JOURNAL OF DIFFERENTIAL EQUATIONS 6, 71–86 (1969)

The Mixed Initial-Boundary Value Problem for the Equations of Nonlinear One-Dimensional Viscoelasticity

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Received March 25, 1968

1. INTRODUCTION

It has been proved (Lax [1], MacCamy–Mizel [2]) that the equations of one-dimensional nonlinear elasticity do not admit, in general, smooth solutions in the large. It is expected though, that if the stress depends on the history of motion in an appropriate fashion, then smooth solutions exist.

The simplest model of a solid with history dependence is provided by one-dimensional viscoelasticity, where the stress σ is a function of the deformation gradient u_x and its time derivative $\dot{u}_x$. Greenberg, MacCamy and Mizel [3] have considered the semilinear case, $\sigma(u_x, \dot{u}_x) \equiv \varphi(u_x) + \dot{u}_x$ where φ is a strictly increasing function. They prove the existence of a unique solution which is asymptotically stable.

In this paper we consider the traction boundary value problem in the general case where $\sigma(u_x, \dot{u}_x)$ may be nonlinear in both $u_x, \dot{u}_x$. The form of the dependence of $\sigma(u_x, \dot{u}_x)$ on $\dot{u}_x$ is restricted by the requirement that the viscosity be bounded away from zero. On the contrary, the dependence on u_x is essentially unrestricted apart from certain requirements of boundedness.

It turns out that the viscoelastic part dominates the elastic part and secures the existence of a unique solution in the large. This solution is smooth enough so that all derivatives entering the equation of motion are Hölder continuous. The tools of the proof are certain "energy" estimates combined with known a priori bounds from the theory of parabolic equations, and the Leray–Schauder fixed point theorem.

In the final part of the paper, we investigate the asymptotic stability of the solution. The mechanism which provokes the decay of the solution is induced by the viscosity. On the other hand, the number and the nature of all possible static configurations depend entirely on the elastic part of the stress. Since we have allowed a broad variety of forms in the elastic part, it is to be expected that the problem of asymptotic stability will be of some interest. Somewhat surprisingly it turns out that the solution is always asymptotically stable in the sense that the stress, the velocity, and the acceleration decay to zero as time grows to infinity. The deformation gradient is not necessarily asymptotically stable and may tend to infinity with time.

Quite similar results can be obtained by the same method for the boundary value problem of place.

2. Formulation of the Mixed Initial-Boundary Value Problem

We consider a one-dimensional homogeneous[1] body whose reference configuration is the interval $[0, 1]$. The motion of the body in the time interval $[0, T]$ is described by the displacement function $u(x, t)$ on the quadrangle $\bar{Q}_T$, where $Q_T \equiv (0, 1) \times (0, T)$. We assume that the stress is a function of u_x, $\dot{u}_x$,

$$\sigma = \sigma(u_x, \dot{u}_x)$$

which satisfies the following conditions:

1. $\sigma(p, q)$ is continuous and continuously differentiable in both arguments. Furthermore, the partial derivatives $\sigma_p(p, q)$, $\sigma_q(p, q)$ are locally Hölder continuous in $\mathbf{R}^2$ with exponent α, $0 < \alpha < 1$.

2. There exist positive constants K, N such that for all p, q,

$$K \leqslant \sigma_q(p, q), \tag{2.1}$$

$$|\sigma_p(p, q)| \leqslant N[\sigma_q(p, q)]^{1/2}. \tag{2.2}$$

3. $$\sigma(0, 0) = 0. \tag{2.3}$$

The equation of linear momentum takes the form

$$\rho \ddot{u} = \sigma(u_x, \dot{u}_x)_x + f(x, t) \tag{2.4}$$

where the density ρ is a positive constant and $f \in C^0(\bar{Q}_T)$ is an assigned body force. By a solution of (2.4) in $\bar{Q}_T$ we mean a function $u(x, t)$, u, $\dot{u}$, u_x,

[1] The assumption of homogeneity is made for the sake of simplicity. The analysis can be immediately extended to nonuniform materials, provided that the dependence of the density and the stress on the reference configuration is sufficiently smooth.

$\dot{u}$, $\dot{u}_x$, u_{xx}, $\dot{u}_{xx} \in C^0(\bar{Q}_T)$, which satisfies (2.4) for all $(x, t) \in \bar{Q}_T$.[2] We are interested here in a solution $u(x, t)$ which assumes boundary conditions

$$\begin{aligned} \sigma(u_x(0, t), \dot{u}_x(0, t)) &= \sigma_0(t), \qquad t \in [0, T] \\ \sigma(u_x(1, t), \dot{u}_x(1, t)) &= \sigma_1(t), \qquad t \in [0, T] \end{aligned} \tag{2.5}$$

and given initial conditions,

$$u(x, 0) = u^0(x), \dot{u}(x, 0) = \dot{u}^0(x), \qquad x \in [0, 1]. \tag{2.6}$$

Obviously, for the existence of a solution in the above sense, it is necessary that

$$\begin{aligned} \sigma({u^0}_x(0), {\dot{u}^0}_x(0)) &= \sigma_0(0), \\ \sigma({u^0}_x(1), {\dot{u}^0}_x(1)) &= \sigma_1(0). \end{aligned} \tag{2.7}$$

Experience with parabolic equations indicates that (2.1) is a natural restriction. This condition refers to the viscoelastic part and can be interpreted as a requirement for the boundedness of the viscosity away from zero. Note also that (2.1) is in accordance with the Clausius–Duhem inequality.

Condition (2.2) which refers to the elastic part is rather general. Even so in various special cases this condition can be relaxed. For example, if $\sigma(u_x, \dot{u}_x) \equiv \varphi(u_x) + \dot{u}_x$, (2.2) may be replaced by the weaker $\sigma_p(p, q) \geqslant -N$.[3] In Section 4, we demonstrate that condition (2.2) alone is not capable of guaranteeing the asymptotic stability of solutions and we propose an additional restriction.

The boundary conditions (2.5) are ordinary differential equations for the functions $u_x(0, t)$, $u_x(1, t)$. Under the current assumptions on $\sigma(p, q)$, there exist unique smooth solutions of $(2.5)_1$, $(2.5)_2$ with initial conditions ${u^0}_x(0)$, ${u^0}_x(1)$ respectively. For simplicity, we will consider the problem of existence only for the special case $\sigma_0(t) = \sigma_1(t) = 0$, $t \in [0, T]$, ${u^0}_x(0) = {u^0}_x(1) = 0$, ${\dot{u}^0}_x(0) = {\dot{u}^0}_x(1) = 0$. Then (2.5) yield

$$u_x(0, t) = u_x(1, t) = 0, \qquad t \in [0, T]. \tag{2.8}$$

The influence of boundary conditions on the asymptotic behavior of solutions is very interesting and hence, in Section 4, we do not confine ourselves to the case (2.8).

[2] The solution will be physically meaningful only if $u_x \neq -1$ in Q_T.

[3] In fact in this case it is possible to establish an a priori inequality of the form $\sup_{Q_T} | u_x | \leqslant c(T)$, and the analysis of Section 3 is then applicable. Thus, if the initial data are sufficiently small in an appropriate sense, the condition $u_x > -1$ will be satisfied in Q_T. If φ is nondecreasing, $c(T)$ can be selected independent of T.

3. Existence and Uniqueness of Solutions

We first prove a uniqueness theorem:

THEOREM 3.1. *For given $f(x, t)$, $u^0(x)$, $\dot{u}^0(x)$, there is at most one solution of* (2.4), (2.5), (2.6) *in* $\bar{Q}_T$.

Proof. Suppose there exist two solutions $u^{(1)}(x, t)$, $u^{(2)}(x, t)$. Set $v(x, t) \equiv u^{(1)}(x, t) - u^{(2)}(x, t)$ and note that

$$\begin{aligned}\rho\ddot{v} &= [\sigma(u_x^{(1)}, \dot{u}_x^{(1)}) - \sigma(u_x^{(2)}, \dot{u}_x^{(2)})]_x \\ &= [\sigma_p(\bar{u}_x, \bar{\dot{u}}_x)\, v_x + \sigma_q(\bar{u}_x, \bar{\dot{u}}_x)\, \dot{v}_x]_x\end{aligned}$$

where use has been made of the mean value theorem. We multiply the above equation by $\dot{v}$ and we integrate over the quadrangle $Q_t \equiv (0, 1) \times (0, t)$, $t \in (0, T]$. After an integration by parts,

$$\int_0^t \int_0^1 \bar{\sigma}_p v_x \dot{v}_x \, dx\, d\tau + \int_0^t \int_0^1 \bar{\sigma}_q \dot{v}_x^2 \, dx\, d\tau = -\tfrac{1}{2} \int_0^1 \dot{v}^2(t)\, dx \leqslant 0.$$

Recalling (2.1), (2.2) and applying Schwarz's inequality we end up with

$$\int_0^t \int_0^1 \dot{v}_x^2 \, dx\, d\tau \leqslant \frac{N^2}{K} \int_0^t \int_0^1 v_x^2 \, dx\, d\tau. \tag{3.1}$$

From the equation $v_x(x, t) = \int_0^t \dot{v}_x(x, \tau)\, d\tau$ and Schwarz's inequality, it follows that

$$\int_0^1 v_x^2(x, t)\, dx \leqslant t \int_0^t \int_0^1 \dot{v}_x^2 \, dx\, d\tau.$$

Combining the above estimate with (3.1) and using Gronwall's inequality (e.g. [*4*], p. 24) we deduce $v(x, t) \equiv 0$ on $\bar{Q}_T$. Q.E.D.

It turns out that the existence theorem can be formulated efficiently in terms of certain Banach spaces which are familiar from the theory of equations of parabolic type. We recall the definition of these spaces.

DEFINITION. Let $0 < \beta < 1$. By $C^{k+\beta}[0, 1]$, $k = 0, 1, \ldots$, we denote the set of functions on $[0, 1]$ which are k times continuously differentiable, their derivatives being Hölder continuous with exponent β. For $w(x) \in C^{k+\beta}[0, 1]$, we set

$$| w |_{k+\beta} \equiv \sum_{s=0}^{k} \left\{ \sup_{[0,1]} \left| \frac{\partial^s}{\partial x^s} w(x) \right| + \sup_{x,x' \in [0,1]} \frac{\left| \frac{\partial^s}{\partial x^s} w(x) - \frac{\partial^s}{\partial x^s} w(x') \right|}{| x - x' |^\beta} \right\}.$$

The set of functions on $\bar{Q}_T$ which are Hölder continuous (exponent β) with respect to the distance function $d\{(x, t), (x', t')\} \equiv \{| x - x' |^2 + | t - t' |\}^{1/2}$ will be denoted by $C^\beta(\bar{Q}_T)$. With every $v(x, t) \in C^\beta(\bar{Q}_T)$, we associate the norm

$$\| v \|_\beta \equiv \sup_{\bar{Q}_T} | v(x, t)| + \sup_{(x,t),(x',t') \in \bar{Q}_T} \frac{| v(x, t) - v(x', t')|}{\{| x - x' |^2 + | t - t' |\}^{\beta/2}}.$$

In terms of $\| \cdot \|_\beta$, we define the norms

$$\| v \|_{1+\beta} \equiv \| v \|_\beta + \| v_x \|_\beta ,$$

$$\| v \|_{2+\beta} \equiv \| v \|_\beta + \| \dot{v} \|_\beta + \| v_x \|_\beta + \| v_{xx} \|_\beta ,$$

$$||| v |||_{1+\beta} \equiv \| v \|_{1+\beta} + \| \dot{v} \|_{1+\beta} ,$$

$$||| v |||_{2+\beta} \equiv \| v \|_{2+\beta} + \| \dot{v} \|_{2+\beta} .$$

By $C^{1+\beta}(\bar{Q}_T)$, $C^{2+\beta}(\bar{Q}_T)$, $B^{1+\beta}(\bar{Q}_T)$, $B^{2+\beta}(\bar{Q}_T)$ we denote the sets of functions $v(x, t)$ for which the norms $\| v \|_{1+\beta}$, $\| v \|_{2+\beta}$, $||| v |||_{1+\beta}$, $||| v |||_{2+\beta}$, respectively, are defined and finite.

All the sets defined above are Banach spaces with respect to the indicated norm. For their properties we refer to Friedman [5].

We now state the existence theorem:

THEOREM 3.2. *Let $u^0(x)$, $\dot{u}^0(x) \in C^{2+\alpha}[0, 1]$, $u^0{}_x(0) = u^0{}_x(1) = 0$, $\dot{u}^0{}_x(0) = \dot{u}^0{}_x(1) = 0$, $f(x, t) \in C^\alpha(\bar{Q}_T)$. Moreover, assume that f possesses a generalized derivative $\dot{f} \in L^2(Q_T)$. Then there exists a solution $u(x, t) \in B^{2+\alpha}(\bar{Q}_T)$ of the equation* (2.4) *with boundary conditions* (2.8) *and initial conditions* (2.6). *Furthermore, u possesses a generalized derivative $\ddot{u}_x \in L^2(Q_T)$.*

The proof of the above theorem is lengthy. For convenience, it will be partitioned into several lemmas.

LEMMA 3.1. *Let $\varphi(x, t)$, $\psi(x, t)$, $f(x, t) \in C^\beta(\bar{Q}_T)$, $0 < \beta < 1$, $\psi(x, t) \geqslant K$. Furthermore, let $u^0(x)$, $\dot{u}^0(x) \in C^{2+\beta}[0, 1]$, $u^0{}_x(0) = u^0{}_x(1) = 0$, $\dot{u}^0{}_x(0) = \dot{u}^0{}_x(1) = 0$. Suppose that $u(x, t) \in B^{2+\beta}(\bar{Q}_T)$ is a solution of the equation*

$$\rho \ddot{u} = \varphi(x, t)\, u_{xx} + \psi(x, t)\, \dot{u}_{xx} + f(x, t) \tag{3.2}$$

in $\bar{Q}_T$ *with boundary conditions* (2.8) *and initial conditions* (2.6). *Then there exists a constant* C_1 *depending only on* β, ρ, K *and on upper bounds of* $\| \varphi \|_\beta$, $\| \psi \|_\beta$, T, *such that*

$$||| u |||_{2+\beta} \leqslant C_1[\| f \|_\beta + | u^0 |_{2+\beta} + | \dot{u}^0 |_{2+\beta}]. \tag{3.3}$$

Proof. By symmetric reflections upon the lines $x = 0, \pm 1, \pm 2, \ldots$ we extend the functions $\varphi(x, t)$, $\psi(x, t)$, $f(x, t)$, $u(x, t)$ onto $(-\infty, \infty) \times [0, T]$ and the functions $u^0(x)$, $\dot{u}^0(x)$ onto $(-\infty, \infty)$. Under this extension, the smoothness of all these functions is preserved and the extended u is a solution of the Cauchy problem for the equation (3.2) with initial conditions the extended $u^0(x)$, $\dot{u}^0(x)$. Equation (3.2) may be visualized as of parabolic type for $\dot{u}$ with source term $\varphi(x, t)\, u_{xx} + f(x, t)$. Then for any rectangle $Q_\tau \equiv (0, 1) \times (0, \tau)$, $\tau \in (0, T]$, the following estimate holds (Friedman [5], p. 121),

$$\| \dot{u} \|_{2+\beta, \bar{Q}_\tau} \leqslant C[\| \varphi(x, t)\, u_{xx} \|_{\beta, \bar{Q}_\tau} + \| f \|_{\beta, \bar{Q}_\tau} + | \dot{u}^0 |_{2+\beta}]. \tag{3.4}$$

The constant C depends only on β, ρ, K, and on upper bounds of $\| \psi \|_{\beta, \bar{Q}_\tau}$ and τ, or, a fortiori, on upper bounds of $\| \psi \|_{\beta, \bar{Q}_T}$ and T. Note that ([5], p. 66)

$$\| \varphi u_{xx} \|_{\beta, \bar{Q}_\tau} \leqslant \| \varphi \|_{\beta, \bar{Q}_\tau} \| u_{xx} \|_{\beta, \bar{Q}_\tau} \leqslant \| \varphi \|_{\beta, \bar{Q}_T} \| u_{xx} \|_{\beta, \bar{Q}_\tau}. \tag{3.5}$$

Furthermore, from the equations

$$\frac{\partial^k}{\partial x^k} u(x, t) = \frac{d^k}{dx^k} u^0(x) + \int_0^t \frac{\partial^k}{\partial x^k} \dot{u}(x, \tau)\, d\tau,$$

$k = 0, 1, 2$, and $\dot{u}(x, t) = \dot{u}^0(x) + \int_0^t \ddot{u}(x, \tau)\, d\tau$, it follows easily that

$$\| u \|_{2+\beta, \bar{Q}_\tau} \leqslant | \dot{u}^0 |_\beta + | u^0 |_{2+\beta} + (\tau + \tau^{1-\beta/2}) \| \dot{u} \|_{2+\beta, \bar{Q}_\tau}. \tag{3.6}$$

There exists τ depending only on β, ρ, K and on upper bounds of $\| \varphi \|_{\beta, \bar{Q}_T}$, $\| \psi \|_{\beta, \bar{Q}_T}$, T such that

$$C(\tau + \tau^{1-\beta/2}) \| \varphi \|_{\beta, \bar{Q}_T} \leqslant \tfrac{1}{2}.$$

For such a τ, a combination of (3.4), (3.5), (3.6) leads to the estimate

$$\| \dot{u} \|_{2+\beta, \bar{Q}_\tau} \leqslant 2C(\| \varphi \|_{\beta, \bar{Q}_T} + 1)[\| f \|_{\beta, \bar{Q}_\tau} + | u^0 |_{2+\beta} + | \dot{u}^0 |_{2+\beta}]. \tag{3.7}$$

From (3.6), (3.7),

$$||| u |||_{2+\beta, \bar{Q}_\tau} \leqslant C_0[\| f \|_{\beta, \bar{Q}_T} + | u^0 |_{2+\beta} + | \dot{u}^0 |_{2+\beta}]$$

where C_0 depends only on β, ρ, K, and on upper bounds of $\| \varphi \|_{\beta,\bar{Q}_T}$, $\| \psi \|_{\beta,\bar{Q}_T}$, T. In particular,

$$| u(\tau)|_{2+\beta} + | \dot{u}(\tau)|_{2+\beta} \leqslant C_0[\| f \|_{\beta,\bar{Q}_T} + | u^0 |_{2+\beta} + | \dot{u}^0 |_{2+\beta}].$$

Repeating the same procedure for the rectangles $(0, 1) \times (\tau, 2\tau)$, $(0, 1) \times (2\tau, 3\tau)$,..., we end up with estimate (3.3) where $C_1 \equiv (1 + C_0)^{1+[T/\tau]}$.

Q.E.D.

LEMMA 3.2. *Under the assumptions of Lemma* 3.1, *there exists a unique solution* $u \in B^{2+\beta}(\bar{Q}_T)$ *of* (3.2) *which assumes boundary conditions* (2.8) *and initial conditions* (2.6).

Proof. The uniqueness follows immediately from (3.3). We will prove existence by the method of continuity. Consider the family of operators

$$L_\lambda[u] \equiv \rho\ddot{u} - [(1 - \lambda) K + \lambda\psi(x, t)] \dot{u}_{xx} + \lambda\varphi(x, t) u_{xx}$$

where λ is a real number. By S we denote the set of λ for which the equation $L_\lambda[u] = f(x, t)$ possesses a solution in $B^{2+\beta}(\bar{Q}_T)$ with boundary conditions (2.8) and initial conditions (2.6) for every $f(x, t) \in C^\beta(\bar{Q}_T)$. We want to show that $1 \in S$.

Note first that $L_0[u]$ is the operator of heat conduction. Hence $0 \in S$.

Suppose now that $\lambda_0 \in S$. Fix some λ. For $u \in B^{2+\beta}(\bar{Q}_T)$ let $v \equiv V_\lambda u$ be the solution of the equation

$$L_{\lambda_0}[v] = L_{\lambda_0}[u] - L_\lambda[u] + f(x, t) \tag{3.8}$$

with boundary conditions (2.8) and initial conditions (2.6). Note that (3.8) is of the form (3.2), hence estimate (3.3) holds for an appropriate constant C_1. Consider the sphere

$$X \equiv \{u \in B^{2+\beta}(\bar{Q}_T) | \,\| u \|_{2+\beta} \leqslant 2C_1[\| f \|_\beta + | u^0 |_{2+\beta} + | \dot{u}^0 |_{2+\beta}]\}$$

and let $\epsilon \equiv [2C_1(\| \psi \|_\beta + \| \varphi \|_\beta - K)]^{-1}$. Using estimate (3.3) it is easy to show that if $| \lambda - \lambda_0 | < \epsilon$, then V_λ maps X into itself. Furthermore, if $u_1, u_2 \in X$,

$$\| V_\lambda u_1 - V_\lambda u_2 \|_{2+\beta} < \tfrac{1}{2} \| u_1 - u_2 \|_{2+\beta} .$$

Thus V_λ is a contraction and a unique fixed point $u = V_\lambda u$ exists in X. From (3.8) it follows that this u satisfies the equation $L_\lambda[u] = f(x, t)$ with boundary conditions (2.8) and initial conditions (2.6). In other words $\lambda \in S$ and S is open.

Assume now that $\lambda \in \bar{S}$. There exists $\{\lambda_n\}$, $\lambda_n \in S$, $\lambda_n \to \lambda$ as $n \to \infty$. By $u_n \in B^{2+\beta}(\bar{Q}_T)$ we denote the solution of $L_{\lambda_n}[u] = f(x, t)$ with boundary conditions (2.8) and initial conditions (2.6). Using estimate (3.3) we can prove that $\{u_n\}$ is Cauchy in $B^{2+\beta}(\bar{Q}_T)$. Let $u_n \to u$, $n \to \infty$. It is easily seen that u is a solution of the equation $L_\lambda[u] = f(x, t)$ with boundary conditions (2.8) and initial conditions (2.6). In other words $\lambda \in S$ and S is closed.

Combining the above information we conclude that $1 \in S$. Q.E.D.

LEMMA 3.3. *Let $v(x, t)$ be a function on $\bar{Q}_T$. Suppose that $v(x, t)$ is Hölder continuous in t with exponent γ. Furthermore, the partial derivative $v_x(x, t)$ exists and is Hölder continuous in x with exponent δ. Then $v_x(x, t)$ satisfies a Hölder condition in t with exponent $\gamma\delta/(1 + \delta)$ and with a coefficient which can be estimated solely in terms of γ, δ and the Hölder coefficients of v in t and v_x in x. The assertion is valid even if v is Lipschitz continuous in t and/or v_x is Lipschitz continuous in x, provided we set $\gamma = 1$ and/or $\delta = 1$.*

Proof. See Ladyzenskaja and Ural'ceva ([6], II, Section 2). Q.E.D.

LEMMA 3.4. *For $\eta, \theta \in (0, 1)$, $B^{2+\eta}(\bar{Q}_T) \supset B^{1+\theta}(\bar{Q}_T)$. Moreover, the natural embedding of $B^{2+\eta}(\bar{Q}_T)$ into $B^{1+\theta}(\bar{Q}_T)$ is compact.*

Proof. It is a corollary of Lemma 3.3 and the fact that the natural embedding of $B^{1+\zeta}(\bar{Q}_T)$ into $B^{1+\theta}(\bar{Q}_T)$, $0 < \theta < \zeta < 1$, is compact (e.g. [5], p. 188). Q.E.D.

Let $\lambda \in [0, 1]$. By U_λ we denote the map

$$U_\lambda : B^{1+\frac{1}{3}}(\bar{Q}_T) \to B^{1+\frac{1}{3}}(\bar{Q}_T)$$

which sends a function $v \in B^{1+\frac{1}{3}}(\bar{Q}_T)$ into the solution $w(x, t)$ of the equation

$$\rho\ddot{w} = \sigma_p(v_x, \dot{v}_x)\, w_{xx} + \sigma_q(v_x, \dot{v}_x)\, \dot{w}_{xx} + \lambda f \tag{3.9}$$

satisfying boundary conditions (2.8) and initial conditions

$$w(x, 0) = \lambda u^0(x), \qquad \dot{w}(x, 0) = \lambda \dot{u}^0(x). \tag{3.10}$$

On account of Lemma 3.2, $w \in B^{2+(\alpha/3)}(\bar{Q}_T)$. Then estimate (3.3) and Lemma 3.4 imply that U_λ is well defined and compact.

LEMMA 3.5. *For v in bounded sets of $B^{1+\frac{1}{3}}(\bar{Q}_T)$, $U_\lambda v$ is uniformly continuous in λ.*

Proof. It is an immediate consequence of estimate (3.3) and Lemma 3.4. Q.E.D.

LEMMA 3.6. *For fixed* λ, U_λ *is a continuous map of* $B^{1+\frac{1}{3}}(\bar{Q}_T)$ *into itself.*

Proof. Fix $v \in B^{1+\frac{1}{3}}(\bar{Q}_T)$ and set $w \equiv U_\lambda v$. Conisder any sequence $\{v_n\}$, $v_n \in B^{1+\frac{1}{3}}(\bar{Q}_T)$, $v_n \xrightarrow{B^{1+\frac{1}{3}}(\bar{Q}_T)} v$, $n \to \infty$. Let $w_n \equiv U_\lambda v_n$. Estimate (3.3) implies that w_n is bounded in $B^{2+(\alpha/3)}(\bar{Q}_T)$. Then every subsequence of w_n contains a subsequence which is convergent in $B^{2+\beta}(\bar{Q}_T)$, $0 < \beta < \alpha/3$. Since (3.9) admits a unique solution, every convergent subsequence of $\{w_n\}$ converges to w. Thus $w_n \xrightarrow{B^{2+\beta}(\bar{Q}_T)} w$, $n \to \infty$, and the continuity of U_λ follows with the help of Lemma 3.4. Q.E.D.

Let $v(x, t)$ be a fixed point of the transformation U_λ for some $\lambda \in [0, 1]$. The following Lemmas 3.7–3.9 provide a priori bounds of v. The constants $M_1, M_2, \ldots$, which enter in those lemmas, can be estimated a priori in terms of ρ, known bounds of σ_p, σ_q, the initial conditions u^0, $\dot{u}^0$, the body force f, and T. Explicit values for the M's will be obtained in the course of the proofs.

LEMMA 3.7. *The function* $v(x, t)$ *possesses a generalized derivative* $\ddot{v}_x \in L^2(Q_T)$. *Moreover,*

$$\int_0^T \int_0^1 \ddot{v}_x^2 \, dx \, d\tau \leqslant M_1, \tag{3.11}$$

$$\max_{[0,T]} \int_0^1 \ddot{v}^2(x, t) \, dx \leqslant M_2. \tag{3.12}$$

Proof. For $h \in (0, T)$ and any function $w(x, t)$ defined on $\bar{Q}_T$, we set

$$w_h(x, t) \equiv \frac{w(x, t + h) - w(x, t)}{h}, \qquad t \in [0, T - h].$$

From (3.9) we obtain

$$\rho \ddot{v}_h = \sigma_{hx} + \lambda f_h, \qquad t \in [0, T - h]. \tag{3.13}$$

With the help of the mean value theorem, the above equation may be written in the form

$$\rho \ddot{v}_h = [\bar{\sigma}_p v_{xh} + \bar{\sigma}_q \dot{v}_{xh}]_x + \lambda f_h, \qquad t \in [0, T - h] \tag{3.14}$$

where $\bar{\sigma}_p$, $\bar{\sigma}_q$ stand for $\sigma_p(p, q)$, $\sigma_q(p, q)$ evaluated at an appropriate point $(\bar{p}, \bar{q})$. We multiply (3.14) by $\dot{v}_h$ and we integrate over the rectangle $Q_t \equiv (0, 1) \times (0, t)$, $t \in (0, T - h]$, thus obtaining

$$\tfrac{1}{2}\rho \int_0^1 \dot{v}_h^2 \, dx \Big|_0^t + \int_0^t \int_0^1 [\bar{\sigma}_p v_{hx} \dot{v}_{hx} + \bar{\sigma}_q \dot{v}_{hx}^2] \, dx \, d\tau = \lambda \int_0^t \int_0^1 f_h \dot{v}_h \, dx \, d\tau. \tag{3.15}$$

Integrating (3.9) (with $w \equiv v$) over $(0, 1) \times (\tau, \tau + h)$ and recalling the boundary conditions one obtains

$$\rho \int_0^1 [\dot{v}(x, \tau + h) - \dot{v}(x, \tau)]\, dx = \lambda \int_\tau^{\tau+h} \int_0^1 f(x, \xi)\, dx\, d\xi.$$

Hence,

$$\left[\int_0^1 \dot{v}_h(x, \tau)\, dx\right]^2 = \left[\frac{\lambda}{\rho} \frac{1}{h} \int_\tau^{\tau+h} \int_0^1 f(x, \xi)\, dx\, d\xi\right]^2$$

$$\leqslant \frac{\lambda^2}{\rho^2} \left[\max_{[0,T]} \int_0^1 |f(x, \xi)|\, dx\right]^2 \leqslant \frac{\lambda^2}{\rho^2} \max_{[0,T]} \int_0^1 f^2(x, \xi)\, dx.$$

Applying the Poincaré inequality,

$$\int_0^t \int_0^1 \dot{v}_h{}^2\, dx\, d\tau \leqslant \frac{\lambda^2 T}{\rho^2} \max_{[0,T]} \int_0^1 f^2(x, \tau)\, dx + \tfrac{1}{2} \int_0^t \int_0^1 \dot{v}_{hx}^2\, dx\, d\tau. \tag{3.16}$$

From (3.15), (2.2), and the Cauchy inequality,

$$\tfrac{1}{2}\rho \int_0^1 \dot{v}_h{}^2\, dx \Big|_0^t + \int_0^t \int_0^1 \bar{\sigma}_q \dot{v}_{hx}^2\, dx\, d\tau$$

$$\leqslant \tfrac{1}{4} \int_0^t \int_0^1 \bar{\sigma}_q \dot{v}_{hx}^2\, dx\, d\tau + N^2 \int_0^t \int_0^1 v_{hx}^2\, dx\, d\tau$$

$$+ \frac{\lambda K}{2} \int_0^t \int_0^1 \dot{v}_h{}^2\, dx\, d\tau + \frac{\lambda}{2K} \int_0^t \int_0^1 f_h{}^2\, dx\, d\tau.$$

Recalling (2.1) and using (3.16),

$$\tfrac{1}{2}\rho \int_0^1 \dot{v}_h{}^2\, dx \Big|_0^t + \frac{K}{2} \int_0^t \int_0^1 \dot{v}_{hx}^2\, dx\, d\tau$$

$$\leqslant N^2 \int_0^t \int_0^1 v_{hx}^2\, dx\, d\tau + \frac{\lambda}{2K} \int_0^t \int_0^1 f_h{}^2\, dx\, d\tau + \frac{KT}{2\rho^2} \max_{[0,T]} \int_0^1 f^2(x, \tau)\, dx. \tag{3.17}$$

Letting h go to zero we conclude that the generalized derivative $\ddot{v}_x$ exists in $L^2(Q_T)$. Moreover, in the place of (3.15), (3.17), and for $t \in (0, T]$, we have

$$\tfrac{1}{2}\rho \int_0^1 \ddot{v}^2\, dx \Big|_0^t + \int_0^t \int_0^1 [\sigma_p \dot{v}_x \ddot{v}_x + \sigma_q \ddot{v}_x{}^2]\, dx\, d\tau = \lambda \int_0^t \int_0^1 \dot{f} \ddot{v}\, dx\, d\tau, \tag{3.18}$$

$$\tfrac{1}{2}\rho \int_0^1 \ddot{v}^2\, dx \Big|_0^t + \frac{K}{2} \int_0^t \int_0^1 \ddot{v}_x{}^2\, dx\, d\tau$$

$$\leqslant N^2 \int_0^t \int_0^1 \dot{v}_x{}^2\, dx\, d\tau + \frac{\lambda}{2K} \int_0^t \int_0^1 \dot{f}^2\, dx\, d\tau + \frac{KT}{2\rho^2} \max_{[0,T]} \int_0^1 f^2(x, \tau)\, dx. \tag{3.19}$$

On the other hand,

$$\int_0^1 \dot{v}_x{}^2(x, t)\, dx \leqslant 2 \int_0^1 \dot{u}_x^{02}(x)\, dx + 2t \int_0^t \int_0^1 \ddot{v}_x{}^2\, dx\, d\tau.$$

Thus (3.19) gives rise to an estimate of the form

$$\int_0^1 \dot{v}_x{}^2(x, t)\, dx \leqslant \frac{4N^2 t}{K} \int_0^t \int_0^1 \dot{v}_x{}^2\, dx\, d\tau + M.$$

Applying Gronwall's inequality ([4], p. 24),

$$\int_0^1 \dot{v}_x{}^2(x, t)\, dx \leqslant M \exp\left(\frac{4N^2 T^2}{K}\right), \qquad t \in [0, T].$$

With the help of the above bound, (3.19) immediately yields estimates of the type (3.11), (3.12). Q.E.D.

LEMMA 3.8.

$$\max_{[0,T]} \int_0^1 \dot{v}_{xx}^2(x, t)\, dx \leqslant M_3. \tag{3.20}$$

Proof. We multiply (3.9) by $\dot{v}_{xx}$ and we integrate over (0, 1). Applying the Cauchy inequality and recalling (2.2) we obtain

$$\int_0^1 \sigma_q \dot{v}_{xx}^2\, dx \leqslant \tfrac{1}{4} \int_0^1 \sigma_q \dot{v}_{xx}^2\, dx + N^2 \int_0^1 v_{xx}^2\, dx + \frac{K}{8} \int_0^1 \dot{v}_{xx}^2\, dx + \frac{2\rho^2}{K} \int_0^1 \ddot{v}^2\, dx$$
$$+ \frac{K}{8} \int_0^1 \dot{v}_{xx}^2\, dx + \frac{2\lambda^2}{K} \int_0^1 f^2\, dx$$

or, using (2.1), (3.12),

$$\int_0^1 \dot{v}_{xx}^2\, dx \leqslant \frac{2N^2}{K} \int_0^1 v_{xx}^2\, dx + \frac{4\rho^2}{K^2} M_2 + \frac{4}{K^2} \int_0^1 f^2\, dx.$$

Combining the above estimate with

$$\int_0^1 v_{xx}^2(x, t)\, dx \leqslant 2t \int_0^t \int_0^1 \dot{v}_{xx}^2\, dx\, d\tau + 2 \int_0^1 u_{xx}^{02}\, dx$$

and Gronwall's inequality, we deduce (3.20). Q.E.D.

LEMMA 3.9.

$$\||\, v \,|\|_{1+\frac{1}{2}, \bar{Q}_T} \leqslant M_4. \tag{3.21}$$

Proof. From (3.11), (3.12) we derive the estimate

$$\int_0^T \ddot{v}^2(x, t)\, dt \leqslant 2TM_2 + 2M_1, \qquad t \in [0, T]. \tag{3.22}$$

A combination of (3.22) with

$$\dot{v}(x, t) = \lambda \dot{u}^0(x) + \int_0^t \ddot{v}(x, \tau)\, d\tau$$

leads to

$$\max_{\bar{Q}_T} |\dot{v}(x, t)| \leqslant \max_{[0,1]} |\dot{u}^0(x)| + (2T^2M_2 + 2TM_1)^{1/2},$$

$$\sup_{0 \leqslant t < t' \leqslant T} \frac{|\dot{v}(x, t) - \dot{v}(x, t')|}{|t - t'|^{1/2}} \leqslant (2TM_2 + 2M_1)^{1/2}, \qquad x \in [0, 1]$$

Similarly, from (3.20), (2.8) one obtains

$$\max_{\bar{Q}_T} |\dot{v}_x(x, t)| \leqslant (M_3)^{1/2},$$

$$\sup_{0 \leqslant x < x' \leqslant 1} \frac{|\dot{v}_x(x, t) - \dot{v}_x(x', t)|}{|x - x'|^{1/2}} \leqslant (M_3)^{1/2}, \qquad \in [0, T].$$

With the help of the above inequalities and Lemma 3.3 we deduce an estimate

$$\| \dot{v} \|_{1+\frac{1}{3}, \bar{Q}_T} \leqslant M'$$

which implies (3.21) for

$$M_4 \equiv | u^0 |_{1+\frac{1}{3}} + (1 + T + T^{5/6})\, M'. \qquad \text{Q.E.D.}$$

Proof of Theorem 2.2. Theorem 2.1 implies that $v \equiv 0$ is the unique fixed point of the transformation U_0. On account of Lemmas 3.5, 3.6, 3.9, and the Leray–Schauder fixed point theorem [7], it follows that U_1 has a fixed point $u \in B^{1+\frac{1}{3}}(\bar{Q}_T)$. From Lemma 3.7, $\ddot{u}_x \in L^2(Q_T)$.

It remains to show that $u \in B^{2+\alpha}(\bar{Q}_T)$. We already know that $u \in B^{2+(\alpha/3)}(\bar{Q}_T)$. In particular, $\dot{u}$ is Lipschitz continuous in t, and $\dot{u}_x$ is Lipschitz continuous in x. Then (Lemma 3.3) $\dot{u}_x$ is Hölder continuous in t with exponent $\frac{1}{2}$. Thus $\sigma_p(u_x, \dot{u}_x)$, $\sigma_q(u_x, \dot{u}_x) \in C^\alpha(\bar{Q}_T)$ in which case Lemma 3.2 implies that $u \in B^{2+\alpha}(\bar{Q}_T)$. Q.E.D.

4. The Asymptotic Stability of Solutions

In the present section we investigate the asymptotic stability of solutions of the equation (2.4) under boundary conditions (2.5) and initial conditions (2.6). For simplicity, we will assume that $f(x, t) \equiv 0$, $\sigma_1(t) \equiv \sigma_2(t) \equiv 0$

although the results may be extended to the general case provided that $f(x, t)$, $\sigma_1(t)$, $\sigma_2(t)$ behave properly as $t \to \infty$. The values of $u^0_x(0)$, $u^0_x(1)$, $\dot{u}^0_x(0)$, $\dot{u}^0_x(1)$ may be different from zero, provided of course that they satisfy the compatibility conditions (2.7).

Without loss of generality, we impose on the initial data the restrictions

$$\int_0^1 u^0(x)\,dx = 0, \qquad \int_0^1 \dot{u}^0(x)\,dx = 0 \tag{4.1}$$

thus ruling out the trivial family of rigid motions. Integrating (2.4) on (0, 1) and taking account of (2.5), (4.1) we obtain

$$\int_0^1 u(x, t)\,dx = 0, \qquad t \in [0, \infty). \tag{4.2}$$

Note that the function $u(x, t) \equiv \cos(\pi x)\, e^{\pi^2 t}$ is a solution of (2.4), (2.5) for $\sigma(u_x, \dot{u}_x) \equiv -2\pi^2 u_x + \dot{u}_x$, $\rho \equiv 1$. It follows that the conditions imposed so far on $\sigma(u_x, \dot{u}_x)$ are not sufficient to guarantee the asymptotic stability of solutions. Let us decompose $\sigma(p, q)$ into an "elastic" and a "viscoelastic" part,

$$\sigma(p, q) = \sigma^e(p) + \sigma^v(p, q)$$

where $\sigma^e(p) \equiv \sigma(p, 0)$. We define the "elastic" energy by

$$W(p) \equiv \int_0^p \sigma^e(\xi)\,d\xi.$$

It is clear that the existence and the nature of static solutions of the equation (2.4) depend exclusively on the form of the function $\sigma^e(p)$ and especially on its roots. By (2.3), $\sigma^e(0) = 0$. The problem of asymptotic stability is particularly interesting in the case where $\sigma^e(p)$ possesses additional zeros.

We now state the following assumption: There exists a number J such that

$$W(p) \geqslant J, \qquad p \in (-\infty, \infty). \tag{4.3}$$

We will prove that the above condition together with our basic assumptions on $\sigma(p, q)$ (Section 2, conditions 1–3) guarantee that all solutions are asymptotically stable in an appropriate sense.

We employ the boundary conditions (2.5) to determine the functions $u_x(0, t)$, $u_x(1, t)$, $t \in [0, \infty)$. Let us denote by $w(t)$ any one of the above two functions. We have

$$\sigma^e(w) + \sigma^v(w, \dot{w}) = 0 \tag{4.4}$$

together with the initial condition $w(0) = w_0$, where w_0 stands for $u^0_x(0)$ or

$u^0{}_x(1)$. It is easy to see that if $\sigma^e(w_0) \leqslant 0$ (resp. $\sigma^e(w_0) \geqslant 0$) then the solution $w(t)$ of (4.4) is an increasing (resp. decreasing) function which approaches the smallest (resp. the greatest) root of $\sigma^e(p)$ which is greater or equal (resp. less or equal) to w_0. If no such root exists, $w(t)$ diverges to $+\infty$ (resp. $-\infty$). The "elastic" energy $W(w(t))$ decreases steadily with time and tends to a finite value W_∞.

From the above discussion it becomes evident that, in general, $u_x(x, t)$ does not approach asymptotically a continuous function. It is even possible that $u_x(x, t)$ diverges to $\pm\infty$ as $t \to \infty$. On the contrary, it is plausible to expect that $\dot{u}(x, t)$, $\dot{u}_x(x, t)$, $\sigma(u_x, \dot{u}_x)$, are asymptotically stable. The following theorem justifies this expectation.

THEOREM 4.1. *The solution $u(x, t)$ of equation* (2.4) ($f(x, t) \equiv 0$) *satisfying boundary conditions* (2.5) ($\sigma_0(t) \equiv \sigma_1(t) \equiv 0$) *and initial conditions* (2.6), *subject to* (4.1), *is asymptotically stable in the following sense*:

$$\dot{u}_x(x, t) \xrightarrow{L^2(0,1)} 0, \qquad t \to \infty, \tag{4.5}$$

$$\ddot{u}(x, t) \xrightarrow{L^2(0,1)} 0, \qquad t \to \infty, \tag{4.6}$$

$$\dot{u}(x, t) \xrightarrow{C^{1/2}[0,1]} 0, \qquad t \to \infty, \tag{4.7}$$

$$\sigma(u_x, \dot{u}_x) \xrightarrow{C^{1/2}[0,1]} 0, \qquad t \to \infty. \tag{4.8}$$

Furthermore, $\int_0^1 W(u_x(x, t))\, dx$ *converges as* $t \to \infty$.

Proof. We multiply (2.4) by $\dot{u}$ and we integrate over the rectangle $Q_t \equiv (0, 1) \times (0, t)$, $t \in (0, \infty)$. An integration by parts yields

$$\int_0^1 [\tfrac{1}{2}\rho\dot{u}^2 + W(u_x)]\, dx \Big|_0^t + \int_0^t \int_0^1 \sigma^v(u_x, \dot{u}_x)\, \dot{u}_x\, dx\, d\tau = 0. \tag{4.9}$$

From (4.9), (2.1) and (4.3) we deduce,

$$\int_0^1 \dot{u}_x^2(x, t)\, dx \in L^1(0, \infty). \qquad 4.10)$$

We rewrite (3.18) for $v \equiv u$, $f \equiv 0$, $\lambda = 1$, $t \in (0, \infty)$:

$$\tfrac{1}{2}\rho \int_0^1 \ddot{u}^2\, dx \Big|_0^t + \int_0^t \int_0^1 [\sigma_p \dot{u}_x \ddot{u}_x + \sigma_q \ddot{u}_x^2]\, dx\, d\tau = 0. \tag{4.11}$$

From (4.11), (4.2), and the Cauchy inequality one obtains

$$\rho \int_0^1 \ddot{u}^2\, dx \Big|_0^t + \int_0^t \int_0^1 \sigma_q \ddot{u}_x^2\, dx\, d\tau \leqslant N^2 \int_0^t \int_0^1 \dot{u}_x^2\, dx\, d\tau$$

which implies that

$$\int_0^1 \sigma_Q \ddot{u}_x^2(x, t)\, dx \in L^1(0, \infty). \tag{4.12}$$

In particular, on account of (2.1),

$$\int_0^1 \ddot{u}_x^2(x, t)\, dx \in L^1(0, \infty). \tag{4.13}$$

From (4.13), (4.2), and the Poincaré inequality it follows that

$$\int_0^1 \ddot{u}^2(x, t)\, dx \in L^1(0, \infty). \tag{4.14}$$

On the other hand, (4.11), combined with (4.10), (4.12), implies that $\int_0^1 \ddot{u}^2(x, t)\, dx$ is uniformly continuous on $[0, \infty)$. Then (4.6) follows from (4.14).

Note now that

$$\frac{d}{dt}\int_0^1 \dot{u}_x^2\, dx = 2\int_0^1 \dot{u}_x \ddot{u}_x\, dx \in L^1(0, \infty).$$

Thus $\int_0^1 \dot{u}_x^2\, dx$ is uniformly continuous on $[0, \infty)$, in which case (4.10) implies (4.5).

The assertions (4.7), (4.8) follow easily from (4.5), (4.6) and (2.4).

Finally, from (4.9), (4.7) we deduce that $\lim_{t\to\infty} \int_0^1 W\, dx$ exists. Q.E.D.

Acknowledgment

This work was supported by a grant from the National Science Foundation. I express here my gratitude to Professor J. L. Ericksen who suggested this problem and provided many valuable discussions and criticisms in the research.

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