisfy (25). This condition (25) is stronger than (6) since

\[
(g = -f \quad \Rightarrow \quad (\text{sign } v)g = -(\text{sign } v)f \leq -f,
\]

where the last inequality comes from the fact that \(-f \geq 0\). As we saw at the end of the previous section, global existence is then rather easily proved from (26). However, the case \( \eta > 0 \) needs more work to get an estimate from below for \( v \).

REFERENCES