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Error estimates in Sobolev spaces for interpolating thin plate splines under tension

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Abstract

This paper discusses L_p-error estimates for interpolation by thin plate spline under tension of a function in the classical Sobolev space on an open bounded set with a Lipschitz-continuous boundary. A property of convergence is also given when the set of interpolating points becomes more and more dense.

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1. Introduction

The concept of curves spline under tension was introduced at first by Schweiket [13]. An empiric generalization of the concept of spline under tension for two variables have been given by Franke [7]. Theoretical properties of thin plate spline under tension in hilbertian space have been studied by Bouhamidi and Le Méhauté [2,3,1]. The idea behind the concept of tension is that the resulting interpolating splines are close to the interpolating pseudo-linear splines if the parameter of tension is large, and are closed to the interpolating pseudo-cubic splines if the parameter of tension is small [2,7]. Splines under tension are useful when the modelled phenomenon has regions with rapid change of gradients. In this case, interpolation by thin plate splines may present some overshoots due to the plate's stiffness. The stiffness can be suppressed by the first derivatives appearing in the semi-norm that leads to thin plate splines under tension [7,11]. Splines under tension were used successively in some experiences such as mathematical geology [14] and Geographical Information System [8].

The aim of this paper is to study the L_p-error estimates in the Sobolev space for interpolating thin plate splines under tension. Some results of error estimates and convergence of thin plate splines as (m, ℓ, s)-splines was given in a recent paper by López de Silanes [9]. Let us recall some results, properties and notations. Let m and d be two nonnegative integers and consider the space $X^m(\mathbb{R}^d)$ given by

$$X^m(\mathbb{R}^d) = \{u \in \mathscr{D}'(\mathbb{R}^d),\ \ D^\alpha u \in L^2(\mathbb{R}^d) \text{ for } |\alpha| = m, m+1\},$$

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where $\mathscr{D}'(\mathbb{R}^d)$ is the space of Schwartz distributions and $D^\alpha u = (\partial^{|\alpha|}/(\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}))u$. Let $\tau > 0$ be a positive parameter. In the space $X^m(\mathbb{R}^d)$ we consider the following semi-scalar product

$$(u|v)_{m,\tau,\mathbb{R}^d} = \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} \int_{\mathbb{R}^d} D^\alpha u(x) D^\alpha v(x)\,\mathrm{d}x + \tau^2 \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^d} D^\alpha u(x) D^\alpha v(x)\,\mathrm{d}x. \tag{1.1}$$

The semi-norm associated to (1.1) is denoted by $|u|_{m,\tau,\mathbb{R}^d} = \sqrt{(u|u)_{m,\tau,\mathbb{R}^d}}$. We assume that $m > (d/2)$ then the space $X^m(\mathbb{R}^d)$ with the semi-scalar product (1.1) is a semi-Hilbert space continuously embedded in the space $\mathscr{C}(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$ [2,3]. The null subspace of the semi-norm $|\cdot|_{m,\tau,\mathbb{R}^d}$ is the space Π_{m-1} of polynomials of d-variables of degree at most $m-1$ whose dimension is denoted by $d(m)$.

Let f be a continuous function on a nonempty subset Ω of $\mathbb{R}^d$ and let $\mathscr{A} = \{x_1, \ldots, x_N\}$ be a finite set of N distinct points of $\overline{\Omega} := closure(\Omega)$. We assume that $\mathscr{A}$ is Π_{m-1}-unisolvent set which means that any polynomial in Π_{m-1} which vanishes on $\mathscr{A}$ is identically zero.

Let Δ^m be the m-times iterated Laplacian operator and let δ be the Dirac measure at the origin. A fundamental solution of the differential operator $(-1)^{m+1}(\Delta^{m+1} - \tau^2\Delta^m)$ is a function $\Phi_{m,d}$ which generates a tempered distribution on $\mathbb{R}^d$ also denoted by $\Phi_{m,d}$ such that

$$(-1)^{m+1}(\Delta^{m+1}\Phi_{m,d} - \tau^2\Delta^m\Phi_{m,d}) = \delta.$$

Let $|x|$ denote the Euclidean norm in $\mathbb{R}^d$ and let K_0 be the classical Bessel function of the second kind. For $d = 1, 2, 3$, the function $\Phi_{m,d}$ has the following expression [2,3,1,4],

$$\Phi_{m,d}(x) = \begin{cases} \dfrac{(-1)^m}{2\tau^{2m+1}} \left(\mathrm{e}^{-\tau|x|} - \displaystyle\sum_{k=0}^{2m-1} \frac{(-\tau|x|)^k}{k!} \right) & \text{for } d = 1 \\ \dfrac{(-1)^m}{2\pi\tau^{2m}} \left(\displaystyle\sum_{k=0}^{m-1} \frac{\tau^{2k}}{4^k(k!)^2} |x|^{2k} \ln(|x|) + K_0(\tau|x|) \right) & \text{for } d = 2 \\ \dfrac{(-1)^m}{4\pi\tau^{2m}|x|} \left(\mathrm{e}^{-\tau|x|} - \displaystyle\sum_{k=0}^{2m-2} \frac{(-\tau|x|)^k}{k!} \right) & \text{for } d = 3. \end{cases} \tag{1.2}$$

Let $I_{\mathscr{A}}(f)$ denote the set $I_{\mathscr{A}}(f) := \{u \in X^m(\mathbb{R}^d) : u(a) = f(a), \forall a \in \mathscr{A}\}$. It has been proved in [2,3,1] that the variational problem

$$\operatorname*{Minimize}_{\substack{u \in X^m(\mathbb{R}^d) \\ u \in I_{\mathscr{A}}(f)}} |u|_{m,\tau,\mathbb{R}^d}, \tag{1.3}$$

has a unique solution $f^{\mathscr{A}}$ given by the following expression

$$f^{\mathscr{A}}(x) = \sum_{i=1}^{N} \lambda_i \Phi_{m,d}(x - x_i) + \sum_{j=1}^{d(m)} \alpha_j q_j(x), \tag{1.4}$$

where $(q_1, \ldots, q_{d(m)})$ denotes a basis of the space Π_{m-1}. Let z, α and λ be the vectors given by $z = (f(x_1), \ldots, f(x_N))^{\mathrm{T}}$, $\alpha = (\alpha_1, \ldots, \alpha_{d(m)})^{\mathrm{T}}$ and $\lambda = (\lambda_1, \ldots, \lambda_N)^{\mathrm{T}}$, respectively. Let K, M and O be the $N \times N$ matrix $K = (\Phi_{m,d}(x_i - x_j))_{1 \leqslant i,j \leqslant N}$, the $d(m) \times N$ matrix $M = (q_i(x_j))_{\substack{1 \leqslant i \leqslant d(m) \\ 1 \leqslant j \leqslant N}}$ and the $d(m) \times d(m)$ zero-matrix, respectively. The notation $()^{\mathrm{T}}$ denotes transposition. The coefficients λ_i and α_j are computed by solving the following nonsingular linear system

$$\begin{pmatrix} K & M^{\mathrm{T}} \\ M & O \end{pmatrix} \begin{pmatrix} \lambda \\ \alpha \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}. \tag{1.5}$$

The system (1.5) is obtained from the interpolating conditions

$$f^{\mathscr{A}}(x_i) = f(x_i), \quad i = 1, \dots, N,$$

together with the orthogonality conditions

$$\sum_{i=1}^{N} \lambda_i q_j(x_i) = 0, \quad j = 1, \dots, d(m).$$

In the remainder of this paper, we assume that $d \geqslant 2$ and we assume that Ω is an open bounded subset of $\mathbb{R}^d$. The fill-distance from $\mathscr{A}$ to Ω is defined by

$$h := h(\mathscr{A}, \Omega) := \sup_{x \in \overline{\Omega}} \inf_{a \in \mathscr{A}} |x - a|.$$

We will use the classical notation $W^{k,p}(\Omega)$ to denote the usual Sobolev space of all distributions f for which all of whose derivatives up to and including order k are in the classical Lebesgue space $L^p(\Omega)$. The classical norm in the Sobolev space is given by

$$\|f\|_{W^{k,p}(\Omega)} := \left[\sum_{|\alpha| \leqslant k} \|D^\alpha f\|^2_{L^p(\Omega)} \right]^{1/2} < \infty,$$

for $1 \leqslant p < +\infty$. Of course, the usual obvious modifications are to be made if $p = \infty$.

Let $|\cdot|_{m,\Omega}$, $|\cdot|_{m+1,\Omega}$ and $|\cdot|_{m,\tau,\Omega}$ denote the semi-norms defined on $W^{m+1,2}(\Omega)$ and associated to the following semi-scalar products

$$\begin{aligned}
(f|g)_{m,\Omega} &= \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega D^\alpha f(x) D^\alpha g(x)\,\mathrm{d}x, \\
(f|g)_{m+1,\Omega} &= \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} \int_\Omega D^\alpha f(x) D^\alpha g(x)\,\mathrm{d}x, \\
(f|g)_{m,\tau,\Omega} &= (f|g)_{m+1,\Omega} + \tau^2 (f|g)_{m,\Omega},
\end{aligned} \tag{1.6}$$

respectively. According to the Sobolev imbedding theorem, if Ω is an open bounded subset of $\mathbb{R}^d$ having a Lipschitz-continuous boundary, then the following continuous injection

$$W^{m,2}(\Omega) \hookrightarrow W^{k,p}(\Omega),$$

holds for a positive integer k with $0 \leqslant k \leqslant m - (d/2) + (d/p)$ and $p \in [2, \infty]$.

The interior cone property is defined as follows:

Definition 1.1. An open Ω of $\mathbb{R}^d$ is said to satisfy the interior cone property with a radius $r > 0$ and an angle $\theta \in (0, \pi/2)$ if for every $t \in \Omega$ there exists a unit vector $\xi(t) \in \mathbb{R}^d$ such that the cone

$$C(t, \xi(t), \theta, r) = \{t + \lambda\eta;\ \eta \in \mathbb{R}^d,\ |\eta| = 1,\ \eta^{\mathrm{T}} \cdot \xi(t) \geqslant \cos\theta,\ 0 \leqslant \lambda \leqslant r\}$$

is entirely contained in Ω.

The following proposition has been proved by Duchon [6].

Proposition 1.1 (*Duchon [6]*). *Let Ω be an open set of $\mathbb{R}^d$ satisfying the cone property with a radius r and an angle θ, then there exists constants $M \geqslant 1$ and M_1 (depending on d and θ) and ε_0 (depending on θ and r) such that for any ε such that $0 < \varepsilon \leqslant \varepsilon_0$ there exists $T_\varepsilon \subset \Omega$ satisfying*

(i) $B(t, \varepsilon) \subset \Omega$ *for all* $t \in T_\varepsilon$,

(ii) $\Omega \subset \bigcup_{t\in T_\varepsilon} B(t, M\varepsilon)$,

(iii) $\sum_{t\in T_\varepsilon} 1_{B(t,M\varepsilon)} \leqslant M_1$.

Here $1_{B(t,M\varepsilon)}$ is the function which has value one on the ball $B(t, M\varepsilon)$ of center t and radius $M\varepsilon$ and zero elsewhere.

Let us remark that Condition (iii) means that any point in Ω belongs to at most M_1 balls $B(t, M\varepsilon)$ with a center $t \in T_\varepsilon$. We recall that an open bounded connected subset Ω of $\mathbb{R}^d$ with a Lipschitz-continuous boundary satisfies the cone property with a radius r and an angle θ.

2. Local error estimates

In this section we provide some results which deal with the local estimates for a finite covering of small balls. We assume that Ω is an open bounded connected subset of $\mathbb{R}^d$ having a Lipschitz boundary (in the sense of Necǎs [12]). The space of restrictions to Ω of functions belonging to $X^m(\mathbb{R}^d)$ (respectively to $W^{m+1,2}(\mathbb{R}^d)$) is denoted by $X^m_\Omega(\mathbb{R}^d)$ (respectively by $W^{m+1,2}_\Omega(\mathbb{R}^d)$). Let R_Ω denote the operator of restriction from $\mathbb{R}^d$ to Ω, we have $X^m_\Omega(\mathbb{R}^d) = R_\Omega[X^m(\mathbb{R}^d)]$ and $W^{m+1,2}_\Omega(\mathbb{R}^d) = R_\Omega[W^{m+1,2}(\mathbb{R}^d)]$. Let $\dot{W}^{m+1,2}(\Omega) := W^{m+1,2}(\Omega)/\Pi_{m-1}$ be the quotient space of $W^{m+1,2}(\Omega)$ by Π_{m-1}. In the space $\dot{W}^{m+1,2}(\Omega)$ we consider the norm defined by

$$\|\dot{f}\|_1 = |f|_{m,\tau,\Omega}, \quad \forall \dot{f} \in \dot{W}^{m+1,2}(\Omega).$$

Since Ω is assumed to be an open bounded connected subset of $\mathbb{R}^d$ having a Lipschitz boundary then the norm $\|\cdot\|_1$ defined in the quotient space $\dot{W}^{m+1,2}(\Omega)$ is equivalent to the usual quotient norm defined on $\dot{W}^{m+1,2}(\Omega)$ by

$$\|\dot{f}\|_q = \inf_{u\in \dot{f}} \|u\|_{W^{m+1,2}(\Omega)}, \quad \forall \dot{f} \in \dot{W}^{m+1,2}(\Omega).$$

See [12, p. 19] for the definition of the quotient norm in $\dot{W}^{m+1,2}(\Omega)$. We recall that $u \in \dot{f}$ means that $u - f$ is a polynomial of d-variables of degree at most $m - 1$. The following proposition is given in [9].

Proposition 2.1. *Let Ω be any open bounded nonempty subset of $\mathbb{R}^d$, then $X^m_\Omega(\mathbb{R}^d) = W^{m+1,2}_\Omega(\mathbb{R}^d)$. Furthermore, if Ω is with a Lipschitz-continuous boundary then the operator R_Ω of restriction to Ω is linear and continuous from $X^m(\mathbb{R}^d)$ onto $W^{m+1,2}(\Omega)$.*

Proof. For the first part, we consider $D^{-k}L^2(\mathbb{R}^d) = \{u \in \mathscr{D}'(\mathbb{R}^d),\ u \in L^2(\mathbb{R}^d) \text{ for } |\alpha| = k\}$ and let $D^{-k}L^2_\Omega(\mathbb{R}^d)$ denote the space of restriction to Ω of the functions belonging to $D^{-k}L^2(\mathbb{R}^d)$. According to [5], $D^{-k}L^2_\Omega(\mathbb{R}^d) = W^{k,2}_\Omega(\mathbb{R}^d)$. Since $X^m_\Omega(\mathbb{R}^d) = D^{-(m+1)}L^2_\Omega(\mathbb{R}^d) \cap D^{-m}L^2_\Omega(\mathbb{R}^d)$, then the required result follows. For the second part, the proof is similar to that of [10, Theorem 1.2] (see also [5]). □

In order to prove the following lemma, it is helpful to introduce the following notations. Let $c(m) := \#\{\alpha \in \mathbb{N}^d, |\alpha| = m, m+1\}$ be the number of multi-indices α such that $|\alpha| = m, m+1$ and consider the space product $Y^m(\mathbb{R}^d) = (L^2(\mathbb{R}^d))^{c(m)}$. A generic element f of $Y^m(\mathbb{R}^d)$ is written as $f = (f_\alpha)_{|\alpha|=m,m+1}$ where each component of $f_\alpha \in L^2(\mathbb{R}^d)$ is indexed by one and only one of all the multi-indices α such that $|\alpha| = m, m+1$. The space $Y^m(\mathbb{R}^d)$ is endowed with the scalar product

$$(f|g)_{Y^m} = \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} \int_{\mathbb{R}^d} f_\alpha(x) g_\alpha(x)\,\mathrm{d}x + \tau^2 \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^d} f_\alpha(x) g_\alpha(x)\,\mathrm{d}x, \tag{2.1}$$

where $f = (f_\alpha)_{|\alpha|=m,m+1}$ and $g = (g_\alpha)_{|\alpha|=m,m+1}$. The norm associated to the scalar product (2.1) is denoted by $\|f\|_{Y^m}$. We consider the linear operator $T : (X^m(\mathbb{R}^d), |\cdot|_{m,\tau,\mathbb{R}^d}) \longrightarrow (Y^m(\mathbb{R}^d), \|\cdot\|_{Y^m})$ defined by

$$Tu = (D^\alpha u)_{|\alpha|=m,m+1}, \quad \forall u \in X^m(\mathbb{R}^d).$$

Since Ω is with a Lipschitz boundary, it follows that Ω satisfies the m-extension property (see [12]). Thus, there exists a continuous linear application E_m from $W^{m+1,2}(\Omega)$ into $X^m(\mathbb{R}^d)$ such that $R_\Omega(E_m u) = u$ for all $u \in W^{m+1,2}(\Omega)$. The mapping E_m is called an extension from $W^{m+1,2}(\Omega)$ into $X^m(\mathbb{R}^d)$. Since $\|Tu\|_{Y^m} = |u|_{m,\tau,\mathbb{R}^d}$, it follows that the operator $T : (X^m(\mathbb{R}^d), |\cdot|_{m,\tau,\mathbb{R}^d}) \longrightarrow (Y^m(\mathbb{R}^d), \|\cdot\|_{Y^m})$ is continuous. From Proposition 2.1, the operator $R_\Omega : (X^m(\mathbb{R}^d), |\cdot|_{m,\tau,\mathbb{R}^d}) \longrightarrow (W^{m+1,2}(\Omega), |\cdot|_{m,\tau,\Omega})$ is continuous. Moreover, the operators T and R_Ω satisfy the following properties,

(1) $R_\Omega(X^m(\mathbb{R}^d)) = W^{m+1,2}(\Omega)$.
(2) $\ker(R_\Omega) \cap \ker(T) = \ker(R_\Omega) \cap \Pi_{m-1} = \{0\}$,
(3) $\ker(R_\Omega) + \ker(T)$ is a closed subspace of $X^m(\mathbb{R}^d)$, because of the finite dimension of the subspace $\ker(T) = \Pi_{m-1}$.

For a function $u \in X^m(\mathbb{R}^d)$ we use the classical notation $u_{|\Omega}$ to denote the restriction of u to Ω, namely $u_{|\Omega} = R_\Omega(u)$. The following lemma is similar to Lemma 3.1 in [6].

Lemma 2.1. *Let Ω an open bounded connected subset of $\mathbb{R}^d$, having the Lipschitz boundary and containing a Π_{m-1}-unisolvent subset. For any $f \in W^{m+1,2}(\Omega)$, the variational problem*

$$\underset{\substack{u \in X^m(\mathbb{R}^d) \\ u_{|\Omega} = f}}{\mathit{Minimize}} \ |u|_{m,\tau,\mathbb{R}^d}, \tag{2.2}$$

has a unique solution f^Ω, and we have

$$|f^\Omega|^2_{m,\tau,\mathbb{R}^d} = |f^{\mathscr{A}}|^2_{m,\tau,\mathbb{R}^d} + |f^\Omega - f^{\mathscr{A}}|^2_{m,\tau,\mathbb{R}^d},$$

where $f^{\mathscr{A}}$ is the solution the variational Problem (1.3). *Moreover, there exists a constant K* (*depending on m and Ω*) *such that*

$$|f^\Omega|_{m,\tau,\mathbb{R}^d} \leqslant K |f|_{m,\tau,\Omega}, \quad \forall f \in W^{m+1,2}(\Omega).$$

Proof. The existence and uniqueness, of the solution of Problem (2.2), result from the properties of the operators T and R_Ω and from the general splines theory. Furthermore, the solution $f^{\mathscr{A}}$ of Problem (1.3) satisfies $(f^{\mathscr{A}}|u)_{m,\tau,\mathbb{R}^d} = 0$ for all $u \in X^m(\mathbb{R}^d)$ vanishing on $\mathscr{A}$. Since the function $u = f^\Omega - f^{\mathscr{A}}$ vanishes on $\mathscr{A}$, we get

$$\begin{aligned} |f^\Omega|^2_{m,\tau,\mathbb{R}^d} &= (f^\Omega - f^{\mathscr{A}} + f^{\mathscr{A}} | f^\Omega - f^{\mathscr{A}} + f^{\mathscr{A}})_{m,\tau,\mathbb{R}^d} \\ &= (f^\Omega - f^{\mathscr{A}} | f^\Omega - f^{\mathscr{A}})_{m,\tau,\mathbb{R}^d} + (f^{\mathscr{A}} | f^{\mathscr{A}})_{m,\tau,\mathbb{R}^d} \\ &= |f^\Omega - f^{\mathscr{A}}|^2_{m,\tau,\mathbb{R}^d} + |f^{\mathscr{A}}|^2_{m,\tau,\mathbb{R}^d}. \end{aligned}$$

In the quotient space $\dot{W}^{m+1,2}(\Omega) := W^{m+1,2}(\Omega)/\Pi_{m-1}$ we consider the norm defined by $\|\dot{f}\|_2 = |f^\Omega|_{m,\tau,\mathbb{R}^d}$ for $\dot{f} \in \dot{W}^{m+1,2}(\Omega)$. The space $\dot{W}^{m+1,2}(\Omega)$ together with the norms $\|\cdot\|_1$ or $\|\cdot\|_2$ is a Hilbert space. Moreover, for all $f \in W^{m+1,2}(\Omega)$, we have $f^\Omega_{|\Omega} = f$. Since

$$\|\dot{f}\|_1 = |f|_{m,\tau,\Omega} \leqslant |f^\Omega|_{m,\tau,\mathbb{R}^d} = \|\dot{f}\|_2, \quad \forall f \in W^{m+1,2}(\Omega),$$

it follows that the injection $j : (\dot{W}^{m+1,2}(\Omega), \|\cdot\|_2) \longrightarrow (\dot{W}^{m+1,2}(\Omega), \|\cdot\|_1)$ is continuous. Using the open mapping theorem, we obtain that j^{-1} is also continuous. Then, there exists $K > 0$ depending on m and Ω such that

$$\|\dot{f}\|_2 = \|j^{-1}(\dot{f})\|_2 = |f^\Omega|_{m,\tau,\mathbb{R}^d} \leqslant K \|\dot{f}\|_1 = K |f|_{m,\tau,\Omega},$$

holds for all $f \in W^{m+1,2}(\Omega)$. $\square$

Let α be a multi-index and $p \in [2, \infty]$ such that $|\alpha| \leqslant m - (d/2) + (d/p)$. Since Ω is an open bounded and connected subset of $\mathbb{R}^d$ having a Lipschitz-continuous boundary, it follows that the inclusions $W^{m+1,2}(\Omega) \subset W^{m,2}(\Omega) \subset W^{|\alpha|,p}(\Omega)$ are with continuous injection. Now the following lemma about the Lagrange polynomial interpolation is from [6] with some slight modifications.

Lemma 2.2. *Let Ω an open bounded and connected subset of $\mathbb{R}^d$ having a Lipschitz-continuous boundary. Let $\mathscr{B} \subset (\mathbb{R}^d)^{d(m)}$ be a compact subset of $d(m)$-tuples $b = (b_1, \dots, b_{d(m)})$ which are Π_{m-1}-unisolvent, let L_b denote the Lagrange Π_{m-1}-interpolation operator defined for $u \in W^{m+1,2}(\Omega)$ $(\subset \mathscr{C}(\overline{\Omega}))$ by*

$$\begin{cases} L_b u \in \Pi_{m-1} \\ L_b u(b^i) = u(b^i) \quad \text{for } i = 1, \dots, d(m). \end{cases}$$

Let α be a multi-index and $p \in [2, \infty]$ such that $|\alpha| \leqslant m - (d/2) + (d/p)$. Then, there exists a constant $C > 0$ (depending on Ω, $\mathscr{B}$, α and p) such that

$$\|D^\alpha(u - L_b u)\|_{L^p(\Omega)} \leqslant C|u|_{m,\Omega},$$

holds, for all $b \in \mathscr{B}$ and all $u \in W^{m+1,2}(\Omega)$.

The following theorem gives a local error estimates.

Theorem 2.1. *For any $M \geqslant 1$, for any $p \in [2, \infty]$ and for any multi-index α such that $|\alpha| \leqslant m - (d/2) + (d/p)$, there exists $R > 0$ (depending on d and m) and there exists $C > 0$ (depending on M, R, d, m α, p and τ) such that for all $h > 0$ and for all $t \in \mathbb{R}^d$ the ball $B(t, Rh)$ of center t and radius Rh contains $d(m)$ balls $B_1, \dots, B_{d(m)}$ of radius h such that, the following inequality*

$$\|D^\alpha u||_{L^p(B(t,MRh))} \leqslant C h^{m-|\alpha|-(d/2)+(d/p)}\, |u|_{m,\tau,B(t,MRh)},$$

holds, for all $u \in W^{m+1,2}(B(t, MRh))$ which vanishes at least on one point of each balls $B_1, \dots, B_{d(m)}$.

Proof. Let $\{b_1^0, \dots, b_{d(m)}^0\}$ be an arbitrary Π_{m-1}-unisolvent set in $\mathbb{R}^d$. Given that the set of all the $d(m)$-tuples of Π_{m-1}-unisolvent points in $\mathbb{R}^d$ is an open subset of $(\mathbb{R}^d)^{d(m)}$ (its complement is the set of the solutions of a system of algebraic equations), it follows that there exists $\delta > 0$ such that if $|b_i - b_i^0| \leqslant \delta$ for $i = 1, \dots, d(m)$ then the set $\{b_1, \dots, b_{d(m)}\}$ is also Π_{m-1}-unisolvent. Dilatation by the factor $1/\delta$ generates a new set of points $a_i = (1/\delta)b_i^0$ for $i = 1, \dots, d(m)$ such that the cartesian product $\mathscr{B} = \overline{B(a_1, 1)} \times \cdots \times \overline{B(a_{d(m)}, 1)}$ is a compact subset of $(\mathbb{R}^d)^{d(m)}$ of $d(m)$-tuples of Π_{m-1}-unisolvent points. The set $\bigcup_{i=1}^{d(m)} \overline{B(a^i, 1)}$ is a bounded subset of $\mathbb{R}^d$, then there exists a radius $R > 0$ (depending on d and m) and there exists a point $a \in \mathbb{R}^d$ such that $\bigcup_{i=1}^{d(m)} \overline{B(a^i, 1)} \subset B(a, R)$. Let $M \geqslant 1$, using Lemma 2.2 for the open ball $B(a, MR)$, there exists a constant $C > 0$ (depending on M, R, α, m, d and p) such that

$$\|D^\alpha(v - L_b v)\|_{L^p(B(a,MR))} \leqslant C|v|_{m,B(a,MR)}, \tag{2.3}$$

for all $v \in W^{m+1,2}(B(a, MR))$ and for all $b = (b_1, \dots, b_{d(m)}) \in \mathscr{B}$. Now, let $h > 0$ and $t \in \mathbb{R}^d$ and consider the transformation $\sigma_t : x \mapsto t + h(x - a)$ which transforms the ball $B(a, MR)$ into the ball $B(t, MRh)$. The transformation σ_t is affine and bijective with the Jacobian matrix is hI_d and $det(hI_d) = h^d$, where I_d denotes the square d-unit matrix. For all $u \in W^{m+1,2}(B(t, MRh))$, we have

$$D^\alpha u(y) = h^{-|\alpha|}[D^\alpha(u \circ \sigma_t)](\sigma_t^{-1}(y)), \tag{2.4}$$

and $v = u \circ \sigma_t \in W^{m+1,2}(B(a, MR))$.

For $2 \leqslant p < \infty$, by using the change of variables $y = \sigma_t(x)$, we obtain

$$\begin{aligned} \|D^\alpha u\|^p_{L^p(B(t,MRh))} &= \int_{B(t,MRh)} |D^\alpha u(y)|^p \,\mathrm{d}y = \int_{\sigma_t(B(a,MR))} |D^\alpha u(y)|^p \,\mathrm{d}y \\ &= h^d \int_{B(a,MR)} h^{-|\alpha|p} |D^\alpha(u \circ \sigma_t)(x)|^p \,\mathrm{d}x \end{aligned}$$

which gives,

$$\|D^\alpha u\|_{L^p(B(t,MRh))} = h^{(d/p)-|\alpha|}\|D^\alpha(u\circ\sigma_t)\|_{L^p(B(a,MR))}. \tag{2.5}$$

For $p=\infty$, we have obviously from (2.4),

$$\|D^\alpha u\|_{L^\infty(B(t,MRh))} = h^{-|\alpha|}\|D^\alpha(u\circ\sigma_t)\|_{L^\infty(B(a,MR))},$$

which is the inequality (2.5) by setting $(d/p)=0$ for $p=\infty$.

In particular for $p=2$ and $|\beta|=m$, we obtain

$$\|D^\beta u\|_{L^2(B(t,MRh))} = h^{(d/2)-m}\|D^\beta(u\circ\sigma_t)\|_{L^2(B(a,MR))},$$

which gives

$$|u\circ\sigma_t|^2_{m,B(a,MR)} = h^{2m-d}|u|^2_{m,B(t,MRh)}. \tag{2.6}$$

The transformation σ_t transforms the ball $B(a,R)$ into the ball $B(t,Rh)$ and the ball $B(a_i,1)$ into the ball $B_i := \sigma_t(B(a_i,1)) = B(t+h(a_i-a),h)$, for $i=1,\dots,d(m)$. Since $B(a_i,1)\subset B(a,R)$ it follows that $B_i\subset\sigma_t(B(a,R)) = B(t,Rh)$, for $i=1,\dots,d(m)$.

Let u be any element of $W^{m+1,2}(B(t,MRh))$ which vanishes at least on one point c_i of each balls B_i for $i=1,\dots,d(m)$. Let $v=u\circ\sigma_t$ and $b_i=\sigma_t^{-1}(c_i)\in B(a_i,1)\subset B(a,MR)$, for $i=1,\dots,d(m)$. Since $u\in W^{m+1,2}(B(t,MRh))$, then v belongs to $W^{m+1,2}(B(a,MR))$ and $v(b_i)=(u\circ\sigma_t)(\sigma_t^{-1}(c_i))=u(c_i)=0$, for $i=1,\dots,d(m)$. Since, the $d(m)$-tuple $b=(b_1,\dots,b_{d(m)})$ belongs to $\mathscr{B}$, then by using (2.3), we obtain

$$\|D^\alpha v\|_{L^p(B(a,MR))}\leqslant C|v|_{m,B(a,MR)}. \tag{2.7}$$

Finally, from (2.5) to (2.7), we get as required the existence of a constant $C>0$ (depending on M, R, α, m, d, p and τ) such that

$$\|D^\alpha u\|_{L^p(B(t,MRh))}\leqslant Ch^{m-|\alpha|-(d/2)+(d/p)}\,|u|_{m,\tau,B(t,MRh)}. \qquad \square$$

3. Global error estimates and convergence

In this section we give some results about the global error estimates and convergence.

Theorem 3.1. *Let Ω be an open bounded connected subset of $\mathbb{R}^d$ having a Lipschitz-continuous boundary. Let m be an integer such that $m>d/2$, then there exists $h_0>0$ (depending on Ω, m and d) such that for any multi index α and for any $p\in[2,\infty]$ with $|\alpha|\leqslant m-(d/2)+(d/p)$, there exists a constant C (depending on Ω, d, m, α, p and τ) such that for every function f belonging to $W^{m+1,2}(\Omega)$ and for every finite Π_{m-1}-unisolvent subset $\mathscr{A}$ of $\overline{\Omega}$ satisfying $h=\sup_{t\in\overline{\Omega}}\inf_{a\in\mathscr{A}}|t-a|\leqslant h_0$, the following inequality holds*

$$\|D^\alpha(f-f^{\mathscr{A}})\|_{L^p(\Omega)}\leqslant Ch^{m-|\alpha|-(d/2)+(d/p)}\,|f^\Omega-f^{\mathscr{A}}|_{m,\tau,\mathbb{R}^d}.$$

Proof. Let Ω be an open bounded connected subset of $\mathbb{R}^d$ having a Lipschitz-continuous boundary. Then Ω satisfies the cone property with a radius r and an angle θ. According to Proposition 1.1, there exists constants $M\geqslant 1$ and M_1 (depending on d and θ) and ε_0 (depending on θ and r) such that for all $\varepsilon\in]0,\varepsilon_0]$ there exists a set $T_\varepsilon\subset\Omega$ satisfying

(i) $B(t,\varepsilon)\subset\Omega$ for all $t\in T_\varepsilon$,

(ii) $\Omega\subset\bigcup_{t\in T_\varepsilon} B(t,M\varepsilon)$,

(iii) $\sum_{t\in T_h} 1_{B(t,M\varepsilon)}\leqslant M_1$.

Let m, p and α given as in the hypothesis of the theorem. By using Theorem 2.1 combining with Proposition 1.1, we get the existence of $R>0$ (depending on d and m) and the existence of $C>0$ (depending on M, d, m, α, p and τ) such that for every finite Π_{m-1}-unisolvent subset $\mathscr{A}$ of $\overline{\Omega}$ satisfying $h=\sup_{t\in\overline{\Omega}}\inf_{a\in\mathscr{A}}|t-a|$ with $h<h_0=\frac{\varepsilon_0}{R}$ there exists a set $T_h:=T_\varepsilon\subset\Omega$ satisfying (i), (ii) and (iii) where we set $\varepsilon=Rh$.

Let $f\in W^{m+1,2}(\Omega)$. The intersection $B(t,h)\cap\mathscr{A}$ is not empty for all $t\in\overline{\Omega}$ and the function $f^\Omega-f^{\mathscr{A}}$ vanishes on $\mathscr{A}$. Then the inequality

$$\|D^\alpha(f^\Omega-f^{\mathscr{A}})\|_{L^p(B(t,MRh))}\leqslant Ch^{m-|\alpha|-(d/2)+(d/p)}\,|f^\Omega-f^{\mathscr{A}}|_{m,\tau,B(t,MRh)},$$

holds for all $t\in T_h$. It follows that

$$\begin{aligned}\|D^\alpha(f^\Omega-f^{\mathscr{A}})\|_{L^p(\Omega)}&\leqslant\left(\sum_{t\in T_h}\int_{B(t,MRh)}|D^\alpha(f^\Omega-f^{\mathscr{A}})|^p\,\mathrm{d}x\right)^{1/p}\\&=\left(\sum_{t\in T_h}\|D^\alpha(f^\Omega-f^{\mathscr{A}})\|^p_{L^p(B(t,MRh))}\right)^{1/p}\\&\leqslant Ch^{m-|\alpha|-(d/2)+(d/p)}\left(\sum_{t\in T_h}|f^\Omega-f^{\mathscr{A}}|^p_{m,\tau,B(t,MRh)}\right)^{1/p}.\end{aligned}$$

For $p\in[2,\infty[$, by using the fact that $(\sum_i|x_i|^p)^{1/p}\leqslant(\sum_i|x_i|^2)^{1/2}$, we get

$$\|D^\alpha(f^\Omega-f^{\mathscr{A}})\|_{L^p(\Omega)}\leqslant Ch^{m-|\alpha|-(d/2)+(d/p)}\left(\sum_{t\in T_h}|f^\Omega-f^{\mathscr{A}}|^2_{m,\tau,B(t,MRh)}\right)^{1/2}.$$

For $p=\infty$, we have

$$\begin{aligned}\|D^\alpha(f^\Omega-f^{\mathscr{A}})\|_{L^\infty(\Omega)}&\leqslant\left(\sum_{t\in T_h}\|D^\alpha(f^\Omega-f^{\mathscr{A}})\|^2_{L^\infty(B(t,MRh))}\right)^{1/2}\\&\leqslant Ch^{m-|\alpha|-(d/2)}\left(\sum_{t\in T_h}|f^\Omega-f^{\mathscr{A}}|^2_{m,\tau,B(t,MRh)}\right)^{1/2}.\end{aligned}$$

But

$$\begin{aligned}|f^\Omega-f^{\mathscr{A}}|^2_{m,\tau,B(t,MRh)}=&\sum_{|\alpha|=m+1}\frac{(m+1)!}{\alpha!}\int_{\mathbb{R}^d}1_{m,\tau,B(t,MRh)}|D^\alpha(f^\Omega-f^{\mathscr{A}})(x)|^2\,\mathrm{d}x\\&+\tau^2\sum_{|\alpha|=m}\frac{m!}{\alpha!}\int_{\mathbb{R}^d}1_{m,\tau,B(t,MRh)}|D^\alpha(f^\Omega-f^{\mathscr{A}})(x)|^2\,\mathrm{d}x,\end{aligned}$$

we get $\sum_{t\in T_h}|f^\Omega-f^{\mathscr{A}}|^2_{m,\tau,B(t,MRh)}\leqslant M_1|f^\Omega-f^{\mathscr{A}}|^2_{m,\tau,\mathbb{R}^d}$. Finally, we have the following result

$$\|D^\alpha(f^\Omega-f^{\mathscr{A}})\|_{L^p(\Omega)}\leqslant C\sqrt{M_1}\,h^{m-|\alpha|-(d/2)+(d/p)}\,|f^\Omega-f^{\mathscr{A}}|_{m,\tau,\mathbb{R}^d}.$$

The required results follows from the fact that $f^\Omega_{|\Omega}=f$. $\square$

Corollary 3.1. *Let Ω be an open bounded connected subset of $\mathbb{R}^d$ having a Lipschitz-continuous boundary. Let m be an integer such that $m > d/2$, then there exists $h_0 > 0$ (depending on Ω, m and d) and there exists K (depending on Ω and m) such that for any multi-index α and for any $p \in [2, \infty]$ with $|\alpha| \leqslant m - (d/2) + (d/p)$, there exists a constant C (depending on Ω, d, m, α, p and τ) such that for every function f belonging to $W^{m+1,2}(\Omega)$ and for every finite Π_{m-1}-unisolvent subset $\mathscr{A}$ of $\overline{\Omega}$ satisfying $h = \sup_{t \in \overline{\Omega}} \inf_{a \in \mathscr{A}} |t - a| \leqslant h_0$, the following inequality holds*

$$\|D^\alpha(f - f^{\mathscr{A}})\|_{L^p(\Omega)} \leqslant KCh^{m-|\alpha|-(d/2)+(d/p)} |f|_{m,\tau,\Omega}.$$

Proof. By combining the results of Theorem 3.1 together with the results of Lemma 2.1 and taking the fact that $f^{\Omega}_{|\Omega} = f$ into account, we obtain the required result. □

We have the following convergence result

Corollary 3.2. *Let Ω be an open bounded connected subset of $\mathbb{R}^d$ having a Lipschitz-continuous boundary. Let m be an integer such that $m > d/2$, let α be a multi-index and $p \in [2, \infty]$ such that $|\alpha| \leqslant m - (d/2) + (d/p)$. For all $\varepsilon > 0$ and for all $f \in W^{m+1,2}(\Omega)$, there exists $h_0 > 0$ such that for every finite Π_{m-1}-unisolvent subset $\mathscr{A}$ of $\overline{\Omega}$ satisfying $h = \sup_{t \in \overline{\Omega}} \inf_{a \in \mathscr{A}_h} |t - a| \leqslant h_0$, the following inequality holds*

$$\|D^\alpha(f - f^{\mathscr{A}})\|_{L^p(\Omega)} \leqslant \varepsilon h^{m-|\alpha|-(d/2)+(d/p)}.$$

Proof. The result is an immediate consequence of Theorems 3.1 and 4.2 in [9]. Let $f \in W^{m+1,2}(\Omega)$. According to Theorem 4.2 in [9], we have $\lim_{h \to 0} |f^\Omega - f^{\mathscr{A}}|_{m,\tau} = 0$. Let $\varepsilon > 0$, there exists h_0 such that for $h = \sup_{t \in \overline{\Omega}} \inf_{a \in \mathscr{A}_h} |t - a| \leqslant h_0$, we have $C|f^\Omega - f^{\mathscr{A}}|_{m,\tau,\mathbb{R}^d} < \varepsilon$, where C is the constant appearing in Theorem 3.1. It follows that

$$\|D^\alpha(f - f^{\mathscr{A}})\|_{L^p(\Omega)} \leqslant \varepsilon h^{m-|\alpha|-(d/2)+(d/p)}. \qquad \square$$

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