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How Tangents Solve Algebraic Equations, or a Remarkable Geometry of Discriminant Varieties

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Abstract: Let $D_{d,k}$ denote the discriminant variety of degree d polynomials in one variable with at least one of its roots being of multiplicity $\geq k$. We prove that the tangent cones to $D_{d,k}$ span $D_{d,k-1}$ thus, revealing an extreme ruled nature of these varieties. The combinatorics of the web of affine tangent spaces to $D_{d,k}$ in $D_{d,k-1}$ is directly linked to the root multiplicities of the relevant polynomials. In fact, solving a polynomial equation $P(z) = 0$ turns out to be equivalent to finding hyperplanes through a given point $P(z) \in D_{d,1} \approx \mathbb{A}^d$ which are tangent to the discriminant hypersurface $D_{d,2}$. We also connect the geometry of the Viète map $\mathbf{V}_d$: $\mathbb{A}^d_{root} \to \mathbb{A}^d_{coef}$, given by the elementary symmetric polynomials, with the tangents to the discriminant varieties $\{D_{d,k}\}$.

Various d-partitions $\{\mu\}$ provide a refinement $\{D^\circ_\mu\}$ of the stratification of $\mathbb{A}^d_{coef}$ by the $D_{d,k}$'s. Our main result, Theorem 7.1, describes an intricate relation between the divisibility of polynomials in one variable and the families of spaces tangent to various strata $\{D^\circ_\mu\}$.

Keywords: discriminant varieties, tangent cones, Young diagrams, polynomial equations.

1. INTRODUCTION

This exposition depicts a beautiful geometry of stratified discriminant varieties which are linked to polynomials in a *single* variable. Perhaps, it was Hilbert's ground breaking paper [Hi] which started the exploration. More general discriminant varieties have been a focus of an active and broad research (cf. [GKZ] which gives a comprehensive account). They are studied using methods of algebraic geometry ([GKZ], [A], [AC], [E], [K]), singularity theory ([A1]–[A3], [Va1]–[Va3], [SW], [SK], [GS]) and representation theory (with a heavy dose of commutative algebra) ([He], [W1], [W2])[1].

Here is a text which does not presume an in-depth familiarity with algebraic geometry, singularity and reprensentation theories. In fact, it is accessible to a graduate student. At the same time, the objects of study are classical and their geometry is fascinating. While many basic facts about such discriminants belong to folklore and are spread all over the mathematical archipelago, I do not know any self-sufficient elementary treatment giving a consistent picture of this small and beautiful island.

The discriminants of polynomials in one variable constitute a very special class among more general discriminants, but it is precisely due to their degenerated nature that they exhibit distinct and unique properties, properties which remain uncovered by general theories.

[1] This list is far from a complete one: it just reflects some sources that I found relevant to this article.

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This paper has its origins in a few observations that I derived from computer-generated images of tangent lines to discriminant plane curves (cf. Figures 1 and 8). The flavor of the observations can be captured in the slogan: "an algebraic problem of solving polynomial equations

$$P(z) = z^d + a_1 z^{d-1} + ... + a_{d-1}z + a_d = 0$$

is equivalent to a geometric problem of finding affine hyperplanes, passing through the point $P = (a_1, a_2, ..., a_d) \in \mathbb{A}^d$ and tangent to the discriminant hypersurface $\mathcal{D} \subset \mathbb{A}^d$" (cf. Corollary 6.1). The discriminant hypersurface is comprised of polynomials $P(z)$ with multiple roots, that is, of polynomials for which the two equations $\{P(z) = 0,\ P'(z) = 0\}$ have a solution $(a_1, a_2, ..., a_d)$.[2]

More generally, one can consider polynomials with roots of multiplicity $\geq k$. They form a $(d - k + 1)$-dimensional affine variety $\mathcal{D}_{d,k} \subset \mathbb{A}^d$. The resulting stratification

$$\mathbb{A}^d = \mathcal{D}_{d,1} \supset \mathcal{D}_{d,2} \supset \mathcal{D}_{d,3} \dots \supset \mathcal{D}_{d,d}$$

terminates with a smooth curve $\mathcal{D}_{d,d}$. This stratification has a remarkable property: the tangent cones to each stratum $\mathcal{D}_{d,k}$ span the previous stratum $\mathcal{D}_{d,k-1}$ (Theorem 6.1). Furthermore, $\mathcal{D}_{d,k-1}$ is comprised of the affine subspaces tangent to $\mathcal{D}_{d,k}$, and the number of such subspaces which hit a given point $P \in \mathcal{D}_{d,k-1}$ is entirely determined by the multiplicities of the $P(z)$-roots.

Surprisingly, the geometry of each stratum $\mathcal{D}_{d,k}$ can be derived from the geometry of a single rational curve $\mathcal{D}_{d,d} \subset \mathbb{A}^d$: its $(d - k + 1)$-st osculating spaces span $\mathcal{D}_{d,k}$ (cf. Theorem 6.2 and [ACGH], pp. 136-137). This leads to a "geometrization" of the Fundamental Theorem of Algebra (Corollary 6.2). Many of these facts are known to experts, but I had a hard time to find out which ones belong to folklore, and which ones were actually written down.

We proceed with a few observations about the $(k-1)$-dimensional varieties $\mathcal{D}^\vee_{d,k}$ which are the projective *duals* of the varieties $\mathcal{D}_{d,k}$. In Corollary 6.4 we prove that, for $k > 2$, $deg(\mathcal{D}^\vee_{d,k}) \leq deg(\mathcal{D}_{d,k-1})$ (we conjecture that this estimate is sharp).

In Theorem 6.3 we investigate the interplay between the geometry of the Viète map $\mathcal{V}_d : \mathbb{A}^d_{root} \to \mathbb{A}^d_{coef}$ (given by the elementary symmetric polynomials) and the tangents to the discriminant varieties $\{\mathcal{D}_{d,k}\}$.

Section 7 is devoted to more refined stratification $\{\mathcal{D}_\mu\}_\mu$ of the coefficient space. The strata $\{\mathcal{D}_\mu\}_\mu$ are indexed by d-partitions $\{\mu\}$. For a partition $\mu = \{\mu_1 + \mu_2 + ... + \mu_r = d\}$, the variety $\mathcal{D}_\mu$ is the closure in $\mathbb{A}^d_{coef}$ of the set $\mathcal{D}^\circ_\mu$ of polynomials with r distinct roots whose multiplicities are prescribed by the μ_i's. When $\mu = \{k + 1 + 1 + ... + 1 = d\}$, $\mathcal{D}_\mu = \mathcal{D}_{d,k}$. However, a generic variety $\mathcal{D}_\mu$ exhibits geometric properties very different from the ones of its ruled relative $\mathcal{D}_{d,k}$.

Our main result is Theorem 7.1. It describes an interesting and intricate relation between the divisibility of polynomials in one variable and the families of spaces tangent to various strata $\mathcal{D}^\circ_\mu$'s. Among other things, Theorem 7.1 depicts the decomposition of the quasi-affine variety $T\mathcal{D}^\circ_\mu$, comprised of spaces tangent to $\mathcal{D}^\circ_\mu$, into various pieces $\{\mathcal{D}^\circ_{\mu'}\}$. Also, it is preoccupied with the multiplicities of

[2]The ground number field is presumed to be $\mathbb{R}$ or $\mathbb{C}$. Most of the time, our arguments are not case-sensitive, but their interpretation is.

the tangent web forming $T\mathcal{D}^\circ_\mu$ (see also Corollary 7.1). Corollary 7.2 describes a remarkable stabilization of tangent spaces $T_Q\mathcal{D}^\circ_\mu$, as a point $Q \in \mathcal{D}^\circ_\mu$ approaches one of the singularities $\mathcal{D}^\circ_\nu \subset \mathcal{D}_\mu$.

We conclude with a few well-known remarks about the topology of the strata $\{\mathcal{D}^\circ_{d,k} := \mathcal{D}_{d,k} \setminus \mathcal{D}_{d,k+1}\}$ and $\{\mathcal{D}^\circ_\mu\}$ in connection to the colored braid groups.

After describing the observations above in a draft, I decided that it is a good time to consult with experts. I am grateful to Boris Shapiro for an eye-opening education. Also, with the help of Harry Tamvakis and Jersey Weyman I have learned about a flourishing research which tackles much more general discriminant varieties. My thanks extend to all these people.

A perceptive reader might wonder why all our references point towards Sections 6 and 7, and what is going on in the other sections of the article. The paper is written to satisfy two types of readership. The readers who are willing to endure the pain of combinatorics and multiple indices can proceed directly to Section 6, devoted to polynomials of a general degree d. The readers who prefer to see basic examples and special cases ($d = 2, 3$) of theorems from Sections 6 and 7, being stripped of combinatorial complexities, could be satisfied by the slow pace of Sections 2—5. In any case, our methods are quite elementary and the proofs are self-contained.

Some of the graphical images were produced using the *3D-FilmStrip*—a Mac-based software tool for a dynamic stereo visualization in geometry. It is developed by Richard Palais to whom I am thankful for help and pleasant conversations.

2. QUADRATIC DISCRIMINANTS

This section describes some "well-known" and some "less-well-known" geometry of the quadratic discriminant. It will provide us with a "baby model" of more general geometric structures to come.

Let u and v be the roots of a monic quadratic polynomial $P(z) = z^2 + bz + c$. The Viète formulas $b = -u - v$, $c = uv$ give rise to a quadratic polynomial map

$$\mathcal{V} : (u, v) \to (-u - v,\ uv)$$

from the *uv-root plane* $\mathbb{A}^2_{root}$ to the *bc-coefficient plane* $\mathbb{A}^2_{coef}$. We call it the *Viète Map*. Points of the root plane are *ordered* pairs of roots. Therefore, generically, $\mathcal{V}$ is a 2-to-1 map: pairs (u, v) and (v, u) generate the same quadratic polynomial. Being restricted to the diagonal line $L = \{u = v\}$, the map $\mathcal{V}$ is 1-to-1.

A simple experiment with a mapping software triggered this investigation. Figure 1 shows the effect of applying the Viète map $\mathcal{V}$ to a grid of vertical and horizontal lines in the uv-plane. At the first glance, the result is quite surprising: not only the images of lines under *quadratic* map $\mathcal{V}$ are *lines*, but these lines seem to be *tangent* to a parabola! In fact, this parabola $\mathcal{D}$ is the $\mathcal{V}$-image of the diagonal $\{u = v\}$. Its parametric equation is $(b, c) = (-2u, u^2)$. Hence, the equation of $\mathcal{D}$ is quite familiar to the frequent users of the quadratic formula: $b^2 - 4c = 0$.

In order to understand the tangency phenomenon, consider the Jacobi matrix of the Viète map

$$D\mathcal{V} = \begin{pmatrix} -1 & -1 \\ v & u \end{pmatrix}.$$

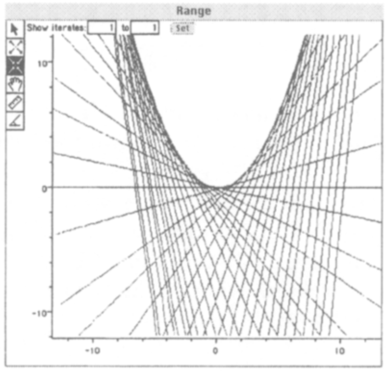


FIGURE 1. Each line tangent to the discriminant parabola represents the set of quadratic polynomials with a fixed root.

Its determinant $J\mathcal{V} = v - u$. It vanishes along the diagonal line $L \subset \mathbb{A}^2_{root}$, where the rank of $D\mathcal{V}$ drops to 1. This reinforces what we already have derived from the symmetry argument: under the Viète map, the root plane is ramified over the coefficient plane along the discriminant parabola $\mathcal{D}$.

The kernel of $D\mathcal{V}|_L = Span\{(1,-1)\}$ does not contain the diagonal. The $\mathcal{V}$-image of any smooth curve, which intersects with the diagonal at a point a and is transversal there to the kernel of $D\mathcal{V}|_L$, is tangent to the discriminant parabola $\mathcal{D} = \mathcal{V}(L)$ at $\mathcal{V}(a)$. In particular, the images of vertical and horizontal lines are tangent to $\mathcal{D}$. However, $\mathcal{V}$ maps each vertical line $l_{u_\star} := \{u = u_\star\}$ to a *line* $(b, c) = (-u_\star - v,\, u_\star v)$. Therefore, the grid of vertical and horizontal lines is mapped by the Viète map to the enveloping family of the discriminant parabola.

Since the line $\mathcal{V}(l_{u_\star})$ is the set of all quadratic polynomials with a fixed root $u_s tar$, its points must satisfy the relation $u_\star^2 + bu_\star + c = 0$. Therefore, the *slope* of the line $\mathcal{V}(l_{u_\star}) = \{c = -u_\star b - u_\star^2\}$ is equal to *minus the root* $u_\star$!

As a result, an algebraic problem of solving a quadratic equation $z^2 + bz + c = 0$ is equivalent to a geometric problem of finding lines passing through the point (b, c) and tangent to the curve $\mathcal{D}$. These observations are summarized in

Proposition 2.1. *Over the complex numbers, through every point $(b, c) \notin \mathcal{D}$, there are exactly two complex lines tangent to $\mathcal{D}$. Through every point $(b, c) \in \mathcal{D}$, the tangent line is unique.*

Over the reals, through each point of the domain $\mathcal{U}_+ = \{c < b^2/4\}$, there exists a pair of tangent lines, while through each point of the domain $\mathcal{U}_- = \{c > b^2/4\}$ no such a line exists.

The slopes of these tangents equal to minus the roots of the quadratic equation $z^2 + bz + c = 0$. □

Figure 2 depicts an analog device which is based on this theorem. It solves quadratic equations over the field $\mathbb{R}$. The discriminant parabola is modeled by a parabolic rim attached to the *bc*-plane. The device consists of two rulers hinged by a pin. We solve an equation by placing the pin at the corresponding point (b, c) and adjusting the rulers to be tangent to the rim. Then we read the measurements of their slopes.

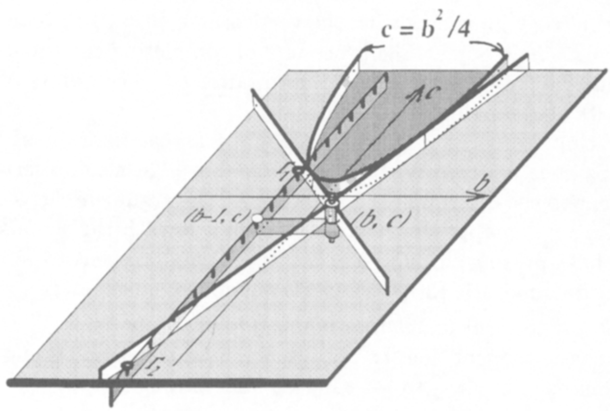


FIGURE 2

"Completing-the-square" magic calls for a substitution $z \Rightarrow z - t$ $(t = -b/2)$, which transforms a given polynomial $P(z) = z^2 + bz + c$ into a polynomial $Q(z) = P(z-t)$ of the form $z^2 - \tilde{c}$. This kind of substitutions defines a t-parametric group of transformations

$$\Phi_t(b,c) = (b - 2t,\ c - bt + t^2) = (P'(-t),\ P(-t)) \tag{2.1}$$

in the bc-plane. The corresponding transformation in the root plane amounts to a simple shift $\Psi_t(u,v) = (u+t,\ v+t)$. In other words, $\mathcal{V}(\Psi_t(u,v)) = \Phi_t(\mathcal{V}(u,v))$.

Evidently, Ψ_t preserves the Jacobian $J\mathcal{V} = v - u$. The Jacobian changes sign under the permutation $(u,v) \Rightarrow (v,u)$. Therefore, it can not be expressed in terms of b and c. However, its square $(J\mathcal{V})^2 = (v-u)^2$ is invariant under the permutation and admits a bc-formulation as $\Delta(b,c) = b^2 - 4c$. Therefore, the *discriminant polynomial* $\Delta(b,c)$ must be *invariant* under the Φ_t-flow. As a result, the Φ_t-trajectories form the family of parabolas $\{\Delta(b,c) = const\}$. In particular, the discriminant parabola is a trajectory of the Φ_t-flow.

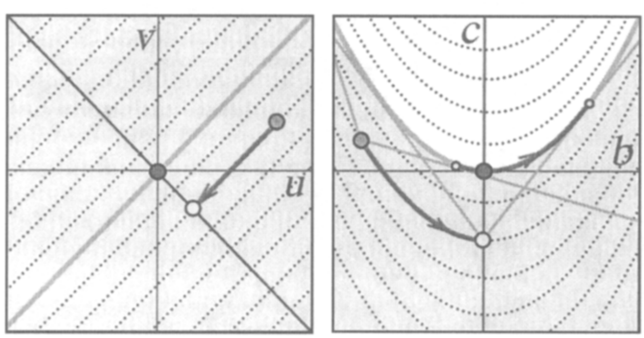


FIGURE 3. The Viète map is equivariant under the flows $\{\Psi_t\}$ and $\{\Phi_t\}$. Transformation Φ_t acts on the enveloping family of the discriminant parabola by "adding t to their slopes".

Each Φ_t-trajectory intersects with the vertical line $\{b = 0\}$ of *reduced* quadratic polynomials at a single point. The procedure of completing the square amounts to traveling along the trajectory $\Phi_t((a, b))$ until, after $t = -b/2$ units of time, it hits the line $\{b = 0\}$ at the point $(0,\, c - b^2/4)$.

We notice that, for a fixed t, Φ_t is an affine transformation of the bc-plane. Hence, it maps lines to lines. By the argument above, Φ_t also preserves the family of lines, tangent to the discriminant parabola. As the argument suggests, a tangent line with a slope k is mapped by Φ_t to a tangent line with the slope $k + t$. In fact, the slopes of lines passing through a point $(0, -d)$ and tangent to $\mathcal{D}$ are $\pm\sqrt{d}$.

Therefore, the quadratic formula reflects the following geometric recipe:

- apply $\Phi_{-b/2}$ to (b, c) to get to a point $Q = (0,\, c - b^2/4)$.
- construct the tangent lines to the discriminant parabola through Q.
- flow them back by $\Phi_{b/2}$ to get tangent lines through (b, c).

Curiously, the flow Φ_t *preserves the euclidean area* in the bc-plane: for a fixed t, formula (2.1) describes Φ_t as a composition of a linear transformation $(b, c) \to (b,\, c - bt)$ with the determinant 1, followed by a shift $(b, c) \to (b - 2t,\, c + t^2)$.

There is an alternative approach which leads to the same geometric observations and does not involve the Viète map. However, it calls for a trip to the 3rd dimension. Consider the surface $S = \{z^2 + bz + c = 0\}$ in the bcz-space (cf. Figure 4). It admits

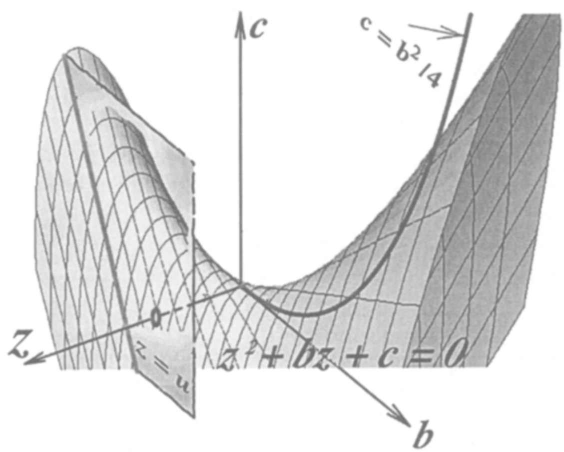


FIGURE 4

a bz-parametrization

$$\mathcal{H} : (b, z) \to (b,\, -bz - z^2,\, z). \tag{2.2}$$

Let $\mathcal{F}$ be the composition of the parametrization $\mathcal{H}$ with the obvious projection $\mathcal{P} : (b, c, z) \to (b, c)$. It is given by the formula

$$\mathcal{F} : (b, z) \to (b,\, -bz - z^2). \tag{2.3}$$

Evidently, the z-function, restricted to the preimage $(\mathcal{P}|_S)^{-1}((b, c))$, gives the roots of $z^2 + bz + c$.

We focus on the singularities of $\mathcal{F}$. Its Jacobi matrix is

$$D\mathcal{F} = \begin{pmatrix} 1 & -z \\ 0 & -b-2z \end{pmatrix}.$$

and its Jacobian $J\mathcal{F} = -b - 2z$. The rank of $D\mathcal{F}$ drops to 1 when $b + 2z = 0$, that is, when $P'(z) = 0$. Here $P(z) = z^2 + bz + c$.

Thus, the set of singular points for the projection $\mathcal{P}|_S$ is a curve C in the bcz-space given by two equations

$$\begin{cases} z^2 + bz + c &= 0 \\ 2z + b &= 0 \end{cases}$$

Expelling z from the system we get the equation $c = b^2/4$ of the ramification locus for the projection $\mathcal{P}$ from the surface S to the bc-plane. Again, as with the Viète map, the discriminant parabola is the ramification locus for the projection $\mathcal{P}$. Of course, this is not surprising: the system of equations $\{P(z) = 0,\ P'(z) = 0\}$ tells us that the polynomial P has a multiple root, that is, $P(z)$ is of the form $(z - u)^2$. In the bc-plane, such polynomials form the discriminant parabola $\mathcal{D}$.

For a fixed a number u, let N^u denote the intersection of the surface S with the plane $\{z = u\}$ in the bcz-space. This intersection is a *line* defined by two equations $\{z = u\}$ and $\{u^2 + bu + c = 0\}$. Hence, S is a *ruled surface* comprised of the distinct lines N^u. We notice that each line $N^u \subset S$ hits the critical curve $C \subset S$ at a single point: there is a single monic quadratic polynomial with a root u of multiplicity 2.

The projection of N^u in the bc-plane is a line $T^u = \{c + ub + u^2 = 0\}$, which, in view of our analysis of the Viète map, is tangent to the discriminant parabola. We also can verify this property directly by comparing the $\mathcal{P}$-images of the lines N^u and the line tangent to the $\mathcal{P}$-critical curve $C \subset S$ at the point $N^u \cap C$. Thus, the enveloping family of the discriminant parabola is the $\mathcal{P}$-image of the u-family of lines $\{N^u\}$ comprising S.

3. Ruled Geometry of Cubic Discriminants

We build on the observations from Section 2 to investigate the discriminant surface for cubic polynomials. This is the simplest case revealing the *stratified* ruled nature of the discriminant varieties.

Facts about the geometry of the cubic discriminant we are going to describe here can be found somewhere else (cf. [BG], 5.36). Often we differ from these sources only in the interpretation. This interpretation will allow us (cf. Sections 6, 7) to investigate the case of discriminant varieties for polynomials of any degree.

Now, our main object of interest is a monic cubic polynomial $P(z) = z^3 + bz^2 + cz + d$. Such polynomials can be coded by points (b, c, d) of the coefficient space $\mathbb{A}^3_{coef}$.[3]

In order to incorporate the roots of polynomials into the picture, consider the hypersurface S_1

$$z^3 + bz^2 + cz + d = 0 \qquad (3.1)$$

in the 4-dimensional space $\mathbb{A}^1 \times \mathbb{A}^3_{coef}$ with the cartesian coordinates z, b, c, d.

[3] As always, the coefficient space comes in two flavors: real and complex.

Put $Q(z,b,c,d) = z^3+bz^2+cz+d$. Since the gradient $\nabla Q = (P'(z),\ z^2,\ z,\ 1) \neq 0$, the hypersurface S_1 is non-singular. It can be viewed as the graph of the function $d(z,b,c) = -z^3 - bz^2 - cz$. Therefore, S_1 admits a (z,b,c)-parametrization by a 1-to-1 polynomial map

$$\mathcal{H}_1 : (z,b,c) \to (z,b,c,d) = (z,\ b,\ c, -z^3 - bz^2 - cz). \tag{3.2}$$

Denote by $\mathcal{P}$ the projection $(z,b,c,d) \to (b,c,d)$. Our immediate goal is to analyze the singularities of this projection, being restricted to the hypersurface S_1. That is, we will investigate the singularities of the composition $\mathcal{F}_1 = \mathcal{P} \circ \mathcal{H}_1$ given by

$$\mathcal{F}_1 : (z,b,c) \to (b,c,d) = (b,\ c, -z^3 - bz^2 - cz). \tag{3.3}$$

The Jacobi matrix $D\mathcal{F}_1$ of $\mathcal{F}_1$ is of the form

$$\begin{pmatrix} 0 & 0 & -3z^2 - 2bz - c \\ 1 & 0 & -z^2 \\ 0 & 1 & -z \end{pmatrix}.$$

Unless the derivative $P'(z) = 3z^2 + 2bz + c = 0$, the rank of $D\mathcal{F}_1$ is 3. When $P'(z) = 0$, it drops to 2. Thus, a point $(z,b,c,d) \in S_1$ is singular for the projection $\mathcal{P}|_{S_1}$, if and only if, two conditions are satisfied: $P(z) = 0$ and $P'(z) = 0$. This happens exactly when z is a root of P of multiplicity ≥ 2. Therefore, the singular locus S_2 of $\mathcal{P}|_{S_1}$ in the $zbcd$-space is the intersection of two hypersurfaces

$$\begin{aligned} z^3 + bz^2 + cz + d = 0 \\ 3z^2 + 2bz + c = 0. \end{aligned} \tag{3.4}$$

Solving this system for c and d, the non-singular surface S_2 can be parametrized by z and b:

$$\mathcal{H}_2 : (z,b) \to (z,b,c,d) = (z,\ b, -3z^2 - 2bz,\ 2z^3 + bz^2). \tag{3.5}$$

Composing $\mathcal{H}_2$ with the projection, we get:

$$\mathcal{F}_2 : (z,b) \to (b,c,d) = (b, -3z^2 - 2bz,\ 2z^3 + bz^2). \tag{3.6}$$

The Jacobi matrix $D\mathcal{F}_2$ of $\mathcal{F}_2$ is

$$\begin{pmatrix} 0 & -6z - 2b & 6z^2 + 2bz \\ 1 & -2z & z^2 \end{pmatrix}$$

Generically, it is of rank is 2. The rank drops to 1 when $P''(z) = 6z + 2b = 0$, that is, when (z,b,c,d) belongs to a curve $S_3 \subset S_2$, defined by the three equations

$$\begin{aligned} P(z) &= z^3 + bz^2 + cz + d = 0 \\ P'(z) &= 3z^2 + 2bz + c = 0 \\ P''(z) &= 6z + 2b = 0 \end{aligned} \tag{3.7}$$

The curve S_3 admits a parametrization by z

$$\mathcal{H}_3 : z \to (z,b,c,d) = (z, -3z,\ 3z^2, -z^3). \tag{3.8}$$

Its non-singular projection $\mathcal{F}_3$ into the bcd-space is given by

$$\mathcal{F}_3 : z \to (b,c,d) = (-3z,\ 3z^2, -z^3), \tag{3.9}$$

In view of (3.7), this curve $\mathcal{D}_3$ represents cubic polynomials with a single root of multiplicity 3.

Let $\mathcal{D}_2$ denote the image $\mathcal{P}(S_2)$ of the surface S_2, and $\mathcal{D}_3$—the image $\mathcal{P}(S_3)$ of the curve S_3 in the *bcd*-space. For the reasons, that will be even more apparent in Section 4, we call these images the *discriminant surface* and the *discriminant curve*. By the definitions, $\mathcal{D}_3 \subset \mathcal{D}_2 \subset \mathbb{A}^3_{coef}$. Figures 5 and 6 show the discriminant surface from different points of view.

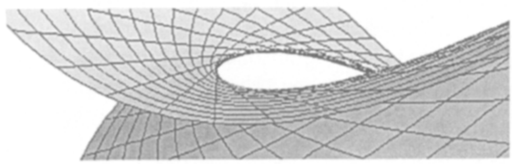


FIGURE 5. $\mathcal{D}_2$ is a ruled surface comprised of lines tangent to the discriminant curve $\mathcal{D}_3$ (perceived as a loop).

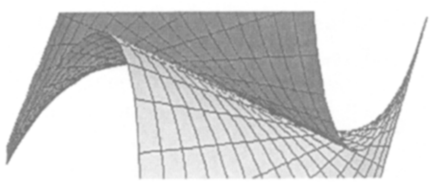


FIGURE 6. This view of the discriminant surface $\mathcal{D}_2$ reveals its symmetry with respect to the involution $(b, c, d) \to (-b,\ c, -d)$.

Looking from a point $P = (b, c, d) \in \mathbb{A}^3_{coef}$ against the projection $\mathcal{P}$, we see all the points of the hypersurface S_1 (cf. (3.1)) suspended over P, in other words, all the roots of the polynomial $P(z) = z^3 + bz^2 + cz + d$. Therefore, the preimage $S_1 \cap \mathcal{P}^{-1}(A)$ can contain 1, 2, or 3 points. Over the complex numbers, the cardinality of the preimage equals 3 when $P \in \mathbb{C}^3 \setminus \mathcal{D}_2$, it is 2 when $P \in \mathcal{D}_2 \setminus \mathcal{D}_3$, and 1 when $P \in \mathcal{D}_3$. Thus, $\mathcal{P}|_{S_1}$ is 3-to-1 map, *ramified* over the discriminant surface $\mathcal{D}_2$. Similarly, $\mathcal{P}|_{S_2}$ is 2-to-1 map, ramified over the discriminant curve $\mathcal{D}_3$. Finally, $\mathcal{P}|_{S_3}$ is 1-to-1 map.

Over the real numbers, the situation is more complex: the surface $\mathcal{D}_2$ divides $\mathbb{R}^3_{coef}$ into chambers, and, by the implicit function theorem, the cardinality of $S_1 \cap \mathcal{P}^{-1}(A)$ remains constant for all the P's in the interior of each chamber. Since a real cubic polynomial with no multiple roots has, alternatively, three real roots or a single real root, the chambers can be only of two types. (In the next section we will see that actually, there is a single chamber of each type.) We notice that, if a *real* cubic polynomial has a multiple complex root, then all its roots are real. Therefore, even over the reals, for $P \in \mathcal{D}_2 \setminus \mathcal{D}_3$, the preimage $S_1 \cap \mathcal{P}^{-1}(A)$ consists of two points.

Now, let's return to the *zbcd*-space. For a given number u, consider the intersection N_1^u of the hyperplane $\{z = u\}$ with the hypersurface S_1. This intersection selects all the quadruples (u, b, c, d) with the property $P(u) = u^3 + bu^2 + cu + d = 0$, where u is fixed. The $\mathcal{P}$-image of N_1^u is the surface T_1^u of all cubic polynomials with the number u as their common root. Evidently, the map $\mathcal{P} : N_1^u \to T_1^u$ is 1-to-1 and onto. One thing is instantly clear: $u^3 + bu^2 + cu + d = 0$ defines a *linear* relation among b, c, d—an *affine plane* in $\mathbb{A}^3_{coef}$. Scanning by u, we see that the hypersurface S_1 is a disjoint union of its u-slices — the planes N_1^u, i.e. it is a *ruled* hypersurface.

It is also clear that each plane N_1^u hits the surface S_2 (of polynomials with multiple roots) along a *line* N_2^u, defined by the equations (3.4) and the equation $\{z = u\}$. Indeed, the set of cubic equations with a root u contains the set of cubic equations with a root u of multiplicity ≥ 2. In turn, the line N_1^u hits the curve S_3 at a single point: there is a single monic cubic polynomial with the root u of multiplicity 3. Therefore, the surface S_2 is a ruled surface comprised of disjoint lines N_2^u. Since $\mathcal{D}_2 = \mathcal{P}(S_2)$ and $\mathcal{P}$ maps lines to lines, $\mathcal{D}_2$ is also a ruled surface comprised of lines defined by

$$\begin{aligned} u^2b + uc + d &= -u^3 \\ 2ub + c &= -3u^2. \end{aligned} \tag{3.10}$$

Let's concentrate on the case when u is a root of multiplicity ≥ 2. Consider a plane N_1^u through a point (u, P), $P = (b, c, d)$, of the surface $S_2 \subset S_1$, and the plane $\tau_{(u,P)}$ *tangent* to S_2 at (u, P). We will show that $T_1^u = \mathcal{P}(N_1^u)$ is *tangent* to $\mathcal{D}_2$ at the point P. It will suffice to check that vectors, tangent to S_2 at (u, P) project into the plane T_1^u.

Using (3.4), the plane $\nu_{(z,P)}$, normal to S_2 at a point (z, P), is spanned by two gradient vectors $\nabla_1(z, P) = (P'(z), z^2, z, 1)$ and $\nabla_2(z, P) = (P''(z), 2z, 1, 0)$. Note that, when $(z, P) \in S_2$, then $\nabla_1(z, P) = (0, z^2, z, 1)$. Denote by $n = n(z)$ the vector $(z^2, z, 1)$. In the new notation, $\nabla_1(z, P) = (P'(z), n(z))$ and $\nabla_2(z, P) = (P''(z), n'(z))$.

Any vector $(a, v) \in \mathbb{A}^1 \times \mathbb{A}^3$, tangent to S_2 at (u, P) must be orthogonal to $\nabla_1(u, P) = (0, n(u))$ and $\nabla_2(u, P) = (P''(u), n'(u))$, in other words, (a, v) must satisfy the system

$$\begin{aligned} v \bullet n(u) &= 0 \\ v \bullet n'(u) &= -aP''(u), \end{aligned} \tag{3.11}$$

where "$\bullet$" stands for the scalar product. We notice that if v satisfies the first equation in (3.11), then one can always to find an appropriate a, provided $P''(u) \neq 0$. On the other hand, when $P''(u) = 0$, that is, when $(u, P) \in S_3$, then (3.11) collapses to $\{v \bullet n(u) = 0,\ v \bullet n'(u) = 0\}$ and the a is free. In such a case, v must belong to a *line* L^u passing through P. This line is the $\mathcal{P}$-image of the tangent plane $\tau_{(u,P)}$ and its parametric equation is of the form $\{P + v\}_v$, where v is a subject to the orthogonality conditions $\{v \bullet n(u) = 0,\ v \bullet n'(u) = 0\}$.

Therefore, if $P \in \mathcal{D}_2^\circ := \mathcal{D}_2 \setminus \mathcal{D}_3$, then any vector v (with its origin at P) orthogonal to $n(u)$ belongs to the plane $\mathcal{P}(\tau_{(u,P)})$. As a result, such a v must be tangent to $\mathcal{D}_2 = \mathcal{P}(S_2)$ at P. On the other hand, the vector $n(u)$ is normal to the plane $T_1^u := \{u^2b + uc + d = -u^3\}$ passing through P. So, as affine planes, $T_1^u = \mathcal{P}(\tau_{(u,P)})$, and therefore, T_1^u must be tangent to $\mathcal{D}_2^\circ$ at P, provided $P(u) = 0$.

Note that, for any $P \in \mathcal{D}_2$, there is a single point (u, P) in $\mathcal{P}^{-1}(P) \cap S_2$: a cubic polynomial can not have more than one multiple root. Therefore, $\mathcal{P} : S_2^\circ \to \mathcal{D}_2^\circ$ is a regular embedding.

In the same spirit, one can check that when $(u, P) \in S_3$, the line $\{P + v\}_v = \mathcal{P}(\tau_{(u,P)})$ determined by $\{v \bullet n(u) = 0,\ v \bullet n'(u) = 0\}$ coincides the line $T_2^u \subset T_1^u$, defined by two equations $\{P(u) = 0,\ P'(u) = 0\}$ as in (3.10). By its definition, $T_2^u \subset \mathcal{D}_2$. Furthermore, each line T_2^u is *tangent* to the curve $\mathcal{D}_3$ at their intersection point P^u which corresponds to a polynomial of the form $P(z) = (z-u)^3$. Indeed, the vector $w(u) = (-3,\ 6u, -3u^2)$, tangent to the curve $\mathcal{D}_3$ at P^u, is orthogonal to $n(u)$ and $n'(u)$. This becomes evident using the identities $\partial_u\{(z-u)^3\} = -3z^2 + 6zu^2 - 3u^2 = (z^2, z, 1) \bullet (-3,\ 6u, -3u^2) = n(z) \bullet w(u)$ and $\partial_z\partial_u\{(z-u)^3\} = -6z + 6u^2 = (2z, 1, 0) \bullet (-3,\ 6u, -3u^2) = n'(z) \bullet w(u)$ — just substitute $z = u$.

One can check that, for *distinct* u, the systems (3.10) do not share a common solution (b, c, d), in other words, all the lines T_2^u are disjoint. In combination with the previous arguments this leads to a conclusion which could be predicted by examining the images in Figures 5 and 6.

Proposition 3.1. *The discriminant surface $\mathcal{D}_2$ is a ruled surface comprised of the disjoint lines $T_2^u \subset \mathbb{A}^3_{coef}$ defined by two constraints $P(u) = 0$, $P'(u) = 0$ as in (3.10). Each line T_2^u is tangent to the discriminant curve $\mathcal{D}_3$ at a point P^u, corresponding to the polynomial $P(z) = (z-u)^3$. In other words, $\mathcal{D}_2$ is spanned by lines tangent to $\mathcal{D}_3$.* □

The same conclusion can be reached following a different approach. Both treatments will play different and complementary roles in Section 6.

The curve $\mathcal{D}_3$ admits a parametrization $A(u) = (-3u,\ 3u^2, -u^3)$. The velocity vector $w(u)$ at $A(u)$ is equal to $\dot{A}(u) = (-3,\ 6u, -3u^2)$. A line, tangent to $\mathcal{D}_3$ at $A(u)$, has a t-parametric equation $A(u) + t\dot{A}(u)$. This line corresponds to the t-family of monic cubic polynomials of the form $P(z) - tP'(z) = (z-u)^3 - 3t(z-u)^2$.

A generic polynomial $Q(z) = (z-u)^2(z-u')$ in $\mathcal{D}_2$, for an appropriate choice of t, can be represented in the form $P(z) + tP'(z)$. To do it, we need to solve for t the z-functional equation $(z-u)^2(z-u') = (z-u)^3 + 3t(z-u)^2$. Miraculously, it has a unique solution $t = (u'-u)/3$! Therefore, any point Q in $\mathcal{D}_2$ lies on a line l^Q tangent to $\mathcal{D}_3$. At the same time, an attempt to solve for t the z-functional equation $(z-u)^2(z-u') = (z-u'')^3 + 3t(z-u'')^2$ fails when $u'' \neq u$. Thus, the tangent line to $\mathcal{D}_3$ through Q is unique. □

Let us make a few crucial observations about the way in which a plane can be tangent to the ruled surface $\mathcal{D}_2$. First we notice that, if a ruled surface $\mathcal{D}$ has a tangent plane T at a non-singular point $P \in \mathcal{D}$, then T must contain all the lines through P from the family which forms $\mathcal{D}$. Therefore, any plane T, tangent to $\mathcal{D}_2$ at $P \in \mathcal{D}_2^\circ$, contains the unique line T_2^u through P. In turn, T_2^u is tangent to $\mathcal{D}_3$ at a different point P^u. This geometry is depicted in Figure 7.

We have seen that $\mathcal{D}_2^\circ$ is a smooth surface. However, the surface $\mathcal{D}_2$ fails to be smooth at the points of the discriminant curve $\mathcal{D}_3$: it has a cusp-shaped fold along $\mathcal{D}_3$ (cf. Corollary 5.1 and Figure 8). Therefore, we need to clarify the notion of a "tangent" plane to $\mathcal{D}_2$ at the points of its singular locus $\mathcal{D}_3$.

We define the "tangent" plane to $\mathcal{D}_2$ at $P \in \mathcal{D}_3$ to be the unique plane spanned by the velocity and acceleration vectors at P of the parametric curve $\mathcal{D}_3$—the, so called, *osculating* plane of the curve. The osculating plane at $P \in \mathcal{D}_3$ happens to be the *limit*, as Q approaches P, of tangent planes at smooth points $Q \in \mathcal{D}_2^\circ$. This is another small miracle of the discriminant surface: although it is singular along $\mathcal{D}_3$,

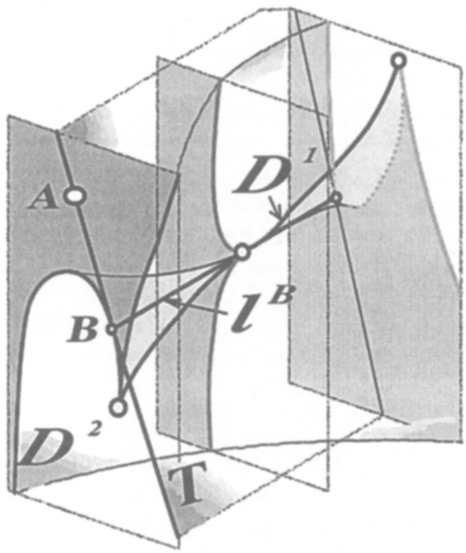


FIGURE 7. The slices of $\mathcal{D}^2$ by planes $\{b = const\}$. Any plane T through a point A and tangent to $\mathcal{D}_2$ is tangent along the whole line l^B. In turn, l^B is tangent to the curve $\mathcal{D}_3$.

the tangent planes of smooth points stabilize towards $\mathcal{D}_3$ (cf. Proposition 3.2)—the tangent bundle of $\mathcal{D}_2^\circ$ extends to a bundle over $\mathcal{D}_2$.

In order to verify these claims, consider a ts-parametric equation of an osculating plane through a point $A(u) = (-3u,\, 3u^2, -u^3)$ on the discriminant curve $\mathcal{D}_3$ — a plane which is spanned by the velocity $\dot{A}(u)$ and the acceleration $\ddot{A}(u)$ vectors:

$$(b, c, d) = (-3u,\, 3u^2, -u^3) + t(-3,\, 6u, -3u^2) + s(0,\, 6, -6u) \tag{3.12}$$

Expelling t and s from these three equations, we get a somewhat familiar relation between b, c, d and u: $\{u^3 + bu^2 + cu + d = 0\}$! Conversely, if u is a root of an equation $z^3 + bz^2 + cz + d = 0$, then (b, c, d) belongs to the osculating plane in (3.12) at $A(u)$: just put $t = u + \frac{1}{3}b$ and $s = \frac{1}{6}c + \frac{1}{3}bu + \frac{1}{2}u^2$.

We notice that the vector $n(u) = (u^2,\, u,\, 1)$, normal to $\mathcal{D}_2^\circ$ at the points of the line T_2^u , is also normal to both vectors: $\dot{A}(u) = (-3,\, 6u, -3u^2)$, $\ddot{A}(u) = (0,\, 6, -6u)$. Therefore, the osculating plane at $A(u) \in \mathcal{D}_3$ coincides with the affine plane T_1^u tangent to $\mathcal{D}_2^\circ$ along the line T_2^u. We have proved the following

Proposition 3.2. *If u is a root of the polynomial $P(z) = z^3 + bz^2 + cz + d$, then the osculating plane of the curve $\mathcal{D}_3$ at the point $A(u) = (-3u,\, 3u^2, -u^3)$ contains the point (b, c, d). That plane coincides with the affine plane T_1^u and thus, is tangent to the surface $\mathcal{D}_2^\circ$ along the line T_2^u. In turn, T_2^u is tangent to the curve at $A(u)$.* □

The embedding $\mathcal{D}_2 \subset \mathbb{A}^3_{coef}$ has a characteristic property described in

Proposition 3.3. Any *plane in $\mathbb{A}^3_{coef}$, passing through the point $P = (b, c, d)$ and tangent to the surface $\mathcal{D}_2$, is of the form $T_1^u := \{bu^2 + cu + d = -u^3\}$, where u is a root of the polynomial $P(z) = z^3 + bz^2 + cz + d$*

Proof. The considerations above already contain the proof. We have seen that the planes $\{T_1^u\}$ are exactly the planes tangent to $\mathcal{D}_2^\circ$. Moreover, by Proposition 3.2, the tangent cones of $\mathcal{D}_2$ at the points of $\mathcal{D}_3$ belong to the same family of planes. On the other hand, every point $P \in \mathbb{A}^3_{coef}$ belongs to each of the planes T_1^u, where u ranges over the roots of $P(z)$. □

Even the existence of *finitely many* tangent planes through a generic P is an extraordinary fact: for a *general* surface S, there exists an 1-parametric family of planes passing through P and tangent to S. What distinguishes $\mathcal{D}_2$ from a general surface, is its ruled geometry— $\mathcal{D}_2$ is formed by the lines tangent to the spatial curve $\mathcal{D}_3$. Via the Gaussian map, the tangent planes of the surface $\mathcal{D}_2$ form an 1-dimensional set in the Grassmanian $Gr(3,2) \approx \mathbb{P}^2$, while a generic surface generates a 2-dimensional Gaussian image.

Now we are ready to restate a fundamental relation between the ruled stratified geometry of the determinant variety $\mathcal{D}_2$ and the roots of the cubic monic polynomials. The proposition below is just a repackaging of the propositions that already have been established.

We start with the complex case which, as usual, is more uniform.

Theorem 3.1.
- *Through any point P of the stratum $\mathbb{C}^3_{coef} \setminus \mathcal{D}_2$, there are exactly 3 planes, tangent to the discriminant surface $\mathcal{D}_2$.*
- *Through any point P of the stratum $\mathcal{D}_2^\circ$, there are exactly 2 planes, tangent to the surface $\mathcal{D}_2$. One of these planes is tangent to $\mathcal{D}_2$ along a line passing through P.*
- *Finally, through any point P of $\mathcal{D}_3$, there is a single plane, tangent to the surface $\mathcal{D}_2$. It is the osculating plane of the curve $\mathcal{D}_3$ at P.*

Each of these planes T_1^u is tangent to the surface $\mathcal{D}_2^\circ$ along a line $T_2^u \subset \mathcal{D}_2$. In turn, the line is tangent to the discriminant curve $\mathcal{D}_3$.

Moreover, if $P = (b, c, d) \in \mathbb{C}^3_{coef}$, then each of the tangent planes T_1^u passing through P is described by an equation of the form $\{u^2 b + uc + d = -u^3\}$, where u runs over the distinct complex roots of the polynomial $P(z) = z^3 + bz^2 + cz + d$.

In fact, each affine plane T_1^u is the osculating plane of the discriminant curve at the point of $\mathcal{D}_3$ corresponding to the polynomial $P(z) = (z-u)^3$. □

The case of cubic polynomials with real coefficients is similar, but has a bit more structure and complexity. At the same time, the proof is virtually the same, as in the complex case. The stratum $\mathbb{R}^3_{coef} \setminus \mathcal{D}_2$ is divided into two chambers $\mathcal{U}_3$ and $\mathcal{U}_1$: the first corresponds to real cubic polynomials with 3 distinct real roots, the second—with a single simple real root.

Theorem 3.2.
- *Through any point $P \in \mathcal{U}_3$, there are exactly 3 planes, tangent to the discriminant surface $\mathcal{D}_2$.*
- *Through any point $P \in \mathcal{U}_1$, there is exactly 1 plane, tangent to $\mathcal{D}_2$.*
- *Through any point $P \in \mathcal{D}_2^\circ$, there are exactly 2 planes, tangent to $\mathcal{D}_2$. One of these planes is tangent to $\mathcal{D}_2$ along a line passing through P.*
- *Finally, through any point P of $\mathcal{D}_3$, there is a single plane, tangent to the surface $\mathcal{D}_2$. It is the osculating plane of the curve $\mathcal{D}_3$ at P.*

Each of these planes T_1^u is tangent to the surface $\mathcal{D}_2^\circ$ along a line $T_2^u \subset \mathcal{D}_2$. In turn, the line is tangent to the discriminant curve $\mathcal{D}_3$.

Moreover, if $P = (b, c, d) \in \mathbb{R}^3_{coef}$, then each of the tangent planes T^u_1 passing through P is described by an equation of the form $\{u^2b + uc + d = -u^3\}$, where u runs over the distinct real roots of the polynomial $P(z) = z^3 + bz^2 + cz + d$.

In fact, each affine plane T^u_1 is the osculating plane of the discriminant curve at the point of $\mathcal{D}_3$ corresponding to the polynomial $P(z) = (z - u)^3$. □

Corollary 3.1. ***(Cardano's formula "via tangents")***
It is possible to reconstruct all the roots of an equation $z^3 + bz^2 + cz + d = 0$ from the planes passing through the point $P = (b, c, d) \in \mathbb{A}^3_{coef}$ and tangent to the stratified discriminant pair $\mathcal{D}_2 \supset \mathcal{D}_3$ — the tangent planes trough P "solve" the cubic equation. Specifically, pick a vector normal to such a tangent plane and having the d-coordinate 1. Then its c-coordinate delivers the corresponding root.
In particular, the tangent planes through $P = (0, 0, d)$ have normal vectors $(\xi^2, \xi, 1)$, where $\xi = \sqrt[3]{d}$ (complex or real). □

Let's take a flight over the discriminant surface to admire its triangular horizon. First, we need a few definitions to inform the trip.

Given a smooth surface $\mathcal{S}$ in $\mathbb{C}^3$ and a point x outside $\mathcal{S}$, one can associate to each point $y \in \mathcal{S}$ the unique line $l_{x,y}$ through x and y. This defines a map $\pi_x : \mathcal{S} \to \mathbb{P}^2_x$ into the projective space $\mathbb{P}^2_x$ of lines through x. We consider the (Zariski) closure $hor(\mathcal{S}, x)$ of the set of critical points for the projection π_x and call it *the horizon* of $\mathcal{S}$ at x. Its interior is formed by points $y \in \mathcal{S}$ for which the line $l_{x,y}$ is tangent to $\mathcal{S}$ at y. The π_x-image of $hor(\mathcal{S}, x)$ in $\subset \mathbb{P}^2_x$, denoted $hor_\pi(\mathcal{S}, x)$, is called *the projective horizon* of $\mathcal{S}$ at x. Over the real numbers, one gets a refined version of these constructions and notions by replacing the space of lines $\mathbb{P}^2_x$ through x by the space of rays. This has an effect of replacing the projective plane by the sphere S^2_x.
If a surface $\mathcal{S}$ has a singular locus K, then we define $hor(\mathcal{S}, x)$ and $hor_\pi(\mathcal{S}, x)$ as the (Zariski) closure of $hor(\mathcal{S} \setminus K, x)$ and $hor_\pi(\mathcal{S} \setminus K, x)$ in $\mathcal{S}$ and $\mathbb{P}^2_x$ respectively.

Recall, that the discriminant surface $\mathcal{D}_2$ has a distinct property: if a plane $T = T^u_1$ is tangent to it at a point Q, then it is tangent to the surface along the entire line $l^Q = T^u_2$ which passes through Q. Therefore, if $Q \in hor(\mathcal{D}_2, P)$, then the line $l^Q \subset hor(\mathcal{D}_2, P)$ and the projective line $\pi_Q(l^Q) \subset hor_\pi(\mathcal{D}_2, P)$. Over the reals, $\pi_Q(l^Q)$ it is a big circle.
We notice that the "naked singularity" $\mathcal{D}_3$, visible from any point P in the coefficient space, it *tangent* to the perceived singularity—the projective horizon. In the complex case if $P \notin \mathcal{D}^2$, or in the real case when $P \in \mathcal{U}_3$, the discriminant curve $\mathcal{D}_3$ is tangent to the horizon at three points (belonging to the three distinct lines which form the horizon). In the real case, when $P \in \mathcal{U}_1$, the discriminant curve is tangent to the horizon line at a single point. Thus, over the complex numbers, the plane curve $\pi_x(\mathcal{D}_3) \subset \mathbb{P}^2_x$ is *inscribed* in the triangular projective horizon. A similar property holds in the real case when $P \in \mathcal{U}_3$. Hence, another distinct property of the discriminant surface:

Corollary 3.2. *For any point $P \in \mathbb{C}^3_{coef} \setminus \mathcal{D}^2$, the horizon $hor(\mathcal{D}_2, P)$ consists of three lines in a general position in $\mathbb{C}^3_{coef}$. The projective horizon $hor_\pi(\mathcal{D}_2, P)$ is a union of three projective lines occupying a general position in $\mathbb{P}^2_P$. The spatial curve $\mathcal{D}_3$ is inscribed in $hor(\mathcal{D}_2, P)$, while the plane curve $\pi_P(\mathcal{D}_3)$ is inscribed in the "triangular" projective horizon $hor_\pi(\mathcal{D}_2, P)$.*

For any point $P \in \mathcal{U}_3$, the horizon $hor(\mathcal{D}^2, P)$ consists of three lines in a general position in $\mathbb{R}^3$. The projective horizon $hor_\pi(\mathcal{D}^2, P)$ is a union of three big circles occupying a general position in S^2_P.

For any point $P \in \mathcal{U}_1$, the horizon $hor(\mathcal{D}^2, P)$ consists of a single line in $\mathbb{R}^3$, while the projective horizon $hor_\pi(\mathcal{D}^2, P)$ is a big circle in S^2_x.

The spatial curve $\mathcal{D}_3$ is inscribed in $hor(\mathcal{D}^2, P)$, while the plane curve $\pi_P(\mathcal{D}_3)$ is inscribed in $hor_\pi(\mathcal{D}_2, P)$. □

In general, one might conjecture that the degree of a spatial algebraic curve $\mathcal{C}$ is perceived as the number of lines in the generic projective horizon of the surface spanned by the lines tangent to $\mathcal{C}$.

4. The cubic Viète map

All the results of Section 3 can be understood from a different perspective. In Section 2, we described the geometry of the quadratic Viète map. Now we will investigate the geometry of the cubic Viète Map.

Let u, v, w be complex roots of a cubic polynomial $P(z) = z^3 + bz^2 + cz + d$. Then $P(z) = (z-u)(z-v)(z-w)$. Multiplying the three linear terms, we get $P(z) = z^3 - (u+v+w)z^2 + (uv+vw+wu)z - uvw$. This gives the Viète formulas

$$\begin{aligned} b &= -u-v-w \\ c &= uv+vw+wu \\ d &= -uvw, \end{aligned} \tag{4.1}$$

linking roots to coefficients. We think about (3.1) as giving rise to a *polynomial map* $\mathcal{V}$ from the uvw-root space $\mathbb{A}^3_{root}$ to the bcd-coefficient space $\mathbb{A}^3_{coef}$. We call it the *Viète map.*

By the Fundamental Theorem of Algebra, for any triple (b, c, d) there exists a triple of complex numbers (u, v, w), which satisfies the system (4.1), in other words, the *complex* Viète map is *onto.* This is not the case for the real Viète map.

Because the factorization of $P(z)$ into a product of monic linear polynomials is unique up to their ordering, the triple (b, c, d) determines the triple (u, v, w) up to permutations in three letters. They form a permutation group S_3 of order six. Generically, over the complex numbers, the preimage $\mathcal{V}^{-1}(b, c, d)$ consists of 6 elements. This happens when $(b, c, d) = \mathcal{V}(u, v, w)$ with u, v, w being distinct. When two of the roots coincide (that is, when the roots of $P(z)$ are of multiplicities 1 and 2), $\mathcal{V}^{-1}(b, c, d)$ consists of three elements. Finally, when the polynomial has a single root of multiplicity 3, $\mathcal{V}^{-1}(b, c, d)$ is a singleton.

The Fundamental Theorem of Algebra has a fancy formulation in terms of symmetric products of the space $\mathbb{C}$ (or even better, of the projective space $\mathbb{P}_1$).

Recall, that the n-th symmetric product S^nX of a set X is defined to be the n-th cartesian product X^n of X, divided by the natural action of the symmetry group S_n. In other words, while points of X^n are *ordered* n-tuples of points from X, points of S^nX are *unordered* n-tuples.

In these terms, the algebraic root-to-coefficient map $\mathcal{V}$ establishes an 1-to-1 and onto correspondence $\tilde{\mathcal{V}} : S^n\mathbb{C}_{root} \to \mathbb{C}^n_{coef}$. In particular, via the Viète map $\mathcal{V}$ in (4.1), $S^3\mathbb{C}$ and $\mathbb{C}^3$ are isomorphic sets.

The obvious forgetful map $f : \mathbb{C}^n \to S^n\mathbb{C}$, which strips an ordered n-tuple of its order, generically, is $(n!)$-to-1.

The embedding $\mathcal{D}_3 \subset \mathbb{A}^3_{coef}$ provides us with a very geometric way of interpreting the Viète map. This interpretation is based on Proposition 3.2 and Theorems 3.1, 3.2.

The discriminant curve $\mathcal{D}_3$ is a rational curve. It admits a 1-to-1 parametrization $A = \mathcal{F}_3 : \mathbb{A}^1 \to \mathcal{D}_3$ as in (3.9). Given any *unordered* triple of distinct points $A(u), A(v), A(w) \in \mathcal{D}_3$, the corresponding osculating planes T^u_1, T^v_1, T^w_1 of the curve at $A(u), A(v), A(w)$ all intersect at a singleton $\Psi(A(u), A(v), A(w))$ representing the polynomial $P(z) = (z-u)(z-v)(z-w)$. If $u = w$, we define $\Psi(A(u), A(v), A(u))$ to be the singleton where the osculating plane T^v_1 hits the tangent line T^u_2. Of course, this point on $\mathcal{D}^\circ_2$ corresponds to the polynomial $P(z) = (z-u)^2(z-v)$. Finally, when $u = v = w$, $\Psi(A(u), A(u), A(u))$ is defined to be $A(u)$ which corresponds to $P(z) = (z-u)^3$.

This gives rise to well-defined algebraic map $\Psi : S^3\mathcal{D}_3 \to \mathbb{C}^3_{coef}$ from the symmetric cube of the discriminant curve [4] onto the coefficient space.

Theorem 4.1. *The Viète map $\mathcal{V} : \mathbb{C}^3_{root} \to \mathbb{C}^3_{coef}$ is a composition of the forgetful map $f : \mathbb{C}^3_{root} \to S^3\mathbb{C}_{root}$, the obvious A-parametrization map $S^3A : S^3\mathbb{C}_{root} \to S^3\mathcal{D}_3$, and the osculating planes map $\Psi : S^3\mathcal{D}_3 \to \mathbb{C}^3_{coef}$. All the three maps are onto and the maps S^3A and Ψ are 1-to-1.* □

In order to describe a crude geometry of the Viète map, we shall concentrate on the loci in the coefficient space, where the cardinality $|\mathcal{V}^{-1}(b,c,d)|$ of the preimage $\mathcal{V}^{-1}(b,c,d)$ jumps, that is, on the *ramification* loci. The previous argument tells us that, over the complex numbers, the condition $|\mathcal{V}^{-1}(b,c,d)| = 1$ picks the set of polynomials with a single root of multiplicity 3, the condition $|\mathcal{V}^{-1}(b,c,d)| = 3$ picks the set of polynomials with one root of multiplicity 2, finally, the condition $|\mathcal{V}^{-1}(b,c,d)| = 6$ selects the set of polynomials with 3 distinct simple roots. These strata of $\mathbb{C}^3_{coef}$ are familiar under the names $\mathcal{D}_3$, $\mathcal{D}^\circ_2$ and $\mathcal{D}^\circ_1 := \mathbb{C}^3 \setminus \mathcal{D}_2$.

Note that, if a real cubic polynomial $P(z) = z^3 + bz^2 + cz + d$ has a single simple root, the triple (b,c,d) is not in the image of the real Viète map.

The Jacobi matrix $D\mathcal{V}$ of the Viète map $\mathcal{V}$ is

$$\begin{pmatrix} -1 & -1 & -1 \\ v+w & w+u & u+v \\ -vw & -wu & -uv \end{pmatrix}$$

and its determinant, the Jacobian $J\mathcal{V}$, is equal to $(v-u)(w-v)(u-w)$. Therefore, away from the three planes $\Pi_{uv} := \{v = u\}$, $\Pi_{vw} := \{w = v\}$, $\Pi_{wu} := \{u = w\}$ the rank of the Viète map is 3. On each the three planes it drops to 2, and at along the diagonal line $L := \{u = v = w\}$—to 1. Because of the S_3-symmetry, all the three planes have identical images under the $\mathcal{V}$.

The Jacobian $J\mathcal{V}$ is not invariant under the permutations of the variables u, v, w. In fact, it changes sign under the transpositions of any two variables. Therefore, $J\mathcal{V}$ can *not* be expressed in terms of the elementary symmetric polynomials in u, v, w, that is, in terms of the coefficients b, c, d. However, its square, *the discriminant,*

$$(J\mathcal{V})^2 = [(v-u)(w-v)(u-w)]^2 \tag{4.2}$$

[4] Points of $S^3(\mathcal{D}_3)$ are effective divisors of degree 3 on the curve $\mathcal{D}_3$.

is invariant, and thus, is a polynomial Δ in b, c, d. A painful calculation (cf. [V]) shows that the discriminant

$$\Delta(b,c,d) = b^2c^2 - 4b^3d + 18bcd - 4c^3 - 27d^2. \tag{4.3}$$

Under the $\mathcal{V}$, the equations $\{\Delta(b,c,d) = 0\}$ and $\{(v-u)(w-v)(u-w) = 0\}$ are equivalent. Evidently, the latter equation selects the case of multiple roots. In other words, the *discriminant surface* of Section 3 can be defined by an equation of degree 4:

$$\mathcal{D}_2 := \{b^2c^2 - 4b^3d + 18bcd - 4c^3 - 27d^2 = 0\} \tag{4.4}$$

Who could imagine from the first glance that this unpleasant formula hides such a nice geometry?

Over $\mathbb{C}$, the surface $\mathcal{D}_2$ coincides with the $\mathcal{V}$-image of each of the planes Π_{uv}, Π_{vw}, Π_{wu}. Over $\mathbb{R}$, by a stroke of good luck, a similar conclusion holds: if a real cubic polynomial has a complex root of multiplicity ≥ 2, then all its roots must be real.

Lemma 4.1. *The surface $\mathcal{D}_2$ in (4.4) admits a uv-parameterization by*

$$(b,c,d) = (-2u - v,\ u^2 + 2uv,\ -u^2v). \tag{4.5}$$

Proof. Under the substitution (4.1), the equations (4.4) and $J\mathcal{V} = 0$ are equivalent. Clearly, the second equation says that one of the roots must be of multiplicity ≥ 2. Then, putting $u = w$, gives the desired parameterization of $\mathcal{D}_2$ by $\mathcal{V}|_{\Pi_{vw}}$. Note that this restriction is an 1-to-1 map and onto, both over $\mathbb{C}$ and $\mathbb{R}$. Indeed, any permutation from S_3 or acts trivially on triples of the form $\{(u,v,u)\}$, or takes them to triples which do not belong to the plane Π_{vw}. □

The discriminant curve $\mathcal{D}_3$ is the image of the diagonal line $L = \{u = v = w\}$ under the Viète map $\mathcal{V}$.

Lemma 4.2. *The surface $\mathcal{D}_2$ divides the space $\mathbb{R}^3_{coef}$ into two chambers $\mathcal{U}_3$ and $\mathcal{U}_1$, one of which represents real cubic polynomials with 3 real roots and the other — with a single real root. The chamber $\mathcal{U}_3$ is characterized by the inequality*

$$\{b^2c^2 - 4b^3d - 4c^3 + 18bcd - 27d^2 > 0\}.$$

In turn, the curve $\mathcal{D}_3$ divides $\mathcal{D}_2$ into two domains $\{\mathcal{D}_2^{\pm}\}$, one of which corresponds to the cubic polynomials of the form $(z-u)^2(z-v)$ with $u < v$, and the other — with $u > v$.

Proof. By its definition, the chamber $\mathcal{U}_3$ is the interior of the image $\mathcal{V}(\mathbb{R}^3_{coef})$. The chamber $\mathcal{U}_1$ is the interior of the image of the set $\{(u,v,\overline{v}) \in \mathbb{C}^3_{root}\}$, where $u \in \mathbb{R}$, under the complex Viète map. Clearly, the two sets $\{(u,v,\overline{v}) \in \mathbb{C}^3_{root}\}$ and $\{(u,v,w) \in \mathbb{R}^3_{root}\}$ intersect along the set of real roots with one of the roots being of multiplicity ≥ 2. By Lemma 4.1, the $\mathcal{V}$-image of those is the surface $\mathcal{D}_2$.

For distinct real roots u, v, w, the discriminant $(J\mathcal{V})^2 > 0$. At the same time, for a single real root u, $(J\mathcal{V})^2 = [(v-u)(\overline{v}-v)(u-\overline{v})]^2 = [(v-u)(\overline{v}-u)]^2(\overline{v}-v)^2 < 0$.

The proof of the claim about the domains $\{\mathcal{D}_2^{\pm}\}$ is even simpler. □

Many geometric properties of the discriminant curve and surface, established in Section 3, can be easily derived employing the Viète map. Here are a few examples.

Let u, v, w be the roots of a polynomial z^3+bz^2+cz+d. Recall that $\mathcal{D}_2 = \mathcal{V}(\Pi_{uw})$. This gives its uv-parameterization (4.5). Putting $u = v$ in (4.5), generates a familiar parameterization $(b, c, d) = A(u) = (-3u,\ 3u^2, -u^3)$ of the discriminant curve $\mathcal{D}_3$.

A t-parametric equation of a generic tangent line to the curve $\mathcal{D}_3$ can be written as $A(u, t) = (-3u,\ 3u^2, -u^3) + t(-3,\ 6u, -3u^2)$. Hence, the formula $(b, c, d) = (-3u-3t,\ 3u^2+6ut,\ -u^3-3u^2t)$ describes a ruled surface, which has to be compared with the discriminant surface $\mathcal{D}_2$ parameterized by (4.5). In order to show that the two surfaces coincide, we need to solve for t the system of equations:

$$\begin{aligned} -2u - v &= -3u - 3t \\ u^2 + 2uv &= 3u^2 + 6ut \\ -u^2v &= -u^3 - 3u^2t. \end{aligned} \tag{4.6}$$

The only solution is given by $t = (v-u)/3$, in other words, $A(u, \frac{1}{3}(v-u)) = \mathcal{V}(u, v)$. We have arrived to a familiar conclusion: $\mathcal{D}_2$ is a ruled surface comprised of lines, tangent to $\mathcal{D}_3$. Each of the lines is produced with the help of $A(u, t)$ by fixing a particular value of u and varying t. Equivalently, it can be produced with the help of $\mathcal{V}(u, v, u)$ by fixing u and varying v (note that (4.6) are linear expressions in t and v). Since $\mathcal{V} : \Pi_{uw} \to \mathcal{D}_2$ is a 1-to-1 map, distinct lines $\{u = u_\star\}$ in the uv-plane must have disjoint images $T_2^{u_\star} \subset \mathbb{A}^3_{coef}$. Therefore, for a given point $P \in \mathcal{D}_2$, there is a single line through P and tangent to $\mathcal{D}_3$.

Next, we will determine the equation of a generic plane T, tangent to the discriminant surface. At the point $\mathcal{V}(u, v, u)$, it is spanned by the two vectors $\partial_u\mathcal{V}(u, v, u) = (-2,\ 2u + 2v, -2uv)$ and $\partial_v\mathcal{V}(u, v, u) = (-1,\ 2u, -u^2)$. Unless $u = v$, the two tangent vectors are independent. As before, the vector $n(u) = (u^2, u, 1)$ is orthogonal to both vectors $\partial_u\mathcal{V}(u, v, u)$ and $\partial_v\mathcal{V}(u, v, u)$ and therefore, to $T = T(u, v)$. Furthermore, since $n(u)$ is v-independent, the normal vector $n(u)$ is constant along the line $T_2^u = \{\mathcal{V}(u, v, u)\}_v \subset \mathcal{D}_2$. Since $T(u, v) \supset T_2^u$, it must be v-independent, and therefore, deserves the familiar name T_1^u. As before, the normal vector field $n(u)$ extends across the singularity $\mathcal{V}(u, u, u)$, and so is the distribution of tangent planes.

Let $w_\star$ be a fixed number. Consider the image of the plane $W^{w_\star} := \{w = w_\star\}$ under the Viète map. Formulas (4.1) gives a uv-parametric description of that image:

$$(b, c, d) = (-[u + v] - w_\star,\ [uv] + w_\star[u + v],\ -w_\star \cdot [uv]). \tag{4.7}$$

Solving for $u + v$ and for uv, gives a very familiar linear relation

$$w_\star^3 + bw_\star^2 + cw_\star + d = 0$$

among the variables b, c, d. This leads to a still somewhat surprising conclusion: the image $\mathcal{V}(W^{w_\star})$ of the plane $W^{w_\star}$ is contained in the *plane*[5] $T_1^{w_\star} \subset \mathbb{A}^3_{coef}$ tangent to the discriminant surface!

The map $\mathcal{V} : W^{w_\star} \to T_1^{w_\star}$ is generically 2-to-1 map: the cyclic permutation group of order 2 acts on the triples of the form $\{(u, v, w_\star)\}_{u,v}$ by switching u and v. The map is 1-to-1 along the diagonal line $\Delta^{w_\star} := \{(u, u, w_\star)\}$ in $W^{w_\star}$.

In the complex case, the map $\mathcal{V} : W^{w_\star} \to T_1^{w_\star}$ is *onto* since it is algebraic and of the rank 2 at a generic point. In the real case, the semi-algebraic set $\mathcal{V}(W^{w_\star})$

[5] Note, that the $\mathcal{V}$-image of a *generic* plane in $\mathbb{A}^3_{root}$ is a surface whose degree is > 1.

occupies a region of the plane $T_1^{w_\star}$, bounded by the curve $\mathcal{V}(\Delta^{w_\star})$. In fact, this curve, given by the parametric equation $(b,c,d) = (-2u - w_\star,\ u^2 + 2w_\star u,\ -w_\star u^2)$, is a parabola, which resonates with our experience with the *quadratic* Viète map and its discriminant curve! Evidently, the region it bounds is the intersection of the chamber $\mathcal{U}_3 \subset \mathbb{R}^3$ (see Lemma 4.2) with the plane $T_1^{w_\star}$. It can be characterized by the linear equation $\{w_\star^2 b + w_\star c + d = -w_\star^3\}$ coupled with the quadric inequality $\{b^2c^2 - 4b^3d + 18bcd - 4c^3 - 27d^2 > 0\}$.

All these observations are assembled in

Theorem 4.2. *The complex Viète map $\mathcal{V}$ takes each plane $W^{w_\star} := \{w = w_\star\}$ in $\mathbb{C}^3_{root}$ onto the plane $T_1^{w_\star} := \{w_\star^2 b + w_\star c + d = -w_\star^3\}$ in $\mathbb{C}^3_{coef}$, which is tangent to the discriminant surface $\mathcal{D}_2$. The map $\mathcal{V} : W^{w_\star} \to T_1^{w_\star}$ is a 2-to-1 map, ramified along the quadratic curve $\mathcal{V}(\Delta^{w_\star}) \subset \mathcal{D}_2 \cap T_1^{w_\star}$.*

The real Viète map $\mathcal{V}$ takes each plane $W^{w_\star} \subset \mathbb{R}^3_{root}$ onto the region of the tangent plane $T_1^{w_\star}$, bounded by the parabola $\mathcal{V}(\Delta^{w_\star})$. As in the complex case, the map $\mathcal{V} : W^{w_\star} \to T_1^{w_\star}$ is a 2-to-1 map, ramified along the parabola. □

Corollary 4.1. *The images of the three planes W^u, W^v, W^w under the Viète map are contained (in the complex case, coincide with) in the the planes tangent to $\mathcal{D}_2$ and passing through the point $P = \mathcal{V}(u,v,w)$.* □

5. A slice of reality and the reduction flow

In the search for formulas solving polynomial equations of degrees $d \leq 4$, the first step is to replace a generic equation by an equation of the *reduced* form. The reduced form is based on polynomials of degree d with no monomials of degree $d-1$.

The substitution $x = z - b/3$ transforms a generic cubic polynomial $P(z) = z^3 + bz^2 + cz + d$ to its reduced form $Q(x) = x^3 + px + q$. In this form the dimensions of the root and coefficient spaces are reduced by one. We can depict them using the comfortable geometry of the plane.

Since, in the reduced case, the sum of the roots $u + v + w = 0$, we can take two of the roots, say u and v, for the independent variables in the root plane. Then the reduced Viète map $\mathcal{V}$ can be written as

$$(c,d) = (uv - [u+v]^2,\ uv[u+v]). \tag{5.1}$$

The discriminant in (4.3) collapses to

$$[(v-u)(w-v)(u-w)]^2 = -4d^3 - 27c^2. \tag{5.2}$$

Motivated by the success of the substitution $z \to z - b/3$, we will examine the geometry of an 1-parametric group of transformations $\{\Phi_t\}$ of the coefficient space, induced by the t-family of substitutions $\{z \to z + t\}$. A typical transformation is described by the formula

$$\Phi_t(b,c,d) = (b - 3t,\ \ c - 2tb + 3t^2,\ \ d - tc + t^2b - t^3). \tag{5.3}$$

Formula (5.3) is the result of a straightforward computation of $P(z+t)$. By the Taylor formula, while $(b,c,d) = (\frac{1}{2}P''(0),\ P'(0),\ P(0))$,

$$\Phi_t(b,c,d) = \left(\frac{1}{2}P''(-t),\ -P'(-t),\ P(-t)\right). \tag{5.4}$$

Lemma 5.1. *The transformation $\Phi_t : \mathbb{A}^3_{coef} \to \mathbb{A}^3_{coef}$ preserves the Hermitian or Euclidean volume in the coefficient space.*

Proof. For each t, the linear part $(b, c, d) \to (b,\ c - 2tb,\ d - tc + t^2 b)$ of the affine transformation Φ_t, defined by (5.3), has a lower-triangular matrix with the units along the diagonal. Thus, its determinant is equal to 1. □

A direct verification proves

Lemma 5.2. *For a fixed point $A = (b, c, d)$, the t-parametric curve $\Phi_t(b, c, d)$ is a solution of a system of linear differential equations:*

$$\dot{A}(t) = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} A(t) + \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}, \tag{5.5}$$

satisfying the initial condition $A(0) = (b, c, d)^*$. □

We call the flow $\{\Phi_t\}$ defined by (5.5) (equivalently, by (5.3) or (5.4)) the *reduction flow.*

Denote by $\Psi_t : \mathbb{A}^3_{root} \to \mathbb{A}^3_{root}$ the translation by the vector (t, t, t). Note that Ψ_t and Φ_t are conjugated via the Viète map: $\mathcal{V} \circ \Psi_t = \Phi_t \circ \mathcal{V}$. Since the diagonal planes Π_{uv}, Π_{vw}, Π_{wu} are obviously invariant under the Ψ_t-flow, their $\mathcal{V}$-images are invariant under the Φ_t-flow. Therefore, employing Lemma 4.1, the Φ_t-flow must preserve the discriminant surface $\mathcal{D}_2$ as well, as the discriminant curve $\mathcal{D}_3$.

This observation can be reinforced. The function $(v - u)(w - v)(u - w)$ is clearly invariant under the flow Ψ_t. Therefore, the discriminant $\Delta(b, c, d)$ must be invariant under Φ_t-flow. In particular, every surface of constant level of the polynomial $\Delta(b, c, d)$ is invariant under this flow. As a result, the whole web of planes tangent to $\mathcal{D}_2$ and lines tangent to $\mathcal{D}_3$ is preserved under the transformation group $\{\Phi_t\}$: each map Φ_t defined by (5.3) is an affine transformation.

The proposition below captures these observations.

Proposition 5.1. *The polynomial $\Delta(b, c, d) = b^2c^2 - 4b^3d + 18bcd - 4c^3 - 27d^2$ is invariant under the 1-parametric group Φ_t defined by (5.3)—(5.5). In particular, the strata $\mathcal{D}_3 \subset \mathcal{D}_2$ are Φ_t-invariant. Therefore, the web of planes tangent to the discriminant surface, as well as the web of lines, tangent to the discriminant curve, is preserved by the Φ_t-action.* □

For a fixed number k, consider the plane $H^k \subset \mathbb{A}^3_{coef}$ defined by $\{b = k\}$. The reduced polynomials form the plane H^0.

We intend to slice the ruled stratification $\mathcal{D}_3 \subset \mathcal{D}_2 \subset \mathbb{A}^3$ by the planes $\{H^k\}$ and to investigate a typical slice together with its evolution under the flow $\{\Phi_t\}$.

We notice that each orbit $\{\Phi_t(P)\}_t$, where $P = (b, c, d)$, hits the plane H^k at a single point. Indeed, (5.3) admits a single t for which the b-coordinate of $\Phi_t(P)$ is k. Moreover, (5.3) implies that the orbit and the plane H^k are transversal at the intersection.

Consider a curve $\mathcal{D}_2^{[k]} = \mathcal{D}_2 \cap H^k$ and a point $\mathcal{D}_3^{[k]} = \mathcal{D}_3 \cap H^k$ in the slice H^k. The curve $\mathcal{D}_2^{[k]}$ is a cubic *cusp*: just add the constraint $b = k$ to the parametrization (4.5) in Lemma 4.1.

For any plane $T \subset \mathbb{A}^3_{coef}$, denote by $T^{[k]}$ its slice $T \cap H^k$.

Note that any plane T_1^u, tangent to $\mathcal{D}_2$, is in general position with the plane H^k: it contains the line T_2^u (tangent to $\mathcal{D}_3$) which is transversal to H^k. Thus, for any u, k, the intersection $T_1^{u[k]} = T_1^u \cap H^k$ is a line. Furthermore, this line $T_1^{u[k]}$ and the curve $\mathcal{D}_2^{[k]}$ must be *tangent*: a transversal slice of two tangent surfaces produces a pair of tangent curves. Their point of tangency is the intersection of a line T_2^u, along which T_1^u and $\mathcal{D}_2$ are tangent, with the slice H^k.

In the complex case, through any point $P \in H^k \setminus \mathcal{D}_2^{[k]}$ there are exactly three tangent planes. Therefore, their k-slice consists of three lines which contain P and are tangent to the discriminant cusp curve $\mathcal{D}_2^{[k]}$. Similarly, when $P \in \mathcal{D}_2^{[k]}$, there are two tangent planes through P, and their slice produces a pair of tangent lines to $\mathcal{D}_2^{[k]}$ (one of which is tangent at P). Finally, when $P \in \mathcal{D}_3^{[k]}$, the tangent plane through P is unique. Its slice is the line in H^k which contains the singularity P of the cusp curve and is the limit of its tangents as they approach P.

The real case has a slightly different description. To visualize it, compare Figures 7 and 8.

The real cusp $\mathcal{D}_2^{[k]}$ divides the plane H^k into two regions $\mathcal{U}_3^{[k]} := \mathcal{U}_3 \cap H^k$ and $\mathcal{U}_1^{[k]} := \mathcal{U}_1 \cap H^k$. There are three lines tangent to the curve $\mathcal{D}_2^{[k]}$ through every point in region $\mathcal{U}_3^{[k]}$, and only one tangent line through every point of $\mathcal{U}_1^{[k]}$. For any point on the cusp curve $\mathcal{D}_2^{\circ[k]}$, there are two tangent lines. Finally, through $\mathcal{D}_3^{[k]}$ there is a single line "tangent" to $\mathcal{D}_2^{[k]}$ at the apex.

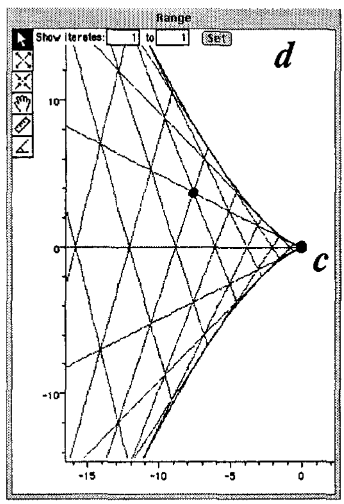


FIGURE 8. This S_3-symmetric pattern is formed by the tangents to the discriminant curve $\mathcal{D}_2^{[0]}$. Their *slopes differ by a fixed amount.* Each point in the domain $4c^3 + 27d^2 < 0$ is hit by tree tangent lines.

In short, the ruled stratified geometry of the slice is the slice of the ambient ruled stratified geometry. Moreover, the reduction flow respects these geometries.

Proposition 5.2. *The section $\mathcal{D}_2^{[k]}$ of the discriminant surface $\mathcal{D}_2$ by the cd-plane $H^k := \{b = k\}$ is a u-parametric cubic curve $\{(c,d) = (-3u^2 - 2ku,\ 2u^3 + ku^2)\}$.*

The roots of the polynomial $P(z) = z^3 + kz^2 + cz + d$ are equal to minus the slopes of the lines tangent to the curve $\mathcal{D}_2^{[k]}$ and passing through the point $P = (k, c, d)$.

The reduction flow Φ_t takes each curve $\mathcal{D}_2^{[k]}$ to the curve $\mathcal{D}_2^{[k-3t]}$ and respects their webs of tangent lines. Specifically, if u is a root of a polynomial $P(z) = z^3 + bz^2 + cz + d$, then the line $\{d = -uc - u^2b - u^3\}$ through $P = (b, c, d)$, residing in the plane H^b and tangent to the curve $\mathcal{D}_2^{[b]}$, is mapped by Φ_t to the line $\{d = -(u+t)c - (u+t)^2b - (u+t)^3\}$ in H^{b-3t}, passing through the point $\Phi_t(P)$ and tangent to the curve $\mathcal{D}_2^{[b-3t]}$. Thus, Φ_t is acting on the tangent lines by subtracting t from their slopes. □

Corollary 5.1. *There is an invertible polynomial transformation $\mathcal{K}$ of the bcd-space $\mathbb{A}^3_{coef}$ mapping the discriminant surface $\mathcal{D}_2$ onto a surface $\tilde{\mathcal{D}}_2$ which is a Cartesian product of the cubic $\{4d^3 + 27c^2 = 0\}$ in the cd-plane and the b-axis $\mathbb{A}^1$. At the same time, $\mathcal{K}$ maps $\mathcal{D}_3$ onto the b-axis.*

Proof. Define a transformation $\mathcal{K}$ of the coefficient space as follows. This $\mathcal{K}$ moves any point $P = (b, c, d) \in H^b$ along its $\{\Phi_t\}$-trajectory until it arrives at a point Q in the plane H^0; then it shifts Q to a point R in the plane H^b with the same cd-coordinates as the ones of Q. The substitution of $t = b/3$ in the formula (5.3) helps to compute $\mathcal{K}(b, c, d)$ explicitly:

$$\mathcal{K}(b,c,d) = (b', c', d') = \left(b,\ \ c - \frac{5}{9}b^2,\ \ d - \frac{1}{3}bc - \frac{2}{27}b^3\right). \tag{5.6}$$

Because of its "upper triangular shape", the polynomial map $\mathcal{K}$ is invertible in the class of polynomial maps: one can uniquely express (b, c, d) in terms of (b', c', d'). By Theorem 5.2, this $\mathcal{K}$ has the desired properties. □

Thus, over the real numbers, there is a smooth homeomorphism of the pairs $(\mathcal{D}_2 \subset \mathbb{R}^3) \approx (\mathbb{R}^2 \subset \mathbb{R}^3)$—the real surface $\mathcal{D}_2$ is topologically flat in the ambient space.

6. Stratified and ruled: the discriminants $\{\mathcal{D}_{d,k}\}$

Notations in this section are similar, but more ornate than the corresponding notations in Sections 2—5 (dealing with polynomials of degrees 2 and 3).

We consider the vector space $\mathbb{A}^d_{coef} = \mathcal{D}_{d,1}$ of monic polynomials

$$P(z) = z^d + a_1 z^{d-1} + \dots + a_{d-1}z + a_d$$

of degree d and its stratification $\{\mathcal{D}_{d,k}\}_{1 \leq k \leq d}$. Each strata $\mathcal{D}_{d,k}$ consists of polynomials with at least one of the roots being of multiplicity $\geq k$. Let $\mathcal{D}^\circ_{d,k} = \mathcal{D}_{d,k} \setminus \mathcal{D}_{d,k+1}$.

As before, we can consider a subvariety $S_{d,k}$ in $\mathbb{A}^1 \times \mathcal{D}_{d,1}$, defined by the system of equations

$$\{P(z) = 0,\ P'(z) = 0,\ P''(z) = 0,\ \dots,\ P^{(k-1)}(z) = 0\} \tag{6.1}$$

Here $\mathbb{A}^1$ stands for the complex or real z-coordinate line and $P^{(j)}(z)$ denotes the j-th derivative of $P(z)$.

Using the "upper triangular" pattern of (6.1), $\{a_d, a_{d-1}, \dots, a_{d-k}\}$ can be uniquely expressed as polynomials in $\{z, a_1, a_2, \dots, a_{d-k-1}\}$. This produces a polynomial 1-to-1 parametrization $\mathcal{H}_{d,k} : \mathbb{A}^{d-k} \to S_{d,k}$ of the smooth variety $S_{d,k}$ of dimension $d - k$.

Evidently, that $\mathcal{D}_{d,k} \subset \mathcal{D}_{d,1}$ is the image of $S_{d,k}$ under the projection $\mathcal{P} : \mathbb{A}^1 \times \mathcal{D}_{d,1} \to \mathcal{D}_{d,1}$. As in the case of quadratic and cubic polynomials, for any $u \in \mathbb{A}^1$, the hyperplane $\{z = u\}$ hits $S_{d,k}$ along an $(d-k-1)$-dimensional *affine space*

$N^u_{d,k}$ — (6.1) are linear equations in the coefficients of $P(z)$. Hence, $S_{d,k}$ is a ruled variety.

The projection $\mathcal{P}$ maps isomorphically $N^u_{d,k}$ onto an affine subspace $T^u_{d,k} \subset \mathcal{D}_{d,k}$ of the coefficient space $\mathcal{D}_{d,1}$. This subspace parameterizes all monic polynomials of degree d having the root u of multiplicity $\geq k$. Since any $P(z) \in \mathcal{D}_{d,k}$ has at least one root of multiplicity $\geq k$, the variety $\mathcal{D}_{d,k}$ is also comprised of the affine spaces $\{T^u_{d,k}\}$ of codimension one. However, they are not necessarily disjoint: a point in $\mathcal{D}_{d,k}$ can belong to many affine hypersurfaces.

Each polynomial $P(z) \in \mathcal{D}_{d,k}$ has *distinct* roots (real or complex) of multiplicities $\{\mu_1 \geq \mu_2 \geq ... \geq \mu_r\}$, with $\mu_1 \geq k$ and $r \leq d$. Note that $P(z) \in \mathcal{D}^\circ_{d,k}$, iff $\mu_1 = k$. In the complex case, the multiplicities $\{\mu_i\}$ define a *partition* $\mu_P := \{\sum_{i=1}^r \mu_i = d\}$ of d. We also interpret μ_P as a non-increasing function $\mu(i) = \mu_i$ on the set $\{1, 2, ..., d\}$, which takes non-negative integral values. In the real case, the same interpretation holds, except that μ_P is a partition of a number which counts only the real roots with their multiplicities.

As a μ-weighted configuration of roots deforms, a root of multiplicity μ_i and a root of multiplicity μ_j can merge producing a single root of multiplicity $\mu_i + \mu_j$. This results in a new partition μ' which we define to be *smaller* than the original partition μ. In a similar manner, several multiple roots can merge into a single one. Thus, the set of d-partitions acquires a *partial ordering*: $\mu \succ \mu'$.

For instance, the d-partition $\mu_{[k]}$, defined by the string of its values $(k, 1, 1, \ldots, 1)$, dominates any other d-partition which starts with k.

In the real case, complex roots occur in conjugate pairs of the same multiplicity, or are confined to the real number line $\mathbb{R}$. In what follows, while discussing the real case, we will use only the "$\mathbb{R}$-visible" part of the root configuration residing in $\mathbb{R}$. Only this part is captured by μ. However, the partial ordering in the set of those μ's is induced in a way similar to the complex case. The only difference is that a pair of "invisible" conjugate roots of a multiplicity μ_i can merge into a "visible" real root of multiplicity $2\mu_i$. For example, for $d = 4$, the "real" $\mu = (2, 0, 0, 0)$, corresponding to the configurations of one real root of multiplicity 2 and a pair of simple conjugate roots, is greater than the real $\mu' = (2, 2, 0, 0)$, corresponding to the configurations of two real roots of multiplicity 2.

By the definition of the projection $\mathcal{P}$, for any $P \in \mathcal{D}_{d,k}$, the cardinality of the preimage $\mathcal{P}^{-1}(P) \subset S_{d,k}$ is the number of distinct roots of multiplicity $\geq k$ possessed by $P(z)$, that is, $|\mu_P^{-1}([k, d])|$. By the same token, the number of spaces $T^u_{d,k}$'s to which $P \in \mathcal{D}^\circ_{d,k}$ belongs is exactly $|\mu_P^{-1}(k)|$. At the same time, the number of hyperspaces $T^u_{d,1}$'s, passing through $P \in \mathcal{D}_{d,1}$, is $|\mu_P|$—the cardinality of the support of the function μ_P. Since for $k > 1$, a *generic* point $P \in \mathcal{D}^\circ_{d,k}$ corresponds to the partition $\mu_{[k]} = (k, 1, 1, \ldots, 1)$, $|\mu_P^{-1}(k)| = 1$ and there is a *single* space $T^u_{d,k}$ passing through P.

Because $\mathcal{P} : N^u_{d,k} \to T^u_{d,k}$ is an isomorphism, the differential $D\mathcal{P}$ of the projection $\mathcal{P} : S_{d,k} \to \mathcal{D}_{d,k}$ can only be of the ranks $d - k$ or $d - k - 1$. The locus of points in $S_{d,k}$ where the rank drops is characterized by the property $D\mathcal{P}(\partial_z) = 0$. This happens when the gradients $\{\nabla_j\}$ of the k functions in (6.1) defining $S_{d,k}$ are orthogonal to the vertical vector ∂_z. The z-component of the gradient vector $\{\nabla_j\}$ is exactly $P^{(j+1)}(z)$. Hence, the locus in question is characterized by a system as (6.1) with $k - 1$ being replaced by k. Therefore, it is the set $S_{d,k+1} \subset S_{d,k}$. Put

$S^{\circ}_{d,k} := S_{d,k} \setminus S_{d,k+1}$. As a result, *locally*, $\mathcal{P} : S^{\circ}_{d,k} \to \mathcal{D}^{\circ}_{d,k}$ is a smooth 1-to-1 map of maximal rank (an *immersion*). In particular, $\mathcal{P} : S^{\circ}_{d,1} \to \mathcal{D}^{\circ}_{d,1}$ is a covering map with a fiber of the cardinality n. Hence, the singular locus of $\mathcal{D}_{d,k}$ consists of $\mathcal{D}_{d,k+1}$ together with the self-intersections $\Sigma^{\circ}_{d,k}$ of $\mathcal{D}^{\circ}_{d,k}$, where each branch of $\mathcal{D}^{\circ}_{d,k}$ has a well-defined tangent space. In fact, $\Sigma^{\circ}_{d,k}$, consists of polynomials $P(z) \in \mathcal{D}^{\circ}_{d,k}$, for which $|\mu_P^{-1}(k)| > 1, \;\; k > 1$.

We notice that, for $k > d/2$, $|\mu_P^{-1}(k)| = 1$. Therefore, the immersion $\mathcal{P} : S^{\circ}_{d,k} \to \mathcal{D}^{\circ}_{d,k}$ is a regular *embedding*, provided $k > d/2$.

Lemma 6.1. *For $k > d/2$, $\mathcal{D}^{\circ}_{d,k}$ is a smooth quasi-affine*[6] *subvariety of $\mathbb{A}^d_{coef}$. Hence, for $k > d/2$, the singular locus of $\mathcal{D}_{d,k}$ is $\mathcal{D}_{d,k+1}$.* □

Example 6.1. $(\mathbf{d} = 4)$.

$\mathcal{D}^{\circ}_{4,1} \subset \mathbb{C}^4_{coef}$ is comprised of polynomials P with the partition $\mu_P = (1,1,1,1)$. The hypersurface $\mathcal{D}^{\circ}_{4,2}$ is comprised of polynomials with $\mu_P = (2,1,1,0)$ or $(2,2,0,0)$, and its self-crossing $\Sigma^{\circ}_{4,2}$—of polynomials with $\mu_P = (2,2,0,0)$. The nonsingular surface $\mathcal{D}^{\circ}_{4,3}$ is comprised of polynomials with $\mu_P = (3,1,0,0)$. Finally, the smooth curve $\mathcal{D}_{4,4}$ corresponds to the partition $(4,0,0,0)$. Thus, each point of $\mathcal{D}^{\circ}_{4,1}$ belongs to four hyperplanes of the type $T^u_{4,1}$; each point of the space $\mathcal{D}^{\circ}_{4,2} \setminus \Sigma^{\circ}_{4,2}$ —to a single plane $T^u_{4,2}$ and each point of the surface $\Sigma^{\circ}_{4,2}$ —to two planes $T^u_{4,2}$; each point of the surface $\mathcal{D}^{\circ}_{4,3}$ belongs to a single line of the type $T^u_{4,3}$.

The case of real degree 4 polynomials is more intricate. Figure 9 shows a *slice* of this stratification by a hypersurface $\{a_1 = 0\}$ of reduced quadric polynomials.

The partitions μ_P which correspond to the three chambers of $\mathcal{D}^{\circ}_{4,1} \subset \mathbb{R}^4$ are: $(1,1,1,1)$ (four distinct real roots—the "triangular" chamber in Figure 9), $(1,1,0,0)$ (two distinct simple real roots—the chamber "below" the surface in Figure 9) and $(0,0,0,0)$ (no real roots—the chamber "above" the surface). The space $\mathcal{D}^{\circ}_{4,2}$ is comprised of four chambers. Three walls, bounding in $\mathcal{D}^{\circ}_{4,1}$ the chamber of four distinct real roots, all correspond to $\mu_P = (2,1,1,0)$. The three walls are distinguished by the three orderings in which a root of multiplicity 2 and two simple roots can be arranged on the number line $\mathbb{R}$. The fourth chamber of the hypersurface $\mathcal{D}^{\circ}_{4,2}$ corresponds to $\mu_P = (2,0,0,0)$ (in Figure 9, the two wings which, behind the triangular tail, merge into a smooth surface). Points of the surface $\Sigma^{\circ}_{4,2}$ — the transversal self-intersection of $\mathcal{D}^{\circ}_{4,2}$ — correspond to $\mu_P = (2,2,0,0)$. In Figure 9 they form the upper edge of the triangular chamber. Points of the surface $\mathcal{D}^{\circ}_{4,3}$ correspond to $\mu_P = (3,1,0,0)$. In Figure 9 they form the two lower cuspidal edges of the triangular chamber. Finally, the curve $\mathcal{D}^{\circ}_{4,4}$ corresponds to $\mu_P = (4,0,0,0)$. In Figure 9 it is the apex of the tail. □

Example 6.2. $(\mathbf{d} = 5)$.

To give a taste of structures to come, Figure 10 depicts a stratification of the space $\mathbb{C}^5_{coef}$ by the 5-partitions $\{\mu_P\}$ (this time represented by the Young type tableau—the graphs of the 5-partition functions). They form a partially ordered set with its elements decreasing from the left to the right. By definition, $\mu \succ \mu'$, if a tableau μ' can be built from a tableau μ by moving a few vertical bars to the *left*. Each μ is indexing a quasi-affine variety $\mathcal{D}^{\circ}_{\mu}$ in $\mathbb{C}^5_{coef}$ formed by polynomials whose roots have the multiplicities prescribed by μ. Its closure $\mathcal{D}_{\mu}$ is a union $\cup_{\mu' \preceq \mu} \mathcal{D}^{\circ}_{\mu'}$.

[6] that is, a Zariski-open set of an affine variety

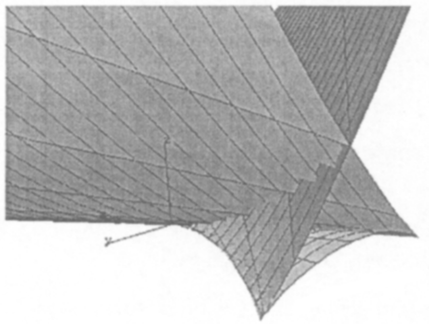


FIGURE 9. This swallow's tail is the section of the real $\mathcal{D}_{4,2}$ by the hypersurface $\{a_1 = 0\}$

The μ's with hook shaped tableaus produce the familiar stratification $\{\mathcal{D}_{5,k}\}$. The dimension of $\mathcal{D}_\mu$ is the number of columns in the tableau μ. We will revisit this example many times. $\square$

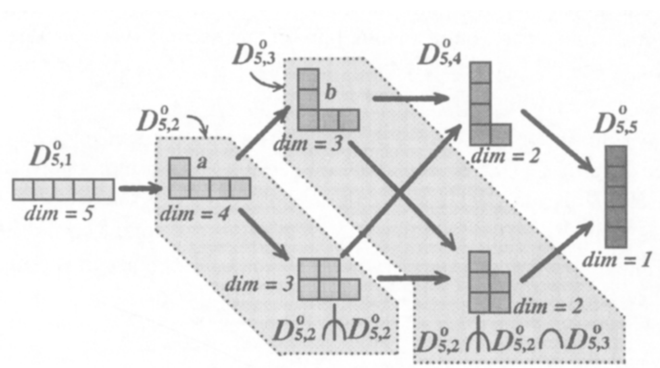


FIGURE 10. The stratification of $\mathbb{A}^5_{coef}$ by the 5-partitions $\{\mu_P\}$.

We intend to show that each affine space $T^u_{d,k}$ is *tangent* to the stratum $\mathcal{D}_{d,k+1}$ at some smooth point. Recall that, the intersection multiplicity of this web of tangent spaces at a point $P \in \mathcal{D}^\circ_{d,k}$ is $|\mu_P^{-1}(k)|$. Moreover, we will see that the tangent cones to $\mathcal{D}_{d,k+1}$ span $\mathcal{D}_{d,k}$.

A normal space $\nu(S_{d,k})$ to the nonsingular variety $S_{d,k} \subset \mathbb{A}^1 \times \mathcal{D}_{d,1}$ defined by (6.1), is spanned by k independent gradient vectors $\{\nabla_j\}, 1 \leq j \leq k$. As (6.1) implies, at each point $(z, P) \in \mathbb{A}^1 \times \mathcal{D}_{d,1} \approx \mathbb{A}^1 \times \mathbb{A}^d$,

$$\nabla_j(z, P) = (P^{(j)}(z),\ n^{(j-1)}(z)),$$

where $n^{(j-1)}(z)$ stands for the $(j-1)$-st derivative of the vector

$$n(z) = (z^{d-1}, z^{d-2}, \ldots, z, 1). \tag{6.2}$$

However, at a point $(u, P) \in S_{d,k}$, $P^{(j)}(u) = 0$, provided $j < k$. Thus,

$$\begin{aligned} \nabla_j(z, P) &= (0,\ n^{(j-1)}(z)), \text{ when } j < k; \\ \nabla_k(z, P) &= (P^{(k)}(z),\ n^{(k-1)}(z)) \end{aligned} \tag{6.3}$$

Therefore, the tangent space $\tau_x(S_{d,k})$ to $S_{d,k}$ at a point $x = (u, P)$ is spanned by vectors $w = (u, P) + (a, v) \in \mathbb{A}^1 \times \mathbb{A}^d$, subject to constraints

$$\begin{aligned} v \bullet n^{(j-1)}(u) &= 0, \qquad 1 \leq j < k; \\ v \bullet n^{(k-1)}(u) &= -a \cdot P^{(k)}(u). \end{aligned} \tag{6.4}$$

Here "$\bullet$" denotes the standard scalar product of vectors. Evidently, if v is orthogonal to all the vectors $\{n^{(j-1)}(u)\}_{1 \leq j < k}$, then it is possible to find an appropriate a satisfying (6.4), provided $P^{(k)}(u) \neq 0$. When $P^{(k)}(u) = 0$, we have to consider the last equation from (6.4) and to free the a.

At the same time, the the $(d-k)$-dimensional space $T^u_{d,k-1}$ is defined by the linear equations

$$P(u) = 0,\ P^{(1)}(u) = 0,\ \ldots,\ P^{(k-2)}(u) = 0.$$

In terms of a vector $\tilde{v} \in \mathbb{A}^d$ they can be written as

$$\tilde{v} \bullet n^{(j-1)}(u) = -(u^d)^{(j-1)}, \qquad 1 \leq j < k. \tag{6.5}$$

Comparing the first $(k-1)$ equations from (6.4) with (6.5), we see that the first system of equations is the homogeneous part of the second system. Therefore, the images of the space $N^u_{d,k-1}$ and the tangent space $\tau_{(u,P)}(S_{d,k})$ under the projection $\mathcal{P} : \mathbb{A}^1 \times \mathcal{D}_{d,1} \to S_{d,1}$ coincide! Thus, $\mathcal{P}(\tau_{(u,P)}(S_{d,k})) = T^u_{d,k-1}$.

The projection $\mathcal{P} : \mathbb{A}^1 \times \mathcal{D}_{d,k} \to S_{d,k}$ takes the tangent space $\tau_x(S_{d,k})$ into the tangent cone $T_{k,P}$ of $\mathcal{D}_{d,k}$ at P. Since $\mathcal{P} : S^\circ_{d,k} \to \mathcal{D}^\circ_{d,k}$ is an immersion, the tangent cone $T_{k,P}$, $P \in \mathcal{D}^\circ_{d,k}$, is the union of the $\mathcal{P}$-images of the tangent spaces to $S^\circ_{d,k}$ at the points from $\mathcal{P}^{-1}(P)$. Therefore, for $P \in \mathcal{D}^\circ_{d,k}$, the cone $T_{k,P}$ is the union of $|\mu_P^{-1}(k)|$ affine spaces $\{T^u_{d,k-1}\}_{(u,P)}$, where u runs over the set of distinct P-roots of multiplicity k.

The inclusion $T^u_{d,k-1} \subset \mathcal{D}_{d,k-1}$ implies that, for $P \in \mathcal{D}^\circ_{d,k}$, the tangent cone $T_{k,P} \subset \mathcal{D}_{d,k-1}$.

As an affine space, $\{T^u_{d,k-1}\}_{(u,P)}$ is determined by the equations $\{v \bullet n^{(j-1)}(u) = 0\}_{1 \leq j < k}$. Such a set of equations depends only on u, not on P. Therefore, it is shared by all the polynomials $P \in \mathcal{D}^\circ_{d,k}$ which have the *same* root u of multiplicity k. In other words, along an open and dense set $T^u_{d,k} \cap \mathcal{D}^\circ_{d,k}$ in the $(d-k-1)$-space $T^u_{d,k}$, the tangent spaces $\{T^u_{d,k-1}\}_P$ are *parallel* and, therefore, *extend across (stabilize towards) the singularity* $T^u_{d,k} \cap \mathcal{D}_{d,k+1}$! Furthermore, since $T^u_{d,k} \subset T^u_{d,k-1}$, as affine spaces, all the $\{T^u_{d,k-1}\}_{P \in T^u_{d,k}}$'s *coincide.*

For $k > d/2$, by Lemma 6.1, $\mathcal{D}^\circ_{d,k}$ is smooth, and the tangent bundle $\tau(\mathcal{D}^\circ_{d,k})$ extends across the singularity $\mathcal{D}_{d,k+1} \subset \mathcal{D}_{d,k}$ to a vector bundle.

Although by now we understand the structure of the tangent cone $T_{k,P}$ at a generic point $P \in \mathcal{D}_{d,k}$ and the stabilization of its components along some preferred directions towards the singular set $\mathcal{D}_{d,k+1}$, the structure of the tangent cone $T_{k,P}$ at at singular points $P \in \mathcal{D}_{d,k+1}$ still remains uncertain. All what is clear that, for $P \in \mathcal{D}^\circ_{d,k+1}$, the cone $T_{k,P}$ contains the well-understood tangent subcone $T_{k+1,P}$.

With this in mind, let's investigate in a more direct fashion the tangent cone $T_{k,P}$ at a point $P(z) = (z-u)^k \hat{P}(z)$, where $\hat{P}(z)$ denotes a monic polynomial of degree $d-k$. When $P \in \mathcal{D}^{\circ}_{d,k}$, $\hat{P}(u) \neq 0$.

Let $P_t(z)$ be a smooth t-parametric curve in $\mathcal{D}_{d,k}$, emanating from the point $P(z)$. Locally, it can be written in the form $(z-u-a_t)^k[\hat{P}(z) + R_t(z)]$, where $R_t(z)$ is a polynomial of degree $d-k-1$ and $lim_{t\to 0}\, a_t = 0$, $lim_{t\to 0}\, R_t(z) = 0$. The components of the velocity vector $\dot{P}_t$ to the t-parametrized curve $P_t(z) \subset \mathcal{D}_{d,1}$ are the coefficients of the z-polynomial

$$\dot{P}_t(z) = k(z-u-a_t)^{k-1}\,\dot{a}_t[\hat{P}(z) + R_t(z)] + (z-u-a_t)^k \dot{R}_t(z).$$

Since the curve $P_t(z)$ is smooth at the origin,

$$\dot{P}_0(z) = lim_{t\to 0}\dot{P}_t(z) = (z-u)^{k-1}[k\dot{a}_0\hat{P}(z) + (z-u)\dot{R}_0(z)]$$

Thus, a τ-parametric equation of any *line* from the tangent cone at $P(z)$ has a form

$$\begin{aligned} &P(z) + \tau\dot{P}_0(z) = \\ (6.6)\qquad &(z-u)^k\hat{P}(z) + \tau(z-u)^{k-1}[k\dot{a}_0\hat{P}(z) + (z-u)\dot{R}_0(z)]. \end{aligned}$$

As a z-polynomial, it is divisible by $(z-u)^{k-1}$—the tangent line resides in $\mathcal{D}_{d,k-1}$. Therefore, for *any* $P \in \mathcal{D}_{d,k}$, $T_{k,P} \subset \mathcal{D}_{d,k-1}$.

Taking the (Zariski) closures, $\mathcal{D}_{d,k-1}$ contains the union of all tangent cones to $\mathcal{D}_{d,k}$. On the other hand, since any $P \in \mathcal{D}_{d,k-1}$ is contained in some $T^u_{d,k-1}$ which is tangent to $\mathcal{D}^{\circ}_{d,k}$, we conclude that $\mathcal{D}_{d,k-1} = \tau(\mathcal{D}_{d,k})$ — the union of all tangent cones to $\mathcal{D}_{d,k}$.

For $k > d/2$, through each point $Q \in \mathcal{D}_{d,k}$ there is a single space T^u_k tangent to $\mathcal{D}_{d,k+1}$. In particular, for any point $P \in \mathcal{D}_{d,k+1}$, there exist a single pair $T^u_{k+1} \subset T^u_k$ containing P. Consider an 1-dimensional space $L^u_k = L^P_k$ which contains P and is orthogonal to T^u_{k+1} in T^u_k. We claim that the union $\cup_{P\in\mathcal{D}_{d,k+1}} L^P_k = \mathcal{D}_{d,k}$. Furthermore, $\mathcal{D}_{d,k}$ is the space of a line bundle over $\mathcal{D}_{d,k+1}$ with a typical fiber L^P_k. Indeed, any $Q \in \mathcal{D}_{d,k}$ belongs to a unique affine space $T^u_k \supset T^u_{k+1}$. Take the line in T^u_k through Q orthogonal to T^u_{k+1}. It hits T^u_{k+1} at a point $P \in \mathcal{D}_{d,k+1}$. Thus, $Q \in L^P_k$. Since $k > d/2$, all the spaces $\{T^u_k\}$ are distinct and so are the lines $\{L^P_k\}$.

The preceding conclusions are summarized in the main result of this section—Theorem 6.1. In a way, it is a special case of our main result —Theorem 7.1, but has a different flavor. Therefore, it is presented here for the benefit of the reader.

Theorem 6.1. *Let $\mathbb{A}$ stand for the number field $\mathbb{C}$ or $\mathbb{R}$. Denote by $P \in \mathbb{A}^d_{coef}$ the point corresponding to a monic polynomial $P(z)$ of degree d.*

- *For any $1 \leq k \leq d$, the stratum $\mathcal{D}_{d,k}$ is a union of tangent cones to the stratum $\mathcal{D}_{d,k+1}$.*
- *Each stratum $\mathcal{D}^{\circ}_{d,k} \subset \mathbb{A}^d_{coef}$ is an immersed smooth manifold. For $k > d/2$, $\mathcal{D}^{\circ}_{d,k}$ is a smooth quasi-affine subvariety. Moreover, such a $\mathcal{D}_{d,k}$ is the space of a line bundle over $\mathcal{D}_{d,k+1}$.*
- *Through each point $P \in \mathcal{D}^{\circ}_{d,k}$, there are exactly $|\mu_P^{-1}(k)|$ affine spaces $\{T^u_{d,k}\}_u$ tangent to the stratum $\mathcal{D}_{d,k+1}$. The spaces $T^u_{d,k}$ are indexed by the distinct $P(z)$-roots $\{u\}$ of multiplicity k over the field $\mathbb{A}$. Each space $T^u_{d,k}$ is defined*

by the linear constraints $\{P(u) = 0,\ P^{(1)}(u) = 0,\ \ldots,\ P^{(k-1)}(u) = 0\}$ *imposed on the coefficients of* $P(z)$.

For $k > 1$, *through a generic point* $P \in \mathcal{D}^\circ_{d,k}$ *there is a single tangent space* $T^u_{d,k}$. *For* $k > d/2$, *every point* $P \in \mathcal{D}^\circ_{d,k}$ *belongs to a single space* $T^u_{d,k}$.

- *Each space* $T^u_{d,k}$ *is tangent to* $\mathcal{D}_{d,k+1}$ *along the subspace* $T^u_{d,k+1}$.
- *On the other side of the same coin, the tangent cone* $T_{k,P}$ *to* $\mathcal{D}^\circ_{d,k}$ *at a point* P *is the union of the affine spaces* $\{T^u_{d,k-1}\}_u$, *where* u *is ranging over the distinct* $P(z)$*-roots of multiplicity* k *over* $\mathbb{A}$. □

Corollary 6.1. *The problem of solving a polynomial equation* $P(z) = 0$ *over* $\mathbb{A}$ *is equivalent to the problem of finding all hyperplanes* T *passing through the corresponding point* $P \in \mathbb{A}^d_{coef}$ *and tangent*[7] *to the discriminant variety* $\mathcal{D}_{d,2}$.

Specifically, consider the normal vector to such a hyperplane, normalized by the condition that its d-th component equals 1. Then, its $(d-1)$*-st component gives a root* u *of* $P(z)$. *Via this construction, distinct roots* $\{u\}$ *of* $P(z)$ *over* $\mathbb{A}$ *and tangent hyperplanes* $\{T\}$ *through* P *are in 1-to-1 correspondence.* □

Let's return to Example 6.2 and Figure 10 to illustrate the claims of Theorem 6.1. The open strata $\mathcal{D}^\circ_{5,1}$, $\mathcal{D}^\circ_{5,3}$, $\mathcal{D}^\circ_{5,4}$, $\mathcal{D}_{5,5}$ are smooth, while the stratum $\mathcal{D}^\circ_{5,2}$ has a transversal self-intersection along a 3-fold $\mathcal{D}^\circ_{5,2} \cap \mathcal{D}^\circ_{5,2}$ which consists of points P with $\mu_P = (2,2,1,0,0)$. The 3-dimensional strata $\mathcal{D}_{5,2} \cap \mathcal{D}_{5,2}$ and $\mathcal{D}_{5,3}$ are not in general position even at a generic intersection point: their intersection is a surface, not a curve. A generic point of $\mathcal{D}_{5,2} \cap \mathcal{D}_{5,2} \cap \mathcal{D}_{5,3}$ corresponds to the partition $\mu_P = (3,2,0,0,0)$. Similarly, the intersection of the surfaces $\mathcal{D}_{5,4}$ and $\mathcal{D}_{5,2} \cap \mathcal{D}_{5,2} \cap \mathcal{D}_{5,3}$ is the curve $\mathcal{D}_{5,5}$. In short, the more refined stratification $\{\mathcal{D}_\mu\}$ corresponding to the 5-partitions can be recovered from the geometry of the crude stratification $\mathcal{D}_{5,1} \supset \mathcal{D}_{5,2} \supset \mathcal{D}_{5,3} \supset \mathcal{D}_{5,4} \supset \mathcal{D}_{5,5}$.

There are 5 hyperplanes tangent to $\mathcal{D}_{5,2}$ through every point of $\mathcal{D}^\circ_{5,1}$, 4 hyperplanes through every point P of $\mathcal{D}^\circ_{5,2}$ with $\mu_P = (2,1,1,1,0)$ (that is, through every point of $\mathcal{D}^\circ_{5,2} \setminus (\mathcal{D}^\circ_{5,2} \cap \mathcal{D}^\circ_{5,2})$), 3 hyperplanes through every point of $\mathcal{D}^\circ_{5,3}$ with $\mu_P = (3,1,1,0,0)$ or through every point of $\mathcal{D}^\circ_{5,2} \cap \mathcal{D}^\circ_{5,2}$ with the $\mu_P = (2,2,1,0,0)$, 2 hyperplanes through every point of $\mathcal{D}^\circ_{5,4}$ or through every point in $\mathcal{D}_{5,3}$ with $\mu_P = (3,2,0,0,0)$, and finally, 1 hyperplane through every point of $\mathcal{D}_{5,5}$. In short, the multiplicity of the web of tangent *hyper*planes to the discriminant hypersurface at P is the cardinality of the support of μ_P (which also happens to be the dimension of the stratum $\mathcal{D}_{\mu_P}$).

At the same time, there is a single 3-space tangent to $\mathcal{D}_{5,3}$ through each point of $\mathcal{D}^\circ_{5,2} \setminus (\mathcal{D}^\circ_{5,2} \cap \mathcal{D}^\circ_{5,2})$, two 3-spaces through each point of $\mathcal{D}^\circ_{5,2} \cap \mathcal{D}^\circ_{5,2}$, a single plane tangent to $\mathcal{D}_{5,4}$ through each point of $\mathcal{D}^\circ_{5,3}$, and a single line tangent to $\mathcal{D}_{5,5}$ through each point of $\mathcal{D}^\circ_{5,4}$. □

As in the discussion preceding Corollary 3.2, we can introduce the notions of a horizon and a projective horizon of a variety in $\mathbb{A}^d_{coef}$, as viewed from a point in its complement. Let's glance at a horizon of the discriminant hypersurface in $\mathbb{A}^d_{coef}$.

Corollary 6.2. *Over* $\mathbb{C}$, *any point* $P \in \mathcal{D}^\circ_{d,1}$ *has a horizon* $hor(\mathcal{D}^\circ_{d,2}, P)$ *comprised of* d *codimension 2 affine spaces in* $\mathbb{C}^d_{coef}$ *which are in a general position. The*

[7]that is, belonging to a tangent space of a smooth point in $\mathcal{D}_{d,2}$.

variety $\mathcal{D}_{d,3}$ is inscribed in the horizon $hor(\mathcal{D}^\circ_{d,2}, P)$. Similarly, for any $P \in \mathcal{D}^\circ_{d,1}$, the projective horizon $hor_\pi(\mathcal{D}^\circ_{d,2}, P) \subset \mathbb{P}^{d-1}_P$ consists of d hyperplanes in a general position. The variety $\pi_P(\mathcal{D}_{d,3})$ is inscribed in $hor_\pi(\mathcal{D}^\circ_{d,2}, P)$. □

Now consider the smallest stratum — $\mathcal{D}_{d,d}$. It is a smooth curve $\kappa(u)$ in $\mathcal{D}_{d,1}$ whose points correspond to polynomials of the form $(z-u)^d$. Its u-parametric representation is given by

$$\{a_k = (-1)^k \binom{d}{k} u^k\}_{1 \le k \le d} \tag{6.7}$$

With any point $\kappa(u)$ on the curve we associate a flag of vector subspaces $V^1_u \subset V^2_u \subset \dots \subset V^{d-1}_u \subset \mathbb{A}^d$ with the origins at $\kappa(u)$. Each osculating space V^k_u is spanned by the linearly independent vectors $\kappa^{(1)}(u), \kappa^{(2)}(u), \dots, \kappa^{(k)}(u)$ emanating from $\kappa(u)$. Here $\kappa^{(j)}(u)$ stands for the j-th derivative of $\kappa(u)$ with respect to u.

Remarkably, each vector $v \in V^{d-k}_u$, emanating from $\kappa(u)$, satisfies the first k orthogonality conditions from (6.4), that is,

$$v \bullet n^{(j)}(u) \ = \ 0, \qquad 0 \le j \le k-1. \tag{6.8}$$

In other words, as affine spaces, $V^{d-k}_u = T^u_{d,k}$!

In order to verify this claim, we have to check that $\kappa^{(q)}(u) \bullet n^{(p)}(u) = 0$ for each pair (q,p), subject to $1 \le q \le d-k$, $0 \le p \le k-1$. The identity is a repackaging of the obvious identities $\partial^p_z \partial^q_u \{(z-u)^d\}|_{\{z=u\}} = 0$, being interpreted as scalar products of two vectors. Here $p+q \ne d$.

These considerations, combined with Theorem 6.1, lead to Theorem 6.2 below. Similar statements can be found in [ACGH], pp. 136-137. Theorem 6.2 testifies that all the geometric and combinatorial complexity of the discriminant varieties $\{\mathcal{D}_{d,k}\}_k$ can be derived from the geometry of a single curve $\mathcal{D}_{d,d} \subset \mathbb{A}^d$!

Theorem 6.2. *Each affine space, tangent to the variety $\mathcal{D}_{d,k}$, for an appropriate u, is of the form V^{d-k+1}_u. In different words, it is the $(d-k+1)$-th osculating space of the rational curve $\mathcal{D}_{d,d}$ at the point $\kappa(u)$. Therefore,*

- *the ruled variety $\mathcal{D}_{d,k}$ is the union of all osculating spaces $\{V^{d-k}_u\}_u$ at the points of the curve $\mathcal{D}_{d,d}$;*
- *the tangent cone $T_{k,P}$ to $\mathcal{D}_{d,k}$ at $P \in \mathcal{D}^\circ_{d,k}$ is the union of $|\mu^{-1}(k)|$ affine spaces $\{V^{d-k+1}_u\}_u$, where u runs over the $P(z)$-roots of multiplicity k.* □

With Theorem 6.2 in place, the Fundamental Theorem of Algebra acquires a new geometric life.

Corollary 6.3. (A geometrization of the Fundamental Theorem of Algebra)

There exists an 1-to-1 algebraic map $\mathcal{W} : S^d(\mathcal{D}_{d,d}) \to \mathcal{D}_{d,1}$ from the d-th symmetric product of the complex rational curve $\mathcal{D}_{d,d} \subset \mathcal{D}_{d,1}$ onto the space $\mathcal{D}_{d,1} \approx \mathbb{C}^d$. It is defined by the following geometric operation.

For any point $X \in S^d(\mathcal{D}_{d,d})$, viewed as an unordered collection of points $\{\kappa(u) \in \mathcal{D}_{d,1}\}$ with their multiplicities $\{\mu_u \ge 1\}$[8]*, $\mathcal{W}(X)$ is defined to be the unique* intersection point *of the osculating spaces $\{V^{d-\mu_u}_u\}_u$, taken at the points $\{\kappa(u)\}$.*

[8]in other words, as an effective divisor $\sum_u \mu_u \kappa(u)$ of degree d on the curve $\mathcal{D}_{d,d}$

Proof. Any polynomial $P(z)$ is uniquely determined my its distinct roots $\{u\}$ with their multiplicities $\{\mu_u\}$. Therefore, there is a single point $P \in \mathcal{D}_{d,1}$ belonging to $\cap_u T^u_{d,\mu_u}$. Moreover, any $P \in \mathcal{D}_{d,1}$ is such an intersection. Now apply Theorem 6.2 which identifies T^u_{d,μ_u} with $V_u^{d-\mu_u}$. □

Let $P(V)$ denote the projective space associated with a finite dimensional vector space V. Denote by V^* the dual vector space. Every hyperplane $H \subset P(V)$ can be viewed a point $H^\vee \in P(V^*)$. Recall, that the *projective dual* $\mathcal{D}^\vee$ of a variety $\mathcal{D} \subset P(V)$ is a subvariety of $P(V^*)$, defined as a closure of the set $\{H^\vee\}$ formed by the hyperplanes $H \subset P(V)$ *tangent* to $\mathcal{D}$ at one of its smooth points. By definition, H is tangent to $\mathcal{D}$ at a smooth point x, if H contains the tangent space $\tau_x(\mathcal{D})$ of $\mathcal{D}$ at x. For any projective variety $\mathcal{D}$, one has $(\mathcal{D}^\vee)^\vee = \mathcal{D}$. Furthermore, if x is a smooth point of $\mathcal{D}$ and $H^\vee$—a smooth point of $\mathcal{D}^\vee$, then H is tangent to $\mathcal{D}$ at x if and only if x, regarded as a hyperplane $x^\vee$ in $P(V^*)$, is tangent to $\mathcal{D}^\vee$ at $H^\vee$ (cf. [GKZ], Theorem 1.1).

Generically, $\mathcal{D}^\vee$ is a hypersurface. When $codim(\mathcal{D}^\vee) > 1$, the original $\mathcal{D}$ is a ruled variety. In the case of the special determinantal varieties $\{\mathcal{D}_{d,k}\}$, or rather their projectivizations, the dimensions of $\{\mathcal{D}^\vee_{d,k}\}$ drop drastically. Indeed, as Theorem 6.2 implies, the tangent spaces of $\mathcal{D}_{d,k}$ at its smooth points form an 1-*parametric* family. For instance, $\mathcal{D}^\vee_{d,2}$ is just a curve!

Corollary 6.4.
- *The dimension* $dim(\mathcal{D}^\vee_{d,k}) = k-1$.
- *Its degree* $deg(\mathcal{D}^\vee_{d,k}) \leq deg(\mathcal{D}_{d,k-1}) = (k-1)(d-k+2)$, *provided* $k > 2$.
- $\mathcal{D}^\vee_{d,2}$ *is a curve* $\mathcal{C} \subset P((\mathbb{A}^{d+1})^*)$ *of degree d, and* $\mathcal{D}_{d,2} = (\mathcal{C})^\vee$.

Proof. We have seen that the family $\{V_u^{d-k+1}\}_u$ of tangent affine spaces to $\mathcal{D}_{d,k} \subset \mathcal{D}_{d,1}$ is 1-dimensional. Each of these spaces V_u^{d-k+1} is contained in a $(k-2)$-dimensional family $\mathcal{H}_u$ of hyperplanes. Hence, $k-2 \leq dim(\mathcal{D}^\vee_{d,k}) \leq k-1$. Since there are infinitely many spaces $\{V_v^{d-k+1}\}_v$ which are not contained in any of the hyperplanes from $\mathcal{H}_u$, $dim(\mathcal{D}^\vee_{d,k}) = k-1$.

The $deg(\mathcal{D}^\vee_{d,k})$ is the number of transversal intersection points of $\mathcal{D}^\vee_{d,k}$ with a generic affine space W^{d-k+1} contained in an affine chart of $\mathcal{D}^\vee_{d,1}$. Due to the projective duality, this number equals to the number of hyperplanes in $\mathcal{D}_{d,1}$ which contain a generic affine subspace $U^{k-2} \subset \mathcal{D}_{d,1}$ and are tangent to $\mathcal{D}_{d,k}$. We can construct U^{k-2} in a way that links it with $\mathcal{D}_{d,k-1}$.

Let $U^{k-2} \subset \mathcal{D}_{d,1}$ be a generic affine subspace which hits $\mathcal{D}_{d,k-1}$ transversally at $deg(\mathcal{D}_{d,k-1})$ points $\{P_\alpha\}$. By a general position argument (the Bertini Theorem 8.18 in [H]), we can assume that all the P_α's are smooth points in $\mathcal{D}^\circ_{d,k-1}$. For a smooth point P_α, $\mu^{-1}_{P_\alpha}(k-1) = 1$, provided $k > 2$. Therefore, in $\mathcal{D}_{d,k-1}$, there exists a single space $T^{u,\alpha}_{d,k-1}$ of dimension $d-k+1$ tangent to $\mathcal{D}_{d,k}$ and passing through P_α. Denote by H_α the minimal affine subspace in $\mathcal{D}_{d,1}$ which contains the transversal subspaces $T^{u,\alpha}_{d,k-1}$ and U^{k-2} (whose intersection is P_α). By its construction, H_α is a hyperplane which contains U^{k-2} and is tangent to $\mathcal{D}_{d,k}$. In fact, any hyperplane H, which contains U^{k-2} and is tangent to $\mathcal{D}_{d,k}$ at a point P, can be constructed in this way. Indeed, it must contain at least one of the spaces $T^u_{d,k-1}$ tangent to $\mathcal{D}_{d,k}$ at P. Because U^{k-2} has been constructed in general position with $\mathcal{D}_{d,k-1} \supset T^u_{d,k-1}$, U^{k-2} and $T^u_{d,k-1}$ must be in general position in H. Counting dimensions, U^{k-2} and $T^u_{d,k-1}$ have a single point P_α of intersection. Therefore, $deg(\mathcal{D}^\vee_{d,k}) \leq deg(\mathcal{D}_{d,k-1})$. In order to replace the inequality by an equality, one needs to verify that all these tangent hyperspaces are distinct. We conjecture that this is the case. By [Hi] (see also [W1], Theorem 2.2 and Corollary 2.3), $deg\mathcal{D}_{d,k} = k(d-k+1)$. Therefore, $deg(\mathcal{D}^\vee_{d,k}) \leq (k-1)(d-k+2)$. In particular, $deg(\mathcal{D}^\vee_{d,3}) \leq 2d-2$.

By a similar argument, the last claim of the theorem follows from the fact that a generic point of $\mathcal{D}_{d,1}$ is hit by d hyperplanes tangent to $\mathcal{D}_{d,2}$. □

As in Sections 2 and 4, the ruled geometry of the strata $\{\mathcal{D}_{d,k}\}$ can be approached using the Viète map $\mathcal{V}_d : \mathbb{A}^d_{root} \to \mathbb{A}^d_{coef}$. It is defined by the elementary symmetric polynomials $\{\sigma_k(u_1, u_2, \ldots, u_d)\}$ which express the coefficient a_k in terms of the roots $\{u_i\}$. The symmetric group S_d acts on the space $\mathbb{A}^d_{root}$ by permuting the coordinates. Denote by St_U the stabilizer in S_d of a point $U \in \mathbb{A}^d_{root}$. If $U = (u_1, u_2, \ldots, u_d)$, then as before, one can associate with U a non-increasing function (a tableau) $\mu^U : \{1, 2, \ldots, r\} \to \{0, 1, \ldots, d\}$ which counts the numbers of equal coordinates in the string U. In these terms, $St_U \approx \prod_{i=1}^r S_{\mu_i^U}$, where i runs over the support of μ^U. Note that the cardinality of the preimage $\mathcal{V}_d^{-1}(\mathcal{V}_d(U))$ is the order $|S_d/St_U| = d!/\prod_i\{(\mu_i^U)!\}$.

Distinct orbit-types $\{S_d/H\}_{H=St_U}$ give rise to a natural stratification $\{\mathbb{A}^{d,H^\circ}_{root}\}_H$ of the root space $\mathbb{A}^d_{root}$ and, because the Viète map is S_d-equivariant, — to a familiar stratification $\{\mathcal{D}^\circ_{\mu_U} := \mathbb{A}^{d,H^\circ}_{coef}\}_H$ of the coefficient space $\mathbb{A}^d_{coef}$ (cf. Figure 10). The coarse stratification $\{\mathcal{D}_{d,k}\}$ can be assembled from this more refined stratification $\{\mathcal{D}^\circ_\mu\}_\mu$. In fact, over the complex numbers, $\mathcal{D}_{d,k}$ consists of all points $P = \mathcal{V}_d(U)$ for which, up to a conjugation, $St_U \supseteq S_k$. Similarly, $\mathcal{D}^\circ_{d,k}$ is comprised of $P = \mathcal{V}_d(U)$ for which, up to a conjugation, $St_U \supseteq S_k$ and does not contain any subgroup S_j with $j > k$. Over the reals, the situation is more subtle: $\mathcal{V}_d$ fails to be *onto*. For instance, the Viète image of the hyperplane $\mathbb{A}^4_{root} \cap \{u_1 + u_2 + u_3 + u_4 = 0\}$ is not the whole space in Figure 9, but just the triangular chamber corresponding to $\mu_P = (1,1,1,1)$.

For a non-increasing function $\mu : \{1, 2, \ldots, r\} \to \{0, 1, \ldots, d\}$, so that $\sum_{q=1}^r \mu_q \leq d$, denote by K_μ the vector subspace of $\mathbb{A}^d_{root}$ defined by the equations $\{u_i = u_j\}$, where $\sum_{q=1}^p \mu_q \leq i, j \leq \sum_{q=1}^{p+1} \mu_q$ and p ranges over the support of μ. The $\mathcal{V}_d$-image of this K_μ belongs to $\mathcal{D}_{d,\mu_1}$.

For example, if $\mu = (4,2,2,1)$, then K_μ is defined by the equations $\{u_1 = u_2 = u_3 = u_4;\ u_5 = u_6;\ u_7 = u_8\}$. The $\mathcal{V}_9$-image of this K_μ (of codimension 5) belongs to $\mathcal{D}_{9,4}$ and forms there a subvariety of codimension 2. In contrast, if μ corresponds to the partition $(4,1,1,1,1,1)$, then $\mathcal{V}_9(K_\mu)$ has codimension 0 in $\mathcal{D}_{9,4}$.

Recall that $\mu_{[k]}$ is a partition, such that $\mu_1 = k$ and, for $i > 1$, $\mu_i = 1$. In the complex case, this $\mu_{[k]}$ describes a generic point of $\mathcal{D}_{d,k}$. Therefore, over the complex numbers, $\mathcal{V}_d(K_{\mu_{[k]}}) = \mathcal{D}_{d,k}$. Over the reals, simple complex roots generically occur in conjugate pairs, which allows for a greater variety of "generic" μ's. In the previous example, in addition to the partition $9 = 4+1+1+1+1+1$, we must also consider "equally generic" subpartitions $4+1+1+1$ and $4+1$.

For any number u, denote by Π^u_k the affine subspace of $\mathbb{A}^d_{root}$ defined by the equations $\{u_1 = u,\ u_2 = u,\ \ldots,\ u_k = u\}$. Evidently, $\mathcal{V}_d(\Pi^u_k) \subset \mathcal{D}_{d,k}$. Furthermore, by the definitions, $\mathcal{V}_d(\Pi^u_k) \subseteq T^u_{d,k}$ and, over the complex numbers, $\mathcal{V}_d(\Pi^u_k) = T^u_{d,k}$.

In view of the Theorem 6.1, we get the following proposition.

Theorem 6.3. *Over $\mathbb{C}$, the Viète image of the vector subspace $K_{\mu_{[k]}}$ is the discriminant variety $\mathcal{D}_{d,k}$. Any affine space T of dimension $d-k+1$, tangent to $\mathcal{D}_{d,k}$, is the image of some affine space $\Pi^u_{k-1} \subset \mathbb{C}^d_{root}$ under the Viète map $\mathcal{V}_d$.*

Over $\mathbb{R}$, $\mathcal{V}_d(K_{\mu_{[k]}})$ forms a chamber in the discriminant variety $\mathcal{D}_{d,k}$. Any affine space T of dimension $d-k+1$, tangent to $\mathcal{D}_{d,k}$, contains a chamber $\mathcal{V}_d(\Pi^u_{k-1})$. □

As in Section 5, the product $[\prod_{i,j}(u_i - u_j)]$ is invariant under the the 1-parametric family of substitutions $\{u_k \to u_k + t\}$. As a result, the discriminant $\Delta_d(a_1, a_2, ..., a_d)$ is an invariant under the invertible algebraic transformations $\{\Phi_t : \mathbb{A}^d_{coef} \to \mathbb{A}^d_{coef}\}$ induced by the substitutions $\{z \to z + t\}$. Hence, $\{\Phi_t\}$ preserve the hypersurface $\mathcal{D}_{d,2} \subset \mathbb{A}^d_{coef}$. Examining (6.1), we see that each variety $\mathcal{D}_{d,k}$ is invariant under the flow $\{\Phi_t\}$. In fact, each stratum $\mathcal{D}^\circ_\mu$ is invariant as well — the multiplicities of roots do not change under the substitutions $\{z \to z + t\}$. In particular, the curve $\mathcal{D}_{d,d}$ is a trajectory of the flow $\{\Phi_t\}$ which takes the point $(a_1, a_2, ..., a_d)$, representing a polynomial $P(z)$, to the point $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_d)$, where $\tilde{a}_k = \frac{(-1)^{d-k}}{(d-k)!} P^{(d-k)}(-t)$. For a fixed t, Φ_t is just an invertible linear transformation of $\mathbb{A}^d_{coef}$. Expanding the formula for $\tilde{a}_k = \tilde{a}_k(t)$ as a polynomial in t, we see that $\{\tilde{a}_k(t) = a_k + b_k(t)\}$, where $b_k(t)$ is a polynomial of degree k with no free term. Because of the "upper triangular" pattern of these formulas, the transformation Φ_t preserves the Euclidean volume form of the space $\mathbb{A}^d_{coef}$.

Since each trajectory $\{\Phi_t(P)\}_t$ hits the hyperplane $\{a_1 = 0\}$ at a singleton P_{red}, the whole stratification $\{\mathcal{D}^\circ_\mu\}$ acquires a product structure: $\mathcal{D}^\circ_\mu \approx \mathbb{A}^1 \times (\mathcal{D}^\circ_\mu)_{red}$, where $(\mathcal{D}^\circ_\mu)_{red} = \mathcal{D}^\circ_\mu \cap \{a_1 = 0\}$. In particular, the swallow tail surface in Figure 9, being multiplied by $\mathbb{R}^1$, is isomorphic to the real discriminant 3-fold $\mathcal{D}_{4,2}$.

These observations are captured in a well known lemma below which expresses the geometry of general polynomials in terms of the geometry of the reduced ones.

Lemma 6.2. *The reduction flow $\{\Phi_t : \mathbb{A}^d_{coef} \to \mathbb{A}^d_{coef}\}$ preserves the stratification $\{\mathcal{D}^\circ_\mu\}$ of the coefficient space, as well as its Euclidean volume form. In particular, the webs of affine spaces, tangent to the strata $\mathcal{D}_{d,k}$, remain invariant under the flow. $\{\Phi_t\}$ also establishes the algebraic isomorphisms $\mathcal{D}^\circ_\mu \approx \mathbb{A}^1 \times (\mathcal{D}^\circ_\mu)_{red}$.* □

This completes our description of the stratification $\{\mathcal{D}_{d,k}\}$.

7. THE WHOLE SHEBANG: TANGENCY AND DIVISIBILITY

We are ready to extend results of the previous section to a generic stratum $\mathcal{D}_\mu$. By now, we have developed immunity to combinatorial complexities. This resistance will help us to meet the challenge of the $\mathcal{D}_\mu$'s intricate geometry.

Let $|aut(\mu)|$ denote the order the symmetry group of a partition $\mu = \{\mu_1 + \mu_2 + ... + \mu_r\}$, i.e. all the permutations of the columns in the tableau μ which preserve its shape. Thus, $|aut(\mu)| = \prod_l [\#\mu^{-1}(l)]!$, where l runs over the distinct values of the function μ. Put $|\mu| = r$.

Lemma 7.1. *Each stratum $\mathcal{D}^\circ_\mu$ is a smooth quasi-affine variety in $\mathbb{A}^d_{coef}$ of dimension $|\mu|$.* [9] *Its degree $deg(\mathcal{D}_\mu) = r!\,\{\prod_{i=1}^r \mu_i\}/\{\prod_l [\#\mu^{-1}(l)]!\}$.*

Proof. We generalize arguments centered on formulas (6.2)—(6.4).

By definition, any polynomial from $\mathcal{D}^\circ_\mu$ is of the form $P(z) = \prod_{i=1}^r (z - u_i)^{\mu_i}$ with all the roots $\{u_i\}$ being distinct. We can regard $\{u_i\}$ as coordinates in a space $\mathbb{A}^r$. Let $(\mathbb{A}^r)^\odot$ be an open subset of $\mathbb{A}^r$ — the complement to the diagonal sets $\{u_i = u_j\}_{i \neq j}$.

[9] Note that, $\mathcal{D}^\circ_{d,k}$, which can be singular, in general, consists of several $\mathcal{D}^\circ_\mu$'s.

Denote by S_μ an r-dimensional subvariety of $\mathbb{A}^r \times \mathbb{A}^d_{coef}$ defined by the equations $\{P^{(j)}(u_i) = 0\}$, where $1 \le i \le r$ and $0 \le j < \mu_i$ (compare this with (6.1)). The equations claim that u_1 is a root of multiplicity μ_1, u_2 is of multiplicity μ_2, etc.

Put $S^\odot_\mu := S_\mu \cap [(\mathbb{A}^r)^\odot \times \mathbb{A}^d_{coef}]$.

The gradient of the function $P^{(j)}(u_i) : \mathbb{A}^r \times \mathbb{A}^d_{coef} \to \mathbb{A}^1$ is given by the formula $\nabla_j(u_i, P) = (w_i^{(j)}(u_i),\ n^{(j-1)}(u_i))$. Here the vector $w_i^{(j)}(u_i) \in \mathbb{A}^r$ has the number $P^{(j)}(u_i)$ as its i-th component, the rest of its coordinates vanish. The vector $n^{(j-1)}(u) \in \mathbb{A}^d$ is the $(j-1)$-st derivative of the familiar vector $n(u) = (u^{d-1}, u^{d-2}, ..., u, 1)$. At the points of $S^\odot_\mu$ the vectors $\{\nabla_j(u_i, P)\}$ are linearly independent; furthermore, their images $\{n^{(j-1)}(u_i)\}$ under the projection $\mathcal{P} : \mathbb{A}^r \times \mathbb{A}^d_{coef} \to \mathbb{A}^d_{coef}$ are independent as well. Thus, $S^\odot_\mu$ is smooth. A calculation similar to (6.4) shows that a tangent space to $S^\odot_\mu$ has a zero intersection with the appropriate fiber of the projection $\mathcal{P} : \mathbb{A}^r \times \mathbb{A}^d_{coef} \to \mathbb{A}^d_{coef}$.

By definition, the projection $\mathcal{P}$ takes $S^\odot_\mu := S_\mu \cap [(\mathbb{A}^r)^\odot \times \mathbb{A}^d]$ exactly onto $\mathcal{D}^\circ_\mu$. It is an $|aut(\mu)|$-to-1 covering map: each polynomial in $\mathcal{D}^\circ_\mu$ determines the ordered list of its distinct roots $(u_1, ..., u_r)$ up to permutations from $aut(\mu)$. Therefore, both $S^\odot_\mu$ and $\mathcal{D}^\circ_\mu$ are smooth quasi-affine varieties.

The degree $deg(\mathcal{D}_\mu)$ apparently has been computed by Hilbert in [Hi], however, I have to admit that I do not understand his arguments. A much more recent computation can be found in [W1], Theorem 2.2 and Corollary 2.3, pp. 377-78. There, using appropriate resolutions, the Hilbert function of $\mathcal{D}_\mu$ is calculated.

Here is an alternative argument which I found easier to describe.

Denote by $\vec{t}^{[k]}$ a k-vector $(t, t, ..., t)$. Let $P^\mu_j(t_1, t_2, ..., t_r)$ denote the polynomial $\sigma_j(\vec{t_1}^{[\mu_1]}, \vec{t_2}^{[\mu_2]}, ..., \vec{t_r}^{[\mu_r]})$, where σ_j is the j-th elementary symmetric polynomial in d variables. Evidently, $\{P^\mu_j(t_1, t_2, ..., t_r)\}_j$ define a parametrization $\mathcal{V}_\mu$ of $\mathcal{D}_\mu$. Therefore, the number of transversal intersections of a generic affine $(d-r)$-space with $\mathcal{D}^\circ_\mu$ is the number of solutions $(t_1, ..., t_r)$ of a generic linear system $\{\sum_{j=1}^d a_{ij} P^\mu_j(t_1, t_2, ..., t_r) = b_i\}_{1 \le i \le r}$, being divided by $\prod_l (\#\mu^{-1}(l))!$ — the degree of the map $\mathcal{V}_\mu$ (and the order of the μ-stabilizer). Each generic polynomial $\sum_{j=1}^d a_{ij} P^\mu_j(t_1, t_2, ..., t_r) - b_i$ contains the same set of monomials $\{t_1^{\nu_1} t_2^{\nu_2} ... t_r^{\nu_r}\}$, where $0 \le \nu_i \le \mu_i$. Therefore, they all share the same Newton polyhedron of the volume $\mu_1\mu_2 ... \mu_r$. By the Bernstein Theorem (cf. Theorem (5.4) in [CLO]), the number of solutions $(t_1, t_2, ..., t_r)$ ($\{t_i \ne 0\}$) of the generic system above is $r![\mu_1\mu_2 ... \mu_r]$. Without loss of generality, we can assume that all $t_i \ne 0$. Therefore, $deg(\mathcal{D}_\mu) = r![\mu_1\mu_2 ... \mu_r] / \prod_l (\#\mu^{-1}(l))!$. □

Now, we need to introduce a few combinatorial notations. For any d-partition $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ with $\mu_1 \ge \mu_2 \ge ... \ge \mu_r \ge 1$, denote by $\mu \downarrow \mathbf{1}$ a $(d-r)$-partition defined by the sequence $(\mu_1 - 1,\ \mu_2 - 1,\ \dots,\ \mu_r - 1)$. It is supported on a smaller set of indices: $|\mu \downarrow \mathbf{1}| = |\mu| - \#(\mu^{-1}(1))$. Also, let $\mu \uparrow \mathbf{1}$ be a $(d - \#(\mu^{-1}(1)))$-partition defined by the rule: $(\mu \uparrow \mathbf{1})_i = \mu_i + 1$, when $\mu_i \ge 2$ and $(\mu \uparrow \mathbf{1})_i = 0$ otherwise (cf. Figure 11).

Let $P(z) = \prod_i (z - u_i)^{\mu_i}$. Then $P^{\downarrow 1}(z)$ denotes the polynomial $\prod_i (z - u_i)^{\mu_i - 1}$ whose root multiplicities are described by $\mu \downarrow \mathbf{1}$. Also, let $\mathbf{1}_r$ denote the partition $(1, 1, \dots, 1)$ of r.

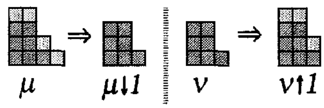


FIGURE 11

One can add partitions (cf. Figure 12): for a d-partition $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ and a d'-partition $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_{r'})$, define a $(d+d')$-partition $\mu \uplus \mu'$ by the formula

$$(\mu_1, \mu_2, \dots, \mu_r, \mu'_1, \mu'_2, \dots, \mu'_{r'}).$$

Of course, the sequence above is no longer a monotone one. To get from it a Young-type tableau we need to reorder its terms.

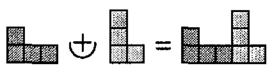


FIGURE 12

Given a partition κ of m and a partition τ of n, $n \geq m$, we introduce $\gamma(\kappa, \tau)$ as the number of distinct monic polynomials of degree m, whose root multiplicities are dictated by κ, and which divide a particular monic polynomial of degree n with the root multiplicities prescribed by τ. In other words, $\gamma(\kappa, \tau)$ counts the number of different functions $\kappa' : \{1, 2, ..., |\tau|\} \to \mathbb{Z}_+$, such that: 1) for every i, $\kappa'_i \leq \tau_i$ and 2) κ' is the form $\sigma(\kappa)$, where $\sigma \in S_{|\tau|}$ is a permutation (cf. Figure 13).

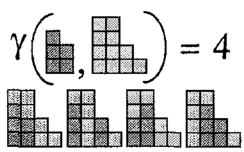


FIGURE 13

A close formula computing $\gamma(\kappa, \tau)$ in terms of the κ_i's and τ_j's is quite unappealing. When $\gamma(\kappa, \tau) \neq 0$, we will write $\kappa \trianglelefteq \tau$.

The proposition below summarizes most of what we know about the geometry of the varieties $\mathcal{D}^\circ_\mu$.

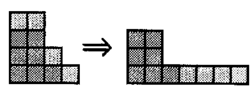
Theorem 7.1. **("Divisibility is tangency")**

- *Over $\mathbb{C}$, for any point $P \in \mathcal{D}^\circ_\mu$ representing a polynomial $P(z) = \prod_i (z - u_i)^{\mu_i}$* [10]*, the tangent space $T_P\mathcal{D}^\circ_\mu \subset \mathbb{C}^d_{coef}$ to $\mathcal{D}^\circ_\mu$ at P consists of all polynomials $Q(z)$ divisible by $P^{\downarrow 1}(z) = \prod_i (z - u_i)^{\mu_i - 1}$. Thus, it is defined in $\mathbb{C}^d_{coef}$ by a system of linear constraints $\{Q^{(j)}(u_i) = 0\}$, where the u_i's range over the* multiple *roots of $P(z)$ and* $0 \leq j \leq \mu_i - 2$.

[10]with all the u_i's being distinct

- *The space $T_P\mathcal{D}^\circ_\mu$ is* tangent *to $\mathcal{D}^\circ_\mu$ along the $|\mu^{-1}(1)|$-dimensional affine space V of polynomials divisible by $\prod_{\{i:\,\mu_i\geq 2\}}(z-u_i)^{\mu_i}$. In turn, V is defined by linear equations $\{Q^{(j)}(u_i)=0\}$, where the u_i's range over the* multiple *roots of $P(z)$ and $0\leq j\leq \mu_i-1$.*

  If the gaps between distinct values of the function μ all are greater than 1, then $T_P\mathcal{D}^\circ_\mu\cap\mathcal{D}_\mu=V$ — the space $T_P\mathcal{D}^\circ_\mu$ is tangent to the variety $\mathcal{D}^\circ_\mu$ at any point of their intersection.
- *The intersection $T_P\mathcal{D}^\circ_\mu\ \cap\ \mathcal{D}^\circ_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})}$ is open and dense in the affine space $T_P\mathcal{D}^\circ_\mu$. Hence, $\overline{T}\mathcal{D}^\circ_\mu$ — the closure of the union of all tangent spaces $\{T_P\mathcal{D}^\circ_\mu\}_P$ — coincides with $\mathcal{D}_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})}$. At the same time, $T\mathcal{D}^\circ_\mu=\coprod_{\{\mu':\,(\mu\downarrow 1)\,\trianglelefteq\,\mu'\}}\mathcal{D}^\circ_{\mu'}$.*
- *Let a partition ν be such that $2\cdot\#(\nu^{-1}(1))\geq|\nu|$. Then, reversing the flow in the previous bullet, $\mathcal{D}^\circ_\nu$ is an open and dense subset of $\overline{T}\mathcal{D}^\circ_\mu$, where $\mu=(\nu\uparrow\mathbf{1})\uplus\mathbf{1}_s$ with $s=2\cdot\#(\nu^{-1}(1))-|\nu|$. Hence, such $\mathcal{D}_\nu$'s are ruled varieties.*
- *For any d-partitions μ,μ' and each point $Q\in\mathcal{D}^\circ_{\mu'}$, there are exactly $\gamma(\mu\downarrow\mathbf{1},\ \mu')$ $|\mu|$-dimensional spaces which are tangent to $\mathcal{D}^\circ_\mu$ and contain Q.*



μ $\quad(\mu\downarrow 1)\uplus 1_{|\mu|}$

FIGURE 14

Remark. The theorem claims that a generic $\mathcal{D}^\circ_\mu$ has a very different geometry than the special $\{\mathcal{D}^\circ_{d,k}\supset\mathcal{D}^\circ_{(k,1,\dots,1)}\}$ indexed by the "hook-shaped" μ's. For example, with $d=4$, the tangent planes $T_P\mathcal{D}^\circ_{(2,2,0,0)}$ sweep $\mathcal{D}_{(1,1,1,1)}\setminus\mathcal{D}_{(4,0,0,0)}$ — the surface $\mathcal{D}_{(2,2,0,0)}$ is "more curved" in $\mathbb{C}^4_{coef}$ than the surface $\mathcal{D}_{(3,1,0,0)}$ whose tangents span just a 3-fold (cf. Theorem 6.1).

Proof. The argument is a refinement of arguments centered on formula (6.6). For any $P\in\mathcal{D}^\circ_\mu$ representing a polynomial $P(z)=\prod_{i=1}^r(z-u_i)^{\mu_i}$ with all its roots $\{u_i\}$ being distinct, consider a smooth t-parametrized curve in $\mathcal{D}^\circ_\mu$ emanating from P and given by the formula: $P_t(z)=\prod_{i=1}^r(z-u_i+a_i(t))^{\mu_i}$. Let $\dot{a}_i:=\frac{d}{dt}a_i(t)|_{t=0}$ and $\dot{P}(z):=\frac{\partial}{\partial t}P_t(z)|_{t=0}$. Then $\dot{P}(z)=\prod_{i=1}^r(z-u_i)^{\mu_i}[\sum_{i=1}^r\mu_i\dot{a}_i(z-u_i)^{-1}]$. So, the τ-parametric tangent line $P+\tau\dot{P}_t$ is represented by the polynomials

$$\prod_{i=1}^{r}(z-u_i)^{\mu_i}[1+\tau\sum_{i=1}^{r}\mu_i\dot{a}_i(z-u_i)^{-1}]. \tag{7.1}$$

They are divisible by $\prod_{i=1}^r(z-u_i)^{\mu_i-1}$ and are not divisible by $(z-u_i)^{\mu_i}$, unless $\dot{a}_i=0$ or $\tau=0$. Polynomials in (7.1) with all the $\dot{a}_i\neq 0$ correspond to partitions of the form $\{(\mu\downarrow\mathbf{1})\uplus(\nu)\}_\nu$, with $(\mu\downarrow\mathbf{1})\uplus(\mathbf{1}_{|\mu|})$ being the maximal element among them. When an $\dot{a}_i$ vanishes, (7.1) becomes divisible by $(z-u_i)^{\mu_i}$.

Let $\kappa:\{i\}\to\{0,1\}$ be a book keeping function registering which $\dot{a}_i$'s vanish. Then (7.1) is divisible by $\prod_{i=1}^r(z-u_i)^{\mu_i-\kappa_i}$ and, *generically*, is not divisible by

$(z-u_i)^{\mu_i}$ for all $\kappa_i \neq 0$ or by $(z-u_i)^{\mu_i+1}$ for all $\kappa_i = 0$. As a result, the corresponding tangent line $P + \tau\dot{P}_t$ is contained in $\mathcal{D}_{(\mu\downarrow\kappa)\uplus(\mathbf{1}_{|\kappa|})} \subset \mathcal{D}_{(\mu\downarrow\mathbf{1})\uplus(\mathbf{1}_{|\mu|})}$. Therefore, $T_P\mathcal{D}^\circ_\mu \subset \mathcal{D}_{(\mu\downarrow\mathbf{1})\uplus(\mathbf{1}_{|\mu|})}$.

On the other hand, any monic polynomial $Q(z)$ of degree d, which is *divisible* by $\prod_{i=1}^r (z-u_i)^{\mu_i-1}$, belongs to a tangent line as in (7.1) from the tangent space of $\mathcal{D}^\circ_\mu$ at a point $P(z) = \prod_{i=1}^r (z-u_i)^{\mu_i}$. Here the *simple* P-roots $\{u_i\}$ are chosen *freely*. Indeed, put $Q(z) = \prod_{i=1}^r (z-u_i)^{\mu_i-1}\hat{Q}(z)$, where $\hat{Q}(z)$ is monic of degree $|\mu|$. Comparing $Q(z)$ with (7.1) leads to an equation

$$\hat{Q}(z) = \prod_{i=1}^{r} (z-u_i)[1 + \tau\sum_{i=1}^{r} \mu_i\dot{a}_i(z-u_i)^{-1}]. \tag{7.2}$$

This forces $\{\dot{a}_i = \hat{Q}(u_i)/(\tau\mu_i\Delta_i)\}$, were $\Delta_i := \prod_{j\neq i}(u_i-u_j) \neq 0$. With this choice of the velocity vector $\dot{a}$ at P, one gets an identity of monic polynomials of degree $|\mu|$, which can be validated by comparing the LHS and RHS of (7.2) at $|\mu|$ distinct points $\{u_i\}$. Hence, $Q \in T_P\mathcal{D}^\circ_\mu$.

We notice that the tangent line $\{P + \tau\dot{P}\}$, $P(z) = \prod_i (z-u_i)^{\mu_i}$, can contain *some* points Q representing polynomials which are divisible by $(z-u_i)^{\mu_i+1}$ or even by higher powers of $(z-u_i)$. These Q's are not in $\mathcal{D}^\circ_{(\mu\downarrow\kappa)\uplus(\mathbf{1}_{|\kappa|})}$, but in its closure.

We have shown that the affine space $T_P\mathcal{D}^\circ_\mu$ *is comprised of polynomials divisible by* $P^{\downarrow\mathbf{1}}(z)$. Therefore, $T_P\mathcal{D}^\circ_\mu$ is defined by linear equations $\{Q^{(j)}(u_i) = 0\}$, where the u_i's range over the *multiple* roots of $P(z)$ and $0 \leq j \leq \mu_i - 2$. Polynomials $Q(z)$ of degree d which are divisible by $R(z) := \prod_{\{i:\, \mu_i \geq 2\}} (z-u_i)^{\mu_i}$ and have the rest of their roots simple, are clearly in $\mathcal{D}^\circ_\mu \cap T_P\mathcal{D}^\circ_\mu$. At the same time, using the previous description of spaces, tangent to $\mathcal{D}^\circ_\mu$, we get $T_P\mathcal{D}^\circ_\mu = T_Q\mathcal{D}^\circ_\mu$ (as affine spaces). Therefore, $T_P\mathcal{D}^\circ_\mu$ is tangent to $\mathcal{D}^\circ_\mu$ along the subset of $\{Q(z)\}$ divisible by $R(z)$. Of course, not any polynomial of degree d which is divisible by $R(z)$ is in $\mathcal{D}^\circ_\mu$, but a generic one is. Note, that polynomials which are divisible by $R(z)$ form an affine space V_R characterized by equations $\{Q^{(j)}(u_i) = 0\}$, where the u_i's range over the *multiple* roots of $P(z)$ and $0 \leq j \leq \mu_i - 1$. Hence, $V_R \cap \mathcal{D}^\circ_\mu \subseteq \mathcal{D}^\circ_\mu \cap T_P\mathcal{D}^\circ_\mu$ is the tangency locus of $\mathcal{D}^\circ_\mu$ and $T_P\mathcal{D}^\circ_\mu$. It is open and dense in V_R. Both sets $\mathcal{D}^\circ_\mu \cap T_P\mathcal{D}^\circ_\mu$ and $\mathcal{D}^\circ_\mu \cap T_P\mathcal{D}^\circ_\mu$ are $|\mu^{-1}(1)|$-dimensional.

For example, when $P(z) = (z-1)^3(z-2)^2(z-3)$, $P^{\downarrow\mathbf{1}}(z) = (z-1)^2(z-2)$, and $T_P\mathcal{D}^\circ_\mu$ is comprised of polynomials of the form $(z-1)^2(z-2)(z-u)(z-v)(z-w)$. Here the roots u, v, w are numbers of our choice. The polynomial $R(z) = (z-1)^3(z-2)^2$, and the line $V_R = \{(z-1)^3(z-2)^2(z-u)\}$ is contained in $\mathcal{D}_\mu$. Its intersection with $\mathcal{D}^\circ_\mu$ is characterized by the inequalities $u \neq 1, u \neq 2$. According to our argument, the 3-space $T_P\mathcal{D}^\circ_\mu$ is tangent to $\mathcal{D}^\circ_\mu$ along this line, pierced at two points. At the same time, $\mathcal{D}^\circ_\mu \cap T_P\mathcal{D}^\circ_\mu$ is a union of that line with the pierced line $\{(z-1)^2(z-2)^3(z-u)\}_{u\neq 1,2}$ and the pierced curve $\{(z-1)^2(z-2)(z-u)^3\}_{u\neq 1,2}$. Note that $T_P\mathcal{D}^\circ_\mu$ is not tangent to $\mathcal{D}^\circ_\mu$ along these two loci.

For many partitions μ, the complex intersection $\mathcal{D}^\circ_\mu \cap T_P\mathcal{D}^\circ_\mu$ simplifies to the linear form $\mathcal{D}^\circ_\mu \cap V_R$. In particular, this happens when the gaps between distinct values of the function μ all are greater than 1 (i.e. the steps in the tableaux μ are higher than 1). For such a μ, any polynomial from $\mathcal{D}^\circ_\mu$ which is divisible by $P^{\downarrow\mathbf{1}}(z)$ is actually divisible by $R(z)$.

Recall, that each space tangent to $\mathcal{D}^\circ_\mu$ consists of polynomials which are divisible by $P^{\downarrow 1}(z)$ for *some* $P \in \mathcal{D}^\circ_\mu$. Therefore, in order to prove the validity of the last statement of the theorem, we notice that the number $\gamma(\mu \downarrow \mathbf{1},\ \mu')$ measures exactly the number of distinct polynomials of the form $P^{\downarrow 1}(z)$ which divide a given polynomial $Q(z),\ Q \in \mathcal{D}^\circ_{\mu'}$. □

For a given partition μ, denote by $\{i_j\}_{1\leq j\leq q}$ a characteristic sequence of indices so that, for each j and all $i \in (i_{j-1}, i_j]$, $\mu_i = \mu_{i_j}$, and $\mu_{i_j} > \mu_{i_{j+1}}$. We say that a partition μ is *steep* if, for each j, subject to $\mu(i_j) \geq 2$, one has:

$$\mu_{i_j} > 1 + i_j + \sum_{k>j} (i_k - i_{k-1})\mu_{i_k}.$$

For example, $\mu = (19, 7, 7, 1, 1)$ is steep. A hook-shaped partition $\mu_{[k]}$ is steep when $2k > d + 2$.

Corollary 7.1. *Assume that all $\mu_i \neq 2$. Then through each point $Q \in \mathcal{D}^\circ_{(\mu\downarrow \mathbf{1})\uplus(\mathbf{1}_{|\mu|})}$ there is a unique $|\mu|$-space tangent to $\mathcal{D}^\circ_\mu$.*

For a steep μ, all the spaces tangent to $\mathcal{D}^\circ_\mu$ are disjoint in $\mathbb{C}^d_{coef}$. Hence, $T\mathcal{D}^\circ_\mu$ is the space of a vector $[|\mu| - \#\mu^{-1}(1)]$-bundle over $\mathcal{D}^\circ_\mu$.

Proof. Each $Q(z)$ as in the corollary has is a single divisor shaped by $\mu \downarrow \mathbf{1}$. Similarly, for a steep μ, any d-polynomial has no more than a single divisor shaped by $\mu \downarrow \mathbf{1}$—the steps in the tableaux $\mu \downarrow \mathbf{1}$ are "too tall". Hence, distinct tangent spaces are *disjoint*. Each of them is tangent to $\mathcal{D}^\circ_\mu$ along a $\#\mu^{-1}(1)$-dimensional subspace. The orthogonal commplements to those (in the tangent spaces) provide $T\mathcal{D}^\circ_\mu$ with the the bundle structure over $\mathcal{D}^\circ_\mu$. □

Given a configuration of distinct points $\{x_j\}$ in the complex plane $\mathbb{C}^1$ (or in the complex projective space $\mathbb{P}^1$), equipped with positive multiplicities ν_j [11], we define its *resolution* to be a new collection points $\{y_{i,j}\}$ with positive multiplicities $\mu_{i,j}$, so that $\nu_j = \sum_i \mu_{i,j}$ and $\{y_{i,j}\}_i$ reside in an ϵ-neighborhood U_j of x_j *free* of the rest of the points. As $\{\mu_{i,j}\}$ define a new partition μ of d, the points $\{y_{i,j}\}_i$ "remember" their parent x_j. Specific locations of $\{y_{i,j}\}_i$ in U_j are irrelevant, all we need is an association between $\{y_{i,j}\}_i$ and x_j provided by U_j. This defines an equivalence relation between resolutions.

Now, fix $\mu \succ \nu$ and consider all equivalence classes of resolutions of $\sum_j \nu_j x_j$ for which $\{\mu_{i,j}\}_{i,j}$ produce μ. We denote them $res(\sum_j \nu_j x_j,\ \mu)$, or alternatively, $res(P, \mu)$, where $P(z) = \prod_j (z - x_j)^{\nu_j}$. One can think of the set $res(P, \mu)$ as indexing *locally* distinct branches of $\mathcal{D}^\circ_\mu$ in the vicinity of the point $P \in \mathcal{D}^\circ_\nu$. As long as $P \in \mathcal{D}^\circ_\nu$, all the sets $res(P, \mu)$ are isomorphic.

By eliminating all the $y_{i,j}$'s with $\mu_{i,j} = 1$ from our list, lowering the rest of $\mu_{i,j}$'s by 1, and still keeping the association of multiple $y_{i,j}$'s with x_j, analogous sets $res^{\downarrow 1}(\sum_j \nu_j x_j,\ \mu) = res^{\downarrow 1}(P,\ \mu)$ can be introduced. Again, the cardinality of $res^{\downarrow 1}(P,\ \mu)$ depends only on μ and ν.

Corollary 7.2. *Let $\nu \prec \mu$ be two d-partitions. Let $\{P_t\}_{0\leq t\leq 1}$ be a path in $\mathcal{D}_\mu$ so that, for $0 \leq t < 1$, $P_t \in \mathcal{D}^\circ_\mu$ and $P_1 \in \mathcal{D}^\circ_\nu$. Then, as $t \to 1$, the tangent spaces $\{T_{P_t}\mathcal{D}^\circ_\mu\}$* stabilize *toward an affine $|\mu|$-dimensional space T containing P_1.*

[11] that is, an effective divisor $\sum_j \nu_j x_j$ of degree d.

Although the limiting space T can depend on the path P_t (which terminates at P_1), the number of such spaces at P_1 is finite. In fact, they are in an 1-to-1 correspondence with the elements of the set $res^{\downarrow 1}(P_1,\ \mu)$. In particular, when $\#(res^{\downarrow 1}(P_1,\ \mu)) = 1$, the tangent bundle of $\mathcal{D}_\mu^\circ$ extends across the singularity $\mathcal{D}_\nu^\circ \subset \mathcal{D}_\mu$ to a vector bundle.

For example, if $\nu = \{3+2+1\}$ and $\mu = \{2+2+1+1\}$, then the only μ-resolution of ν is $\{(2+1)+2+1\}$. Hence, $res^{\downarrow 1}(P,\ \mu)$ consists of a single element $\mu \downarrow 1 = \{2+2\}$, where the first 2 has 3 for a parent and the second 2 has 2 for a parent. As a result, the tangent bundle to $\mathcal{D}^\circ_{(2,2,1,1,0,0)}$ extends across $\mathcal{D}^\circ_{(3,2,1,0,0,0)}$.

Proof. Each space $T_{P_t}\mathcal{D}_\mu^\circ$ consists of polynomials divisible by $P_t^{\downarrow 1}(z) = \prod_i (z-u_i(t))^{\mu_i - 1}$. As some of the distinct roots $\{u_i(t)\}$ merge when $t \to 1$, the set of polynomials divisible by $P_t^{\downarrow 1}(z)$ converges to the set T of polynomials divisible by a polynomial $Q(z) := lim_{t\to 1} P_t^{\downarrow 1}(z)$ (also of degree $d - \#\{\mu^{-1}(1)\})$[12]. Both sets are $|\mu|$-dimensional affine subspaces of $\mathbb{C}^d_{coef}$. Note that $P_1(z)$ is divisible by $Q(z)$. Hence, $P_1 \in T$. Furthermore, the limiting polynomial $Q(z)$ does not depend on the choice of the path P_t, as long as the path is chosen so that the roots of $P_t(z)$, merging into a particular root of $P_1(z)$, are confined to its sufficiently small neighborhood and their multiplicities are prescribed. In this context, "sufficiently small" means that the neighborhoods surrounding the roots of $P_1(z)$ are chosen to be disjoint. This prevents the roots of $P_t(z)$ from loosing focus on a parental root of $P_1(z)$ (that is, from "braiding" from a parental root to a different parental root). This remark justifies our previous definition of combinatorial resolution. Now it becomes clear that the limiting spaces at P_1 are in 1-to-1 correspondence with the elements of $res^{\downarrow 1}(P_1,\ \mu)$, i.e. with the equivalence classes of multiple root configurations governed by the μ and the association with parental roots of P_1. □

Example 7.1.

Perhaps, an additional example can clarify Corollary 7.2 and the argument above. Take $\mu = \{3+2+1+1\}$ and $\nu = \{3+3+1\}$. Put $P_1(z) = (z-4)^3(z-6)^3(z-8)$. We can resolve $P_1(z)$ only in two locally distinct ways: $R(z) = [(z-3.9)^2(z-4.1)] \times$ $\times (z-6)^3(z-8)$ and $S(z) = (z-4)^3[(z-5.9)^2(z-6.1)](z-8)$, each one being consistent with the μ. The tangent space to $\mathcal{D}_\mu^\circ$ at R consists of monic polynomials of degree 7 which are divisible by $R^{\downarrow 1}(z) = (z-3.9)(z-6)^2$ and the one at S — of polynomials divisible by $S^{\downarrow 1}(z) = (z-4)^2(z-5.9)$. The first is close to the limiting 4-space of polynomials divisible by $(z-4)(z-6)^2$, while the second is close to the limiting 4-space of polynomials divisible by $(z-4)^2(z-6)$. The two limiting spaces intersect along a 3-space of polynomials divisible by $(z-4)^2(z-6)^2$, which happens to be the tangent space to $\mathcal{D}_\nu^\circ$ at P_1. □

The next proposition deals with the variety $\mathcal{D}_\mu^\vee$ projectively dual to $\mathcal{D}_\mu$. It resides in $\mathbb{P}(\mathbb{C}^d_{coef} \oplus \mathbb{C}^1)$. Unfortunately, for a general μ, I do not know how to compute the degree of $\mathcal{D}_\mu^\vee$ (cf. Corollary 6.4).

Corollary 7.3. *For any partition μ, $dim(\mathcal{D}_\mu^\vee) = d - 1 - \#\{\mu^{-1}(1)\}$*

Proof. The arguments are similar to the ones in Corollary 6.4. By Theorem 7.1, $\mathcal{D}_{(\mu\downarrow 1)\uplus(1_{|\mu|})}$ is $\bar{T}\mathcal{D}_\mu^\circ$. Therefore, $dim(\bar{T}\mathcal{D}_\mu^\circ) = dim(\mathcal{D}_{(\mu\downarrow 1)\uplus(1_{|\mu|})}) = 2|\mu| -$

[12] $Q(z)$ is different from $P_1^{\downarrow 1}(z)$.

$\#\{\mu^{-1}(1)\}$. Since each tangent space is $|\mu|$-dimensional, the whole family of these spaces must be $(|\mu| - \#\{\mu^{-1}(1)\})$-dimensional. Each of the tangent spaces $T_P\mathcal{D}^\circ_\mu$ is contained in a $d - |\mu| - 1$ dimensional family of hyperplanes. Thus, $dim(\mathcal{D}^\vee_\mu) = (|\mu| - \#\{\mu^{-1}(1)\}) + (d - |\mu| - 1) = d - 1 - \#\{\mu^{-1}(1)\}$. □

The regular embedding $\mathcal{D}^\circ_\mu \subset \mathbb{A}^d_{coef}$ gives rise to a Gaussian map $G_\mu : \mathcal{D}^\circ_\mu \to PGr(|\mu|, d)$, where $PGr(k, d)$ denotes the Grassmanian of k-dimensional projective spaces in a d-dimensional projective space $\mathbb{P}(\mathbb{A}^d_{coef} \oplus \mathbb{A}^1)$. Let $\bar{G}_\mu(\mathcal{D}^\circ_\mu)$ stand for the closure of $G_\mu(\mathcal{D}^\circ_\mu)$ in $PGr(|\mu|, d)$.

Corollary 7.4. *The dimension of the variety $\bar{G}_\mu(\mathcal{D}^\circ_\mu)$ is $|\mu| - \#\{\mu^{-1}(1)\}$, while the dimension of $\mathcal{D}^\circ_\mu$ is $|\mu|$. In particular, for $\mu = (k, 1, ..., 1)$, $dim(\bar{G}_\mu(\mathcal{D}^\circ_{d,k})) = 1$, provided $k > 1$.*

Proof. The union of tangent spaces to $\mathcal{D}^\circ_\mu$ spans an open set in the variety $\mathcal{D}_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})}$ of dimension $|(\mu \downarrow 1) \uplus (\mathbf{1}_{|\mu|})| = \#\{\mu^{-1}([2, d])\} + |\mu|$. Since the dimension of each tangent space is $|\mu|$, $dim(G_\mu(\mathcal{D}^\circ_\mu)) = \#\{\mu^{-1}([2, d])\}$. □

We can generalize the projective duality using Grassmanians instead of projective spaces. Given any projective variety $\mathcal{X} \subset \mathbb{P}(V^{n+1})$ of dimension k, denote by $\mathcal{X}^\vee$ the closure in $PGr(m, n)$, $k \le m < n$, of all m-dimensional projective subspaces tangent (that is, containing a tangent space of $\mathcal{X}$) to $\mathcal{X}$ at its smooth points.

For an appropriate $s \in \mathbb{Z}_+$, the number of m-dimensional projective spaces tangent to $\mathcal{X}$ and containing a fixed (but generic) s-dimensional projective subspace $U \subset \mathbb{P}(V^{n+1})$ is finite. We define the *degree* of $\mathcal{X}^\vee \subset PGr(m, n)$ to be this number.

Theorem 7.2. *For any partition μ, consider the dual variety $\mathcal{D}^\vee_\mu$ residing in the Grassmanian $PGr(d - |\mu| + \#\{\mu^{-1}(1)\}, d)$. Then*

$$deg(\mathcal{D}^\vee_\mu) \le \frac{(|\mu| + \#\{\mu^{-1}(2)\})!}{|\mu|! \cdot (\#\{\mu^{-1}(2)\})!} \cdot deg(\mathcal{D}_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})}) \quad [13]$$
$$= \frac{(2|\mu| - \#\{\mu^{-1}(1)\})!}{|\mu|!} \times \prod_{\{i:\ \mu_i > 2\}} (\mu_i - 1) / \prod_{\{l \ge 2\}} (\#\{\mu^{-1}(l)\}!).$$

Proof. From the definition, $deg(\mathcal{D}^\vee_\mu)$ is the number of affine subspaces of dimension $d - |\mu| + \#\{\mu^{-1}(1)\}$ in $\mathbb{C}^d_{coef}$ which are tangent to $\mathcal{D}^\circ_\mu$ and contain a generic affine space U of dimension $d - 2|\mu| + \#\{\mu^{-1}(1)\}$. Note that $dim(\mathcal{D}_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})})$ and $dim(U)$ are complementary. Pick U to be transversal to $\mathcal{D}^\circ_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})}$ at each of $deg(\mathcal{D}_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})})$ points $\{Q_\alpha\}$ of their intersection. Trough each point Q_α, there are exactly $\gamma(\mu \downarrow \mathbf{1},\ (\mu \downarrow 1) \uplus (\mathbf{1}_{|\mu|}))$ distinct spaces $T_{\alpha,j}$ which are tangent to $\mathcal{D}^\circ_\mu$. Therefore, there are at most $deg(\mathcal{D}_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})}) \times \gamma(\mu \downarrow \mathbf{1},\ (\mu \downarrow 1) \uplus (\mathbf{1}_{|\mu|}))$ tangent $(d - |\mu| + \#\{\mu^{-1}(1)\})$-dimensional spaces which contain U. It remains to notice that $\gamma(\mu \downarrow \mathbf{1},\ (\mu \downarrow 1) \uplus (\mathbf{1}_{|\mu|}))$ is the number of choices of $\#\{\mu^{-1}(2)\}$ objects among $|\mu| + \#\{\mu^{-1}(2)\}$ objects. In particular, when $\#\{\mu^{-1}(2)\} = 0$, $deg(\mathcal{D}^\vee_\mu) \le deg(\mathcal{D}_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})})$. This is the case for any $\mu = (k, 1, ..., 1)$ with $k > 2$.

Finally, by [Hi] (cf. Lemma 7.1),

$$deg(\mathcal{D}_{(\mu\downarrow 1)\uplus(\mathbf{1}_{|\mu|})}) = \theta \times \prod_{\{i:\ \mu_i > 2\}} (\mu_i - 1) / \prod_{\{l \ge 2\}} (\#\{\mu^{-1}(l)\}!),$$

where $\theta = \frac{(2|\mu| - \#\{\mu^{-1}(1)\})!}{(|\mu| + \#\{\mu^{-1}(2)\})!}$, which completes the estimate for $deg(\mathcal{D}^\vee_\mu)$. □

[13] We conjecture that the estimate is sharp.

Although all the previous results were formulated for the affine or quasi-affine varieties $\mathcal{D}_\mu, \mathcal{D}_\mu^\circ$, in fact, many hold for their "projective versions" ${}^\bullet\mathcal{D}_\mu, {}^\bullet\mathcal{D}_\mu^\circ$. These are varieties of μ-weighted configurations of (distinct) points in $\mathbb{P}^1$, in other words, the varieties of degree d effective divisors D of the form $\sum_{i=1}^{|\mu|} \mu_i p_i$, where $p_i = [a_i : b_i] \in \mathbb{P}^1$ are distinct points and $\{\mu_i > 0\}$. While the positively weighted configurations in $\mathbb{A}^1$ can be regarded as zeros of polynomials in one variable z, the positively weighted configurations in $\mathbb{P}^1$ can be regarded as zeros of homogeneous degree d polynomials $P(z_0, z_1) = \prod_i \{-b_i z_0 + a_i z_1\}^{\mu_i}$ in two variables z_0, z_1.

Adjusting the arguments which have established Theorem 7.1 for a t-deformation $P_t(z_0, z_1) = \prod_i \{[-b_i - v_i(t)]z_0 + [a_i + u_i(t)]z_1\}^{\mu_i}$ of $P(z_0, z_1)$, we get

Theorem 7.3.

- *The space $T_D^\bullet \mathcal{D}_\mu^\circ$, tangent to ${}^\bullet\mathcal{D}_\mu^\circ$ at a divisor $D = \sum_{i=1}^{|\mu|} \mu_i p_i$, consists of all the divisors Q of the form $\sum_{i=1}^{|\mu|}(\mu_i - 1)p_i + \hat{Q}$. Here $\hat{Q}$ is any effective divisor of degree $|\mu|$.*
- *In fact, $T_D^\bullet \mathcal{D}_\mu^\circ$ is tangent to ${}^\bullet\mathcal{D}_\mu^\circ$ along an open and dense set of a $\#\{\mu^{-1}(1)\}$-dimensional projective space, formed by the divisors of the form $\sum_{\{i:\, \mu_i > 1\}} \mu_i p_i + \sum_{j=1}^{\#\{\mu^{-1}(1)\}} q_j$, where $\{q_j\}$ are mutually distinct and distinct from the p_i's.*
- *The tangent spaces $\{T_D^\bullet \mathcal{D}_\mu^\circ\}_D$ span a quasi-projective variety $T^\bullet \mathcal{D}_\mu^\circ$ formed by divisors of the form $\sum_{i=1}^{|\mu|}(\mu_i - 1)p_i + \hat{Q}$, with p_i's being distinct and $\hat{Q} > 0$ being of degree $|\mu|$.*

Hence, for a given effective divisor Q of degree d, the number of spaces $\{T_D^\bullet \mathcal{D}_\mu^\circ\}_D$ tangent to ${}^\bullet\mathcal{D}_\mu^\circ$ and containing Q is the number of ways in which $Q = \sum_j \nu_j p_j$ can be represented as $\sum_{i=1}^{|\mu|}(\mu_i - 1)p_i + \hat{Q}$, for some $\hat{Q} > 0$ and a collection of distinct p_i's. As we interpret partitions as non-increasing functions on the index set $\{1, 2, 3, ...\}$, they admit another partial order: we say that $\nu \geq \mu$ when the function $\nu - \mu$ is non-negative (this partial order should not be confused with our old friend —the partial order $\nu \succeq \mu$ induced by merging points in divisors). Evidently, a divisor $Q = \sum \nu_j p_j$ belongs to the tangent space $T^\bullet \mathcal{D}_\mu^\circ$, if and only if $\nu \geq \mu \downarrow \mathbf{1}$. However, to compute the multiplicity of the tangent web $T^\bullet \mathcal{D}_\mu^\circ$ at Q seems to be a tedious combinatorial problem: for given ν, μ, such that $\nu \geq \mu \downarrow \mathbf{1}$, one needs to count the number of permutations $\sigma \in S_{|\nu|}$ which place the function $\sigma(\mu \downarrow \mathbf{1})$ *below* ν, divided by $\prod_l [\#\{(\mu \downarrow \mathbf{1})^{-1}(l)\}!]$—the order of the stabilizer of $\mu \downarrow \mathbf{1}$.

Proof of Theorem 7.3. First, we compute $\frac{d}{dt} P_t(z_0, z_1)|_{t=0}$ which is given by the formula

$$(7.3) \quad \prod_i (-b_i z_0 + a_i z_1)^{\mu_i - 1}\Big[\sum_i \mu_i(-\dot{v}_i z_0 + \dot{u}_i z_1) \prod_{j \neq i}(-b_j z_0 + a_j z_1)\Big],$$

where $\dot{u}_i = \frac{d}{dt} u_i(t)|_{t=0}, \dot{v}_i = \frac{d}{dt} v_i(t)|_{t=0}$. This tells us that the tangent cone to ${}^\bullet\mathcal{D}_\mu^\circ$ at a divisor D is contained in the set of divisors of the form $\sum_{i=1}^{|\mu|}(\mu_i - 1)p_i + \hat{Q}$. Here $\hat{Q}$ being an effective divisor of degree $|\mu|$. On the other hand, any homogeneous polynomial $Q(z_0, z_1)$ of degree d which is divisible by $\prod_i (-b_i z_0 + a_i z_1)^{\mu_i - 1}$ is of the form (7.3) for an appropriate choice of the velocity vectors $\{\dot{u}_i, \dot{v}_i\}$. Indeed, let $Q(z_0, z_1) = \hat{Q}(z_0, z_1) \prod_i (-b_i z_0 + a_i z_1)^{\mu_i - 1}$. Then from (7.3), the proportionality classes $[\dot{u}_i : \dot{v}_i]$ are determined by the equations

$$\hat{Q}(a_i, b_i) = \mu_i \begin{vmatrix} \dot{u}_i & \dot{v}_i \\ a_i & b_i \end{vmatrix} \cdot \prod_{j \neq i} \begin{vmatrix} a_j & b_j \\ a_i & b_i \end{vmatrix}. \tag{7.4}$$

In turn, for such a choice of the velocity vectors, $\sum_i \mu_i(-\dot{v}_i z_0 + \dot{u}_i z_1) \times \prod_{j \neq i}(-b_j z_0 + a_j z_1)$ is proportional to $\hat{Q}(z_0, z_1)$ (since, by (7.4), the two homogeneous polynomials of degree $|\mu|$ agree at $|\mu|$ distinct lines). □

Now we would like to make a few concluding remarks about topology of the complex strata $\{\mathcal{D}_\mu\}$ and $\{\mathcal{D}^\circ_\mu\}$. A wonderfully rich account of the topological properties of disciminants, or rather their complements, can be found in [Va1], [Va2]. Both sources concentrate on more subtle description of real determinantal varieties. A valuable topological information is also contained in [A1]—[A3], [SW], [SK]. These papers tend to focus on calculations of the cohomologies of the complements to discriminant varieties of one kind or another.

First, we notice that each $\mathcal{D}_\mu$ is a *contractible* space. Indeed, the radial retraction of the plane $\mathbb{C}$ to its origin induces a retraction of any configuration to a singleton, taken with multiplicity d.

In contrast, topology of complex strata $\{\mathcal{D}^\circ_\mu\}$ and $\{\mathcal{D}^\circ_{d,k}\}$ is connected to the *colored braid* groups similar to ones of Arnold [A1], [A2]. We think of distinct multiplicities of roots as being *distinct colors.*

Let $\mathcal{U}^\circ_k$ denote the configuration space of of k *ordered* distinct point in $\mathbb{C}$. Its fundamental group is the pure (or colored) braid group F_k of k strings. By [FN], $\mathcal{U}^\circ_k$ is an Eilenberg-MacLane space $K(F_k, 1)$.

For a given partition $\mu : \{1, 2, ..., |\mu|\} \to \mathbb{Z}_+$, the space $\mathcal{U}^\circ_{|\mu|}$ is a finite covering of the space $\mathcal{D}^\circ_\mu$. The covering map identifies each ordered configuration of $|\mu|$ distinct (simple) roots with a root configuration where roots acquire the multiplicities μ_i's and roots of the same multiplicity do not enjoy any order. Therefore, $\mathcal{D}^\circ_\mu$ must be an Eilenberg-MacLane space $K(\pi_1(\mathcal{D}^\circ_\mu), 1)$, where $\pi_1(\mathcal{D}^\circ_\mu)$ can be identified with the μ-colored braid group B_μ. The number of strings in the braids is $|\mu|$. A string starts and ends at roots of the same color-multiplicity μ_i. In fact, B_μ is a subgroup of index $|\mu|! \,/ \prod_l [\#(\mu^{-1}(l))]!$ in $B_{|\mu|}$.

Due to Lemma 6.1, the slice of $\mathcal{D}^\circ_\mu$ by the hyperplane of reduced polynomials also is a $K(B_\mu, 1)$ space.

Revisiting Example 6.2 and Figure 10, the space $\mathcal{D}^\circ_{5,1}$ is a $K(B_5, 1)$-space, where $B_5 := B_{(1,1,1,1,1)}$ is the braid group with 5 strings. The stratum $\mathcal{D}^\circ_{(2,1,1,1,0)}$ is a $K(B_{(2,1,1,1,0)}, 1)$-space, where $B_{(2,1,1,1,0)}$ is the braid group of 4 strings with one of the strings colored with red and 3 strings with blue. Similarly, both $\mathcal{D}^\circ_{(3,1,1,0,0)}$ and $\mathcal{D}^\circ_{(2,2,1,0,0)}$ are $K(B_{(3,1,1,0,0)}, 1)$-spaces, where $B_{(3,1,1,0,0)}$ is the braid group of 3 strings with one of the strings colored with red and 2 strings with blue. Both $\mathcal{D}^\circ_{(4,1,0,0,0)}$ and $\mathcal{D}^\circ_{(3,2,0,0,0)}$ are $K(B_{(4,1,0,0,0)}, 1)$-spaces, where $B_{(4,1,0,0,0)}$ is a pure braid group of 2 strings. Finally, $\mathcal{D}_{(5,0,0,0,0)}$ is contractible. At the same time, $\mathcal{D}^\circ_{5,3}$, $\mathcal{D}^\circ_{5,4}$ both are $K(\mathbb{Z}, 1)$-spaces.

I do not know whether, in general, $\mathcal{D}^\circ_{d,k}$ is an Eilenberg-MacLane space. However, for $k > d/2$, each $\mathcal{D}^\circ_{d,k}$ has a homotopy type of a circle. This is true because each $\mathcal{D}_{d,l}$, $l \geq k$, fibers over $\mathcal{D}_{d,l+1}$ with a fiber $\mathbb{C}$ and each $\mathcal{D}^\circ_{d,l}$, $l \geq k$, fibers over $\mathcal{D}_{d,l+1}$

with a fiber $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Thus, the fundamental group $\pi_1(\mathcal{D}^\circ_{d,k}) \approx \mathbb{Z}$, provided $k > d/2$.

The space $\mathcal{D}^\circ_{d,1}$ admits an *unordered* framing. Locally, it is comprised of d independent vectors fields $\{n_k\}$. At a generic point $P(z) = \prod_k (z - u_k) \in \mathcal{D}^\circ_{d,1}$, n_k is the normal vector $(u_k^{d-1}, u_k^{d-2}, \ldots, u_k, 1)$ to the hyperplane $T^{u_k}_{d,1}$ which passes through P and is tangent to the discriminant variety $\mathcal{D}_{d,2}$. Since all the roots are distinct, the Vandermonde matrix $Van(u_1, u_2, \ldots u_d)$, whose columns are the $\mathbb{C}$-linearly independent vectors $\{n_k\}$, is of the maximal rank d. However, $\{n_k\}$ do not form vector fields: there is no consistent ordering attached to them. Nevertheless, they give rise to a well-defined algebraic embedding

$$\mathcal{V}an^\circ : \mathcal{D}^\circ_{d,1} \to GL(d, \mathbb{C})/S_d,$$

which induces a canonic epimorphism

$$\mathcal{V}an^\circ_* : B(d,1) \to \pi_1(GL(d,\mathbb{C})/S_d) \approx \mathbb{Z} \rtimes S_d.$$

Here the symmetric group S_d acts on $\mathbb{Z}$ according to the parity of its permutations.

The embedding $\mathcal{V}an^\circ$ extends to an embedding $\mathcal{V}an : \mathcal{D}_{d,1} \to Mat(d,\mathbb{C})/S_d$, similarly defined in terms of the matrix $Van(u_1, u_2, \ldots u_d)$.

The determinant gives rise to an obvious map $det : Mat(d,\mathbb{C})/S_d \to \mathbb{C}/\{\pm 1\}$. In fact, $det[Van(u_1, u_2, \ldots u_d)] = \prod_{i<j}(u_i - u_j)$. Evidently, $\mathcal{D}_{d,2} = (det \circ \mathcal{V}an)^{-1}(0)$, which also is the $\mathcal{V}an$-preimage of matrices of rank $d-1$. It is tempting to conjecture that the stratification $\{\mathcal{D}_{d,k}\}_k$ is a pull-back, under the embedding $\mathcal{V}an$, of the natural stratification $\{Mat_l(d,\mathbb{C})\}_l$ of $(d \times d)$-matrices by rank l. However, the reality is different. Since the rank of a Vandermonde matrix $Van(u_1, u_2, \ldots u_d)$ is the number of distinct u_i's, $\{\mathcal{V}an^{-1}(Mat_l(d,\mathbb{C}))\}_l$ defines a stratification $\{\mathcal{R}_{d,l}\}_l$ in $\mathbb{C}^d_{coef}$ by the number of *distinct* roots, not by their maximal multiplicity as $\{\mathcal{D}_{d,k}\}_k$ does. In terms of the partitions μ associated with the roots, $\mathcal{D}_{d,k}$ is comprised of polynomials P with $max(\mu_P) \geq k$, while $\mathcal{R}_{d,l}$ is comprised of polynomials P with $supp(\mu_P) \leq l$. For example, for $d = 4$, polynomials with $\mu = (3,1,0,0)$ and with $\mu' = (2,2,0,0)$ belong to $\mathcal{R}_{4,2}$. At the same time, polynomials with $\mu = (3,1,0,0)$ belong to $\mathcal{D}_{4,3}$, while polynomials with $\mu' = (2,2,0,0)$ belong to the larger stratum $\mathcal{D}_{4,2}$. In general, we only can claim that $\mathcal{D}_{d,k} \subset \mathcal{R}_{d,d-k+1}$.

While the Viète map $\mathcal{V} : \mathbb{C}^d_{root} \to \mathbb{C}^d_{coef}$ transforms a simple linear stratification $\{\mathcal{U}_{d,k} := (u_1 = u_2 = \ldots = u_k)\}$ in $\mathbb{C}^d_{root}$ into a "nonlinear" stratification $\{\mathcal{D}_{d,k}\}$ in $\mathbb{C}^d_{coef}$, the map $\mathcal{V}an$ has an "opposite" effect: it pulls back a linear stratification $\{\mathcal{Z}_{d,k}\}$ in $Mat(d,\mathbb{C})$ to produce $\{\mathcal{D}_{d,k}\}$. A matrix $M = (m_{ij}) \in \mathcal{Z}_{d,k}$ when the first k numbers among $\{m_{d-1,j}\}_j$ are equal (if two such elements are equal, then so are the two columns of the Vandermonde matrix). In short, each stratum $\mathcal{U}_{d,k}$ is a linear subspace of $\mathbb{C}^d_{root}$, each stratum $\mathcal{Z}_{d,k}$ is an linear subspace of $Mat(d,\mathbb{C})$.

It is interesting to contemplate, with the help of $\mathcal{V}an^\circ$, how topologies of $GL(d,\mathbb{C})/S_d$ and $\mathcal{D}^\circ_{d,1}$ interact. It seems that the induced cohomology homomorphism $\mathcal{V}an^{\circ *} : H^*(GL(d,\mathbb{C})/S_d) \to H^*(B_d) = H^*(\mathcal{D}^\circ_{d,1})$ is an epimorphism, at least rationally. Perhaps, this interaction is a subject for a different paper.

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