Localization in elasto-plastic materials: Influence of an evolving yield surface in biaxial loading conditions

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Abstract

The influence of the plasticity yield surface – and of its evolution with plastic deformation – on the development of instabilities in metals is analyzed. Conditions for the activation of slip bands are taken as an instability criterion. They are exhibited in stress states identical to the ones encountered in a flat plate in biaxial tension. The classical bifurcation criterion is replaced by a criterion on the growth of a perturbation at a time scale comparable to the one of the homogeneous solution. This second criterion reveals less severe than the bifurcation one which is reached for the limit case of an infinite growth rate in the perturbation approach. The growth rate is a decreasing function of the biaxiality of the loading which is in agreement with previous studies. The possible destabilizing effect of texture evolution is also exhibited by using an evolving yield surface the curvature of which increases in the neighborhood of the homogeneous solution.

1. Introduction

Localization phenomena are usually observed in plates in tension, in every part where the loading is locally equivalent to uniaxial or even biaxial stretching in the plane of the plate. They take the form of multiple necks or networks of shear bands and generally lead to the fragmentation of the plates. In uniaxial loading, lines perpendicular to the direction of tension are observed on the surface of the plate whereas in equi-biaxial loading the pattern is roughly isotropic.

For instance, the phenomenon has been studied in the last decades in order to evaluate the formability of materials and to display “forming limit diagrams”. Localization is usually supposed to be initiated by elasto-plastic instabilities at the macroscopic scale. Since the work of Hill (1952), the initiation and growth of these instabilities is classically analyzed for thin sheets in a plane stress hypothesis. They are represented as bands in the plane of the sheet (which would in fact give a neck in the thickness in the case when the deformation in the plane is not pure shear). In the framework of Hill study, localization is well understood and justified in stress states varying from uniaxial tension (in the plane of the plate) to the one met in plane strain conditions. It is then possible to determine a more favorable orientation for the band.

However, for the classical Von Mises plasticity model, neither the simulations nor the analytical developments display localization in states between plane strain and biaxial tension. The models were later improved in different ways in order to explain localization in such conditions. The first kind of approaches replaces the Von Mises yield surface by a regular one intermediate between the Von Mises and the Tresca surface (Barlat, 1987). Such surfaces are exhibited by poly-crystal plasticity simulations of annealed bcc and fcc isotropic metals. In this case, Barlat (1987) found noticeably lower strains for localization than with the Von Mises surface. The second kind of approaches allows deviation from the normality rule. Among them are the so-called “J2-deformation theory” of plasticity (Storen and Rice, 1975; Needleman and Rice, 1978) and other theories in the same spirit which keep the Von Mises plasticity surface but introduce in the plastic strain rate a component colinear to the time derivative of the stress deviator. Such models are approximations of yield surfaces developing vertices during plastic deformation which are also displayed by poly-crystal models (Hutchinson, 1970). The use of deformation theories also gives localization in biaxial tension (Hutchinson and Neale, 1978).

In a recent work (Dequiedt, 2010), the problem of localization in uniaxial and biaxial stretching was treated in a slightly different formalism, by exhibiting a condition for the development of a band of localization in an infinite tri-dimensional elasto-plastic medium in small strains and quasi-static loading. This medium was submitted to a stress state equivalent to the one met in a flat plate in biaxial tension. The development of the band was treated as a bifurcation from the homogeneous solution corresponding to a simple glide in a thin layer (cf. Rice, 1976).

This criterion linked the strain hardening coefficient to the shape of the plasticity yield surface and the biaxiality rate of the
stress state in an explicit form. In agreement with previous works, the Von Mises yield surface led to a limit hardening coefficient which is strongly negative in biaxial tension. In other words, materials would not localize in such loading conditions except for strongly softening behaviors which are probably never encountered in real materials. On the contrary, the Tresca yield surface led to a null limit hardening coefficient for any biaxiality rate, which is far more favorable to localization. Moreover, the existence of a vertex point in equi-biaxial tension for the Tresca surface gives several possible orientations for the band. In this former work (Dequiedt, 2010), it was also proved that the Tresca results were closer to what is observed experimentally than the Von Mises ones.

The use of the Tresca yield surface is probably an overestimation of the “vertex effect” in equi-biaxial loading where the model of Barlat only exhibits a surface with a curvature higher than the Von Mises one. However, according to the poly-crystal simulations of Toth et al. (1996), it seems reasonable to admit that this curvature may increase with plastic deformation in biaxial loading due to the evolution of the material texture.

In this paper, we study the development of bands by adding two improvements to the bifurcation analysis of the former work.

First, the bifurcation criterion is replaced by a criterion of growth of a small perturbation of the homogeneous solution. This approach was developed by Dudzinski and Molinari (1991) for a thin sheet in plane stress for rigid visco-plastic material. In their study, the evolution of the homogeneous solution was neglected during the time of development of the perturbation; in other words, their analysis applies to fast growing perturbations. However, they found that the perturbation growth criterion is weaker than the bifurcation one: positive growth rates are exhibited in situations in which no bifurcation occurs in Hill analysis. Thermal effects as long as anisotropy were proved to favor instability.

In the absence of viscosity, Dudzinski and Molinari also proved that an infinite growth rate is found when the bifurcation conditions are fulfilled. The same result was established for bands of localization in a tri-dimensional medium by Barbier et al. (1998), independently of any particular constitutive law. Perturbation growth conditions were also formulated for bands in an infinite tri-dimensional medium by Rousselier (1995) for a material having a rigid plastic behavior with a possible dependence on pressure (in the case when ductile fracture is activated for instance).

In Section 2, the equations for the evolution of a perturbation are exhibited for an elasto-plastic material in small strains and they are particularized to linear isotropic elasticity. The hypothesis of a constant homogeneous solution is relaxed in order to catch perturbations growing with a characteristic time comparable to the one of this solution. The differences with the bifurcation criterion and the dependence on the plasticity yield surface and its evolution are underlined. In Section 3, after recalling the bifurcation conditions, the growth rates of perturbations are exhibited for the Von Mises case in biaxial tension and proportional loading path (the homogeneous solution keeps a constant direction in the stress space).

In a second time, the effect of the yield surface shape and its evolution is added to the former developments: the normal to the yield surface is kept unchanged in the direction of loading but its curvature evolves with plastic deformation (in order to model the progressive formation of a corner for example). As was suggested by Toth et al. (1996), at each time, the surface is approximated by a Hill-ellipsoid in the neighborhood of the homogeneous solution; the anisotropy parameter is a function of plastic deformation. In other words, when this parameter increases, the curvature of the surface also increases in the direction of the homogeneous solution and a vertex tends to appear. In Section 4, the growth rate evaluation of Section 3 is repeated by replacing the Von Mises yield surface by this evolving Hill surface in such a way to quantify the influence of the evolving curvature.

### 2. Bifurcation and perturbation growth in elasto-plasticity

#### 2.1. General formulation of the criteria

Let us consider an elasto-plastic material in a hypothesis of small strains and small rotations. Its mechanical state is characterized by its deformation ε, its plastic deformation εp and an internal state variable z which describes the strain hardening associated with plastic deformation. We suppose that the elastic behavior and the plastic flow rule are such that, so long as the material deforms plastically, it is possible to write the evolution of the stress tensor and of the internal state variable as functions of the strain rate:

\[
\sigma = \mathbf{L}(\sigma, \varepsilon) : \dot{\varepsilon} \quad \text{and} \quad z = \mathbf{B}(\sigma, \varepsilon) : \dot{\varepsilon}.
\]  

(1)

A band of localization is modeled as a perturbation of the homogeneous solution to a quasi-static mechanical problem. This perturbation consists in a simple glide in a thin layer of normal \( \mathbf{N} \) and the associated displacement field writes:

\[
\Delta\mathbf{u} = \Delta\mathbf{u}(x_0)\mathbf{M}
\]

(2)

with:

\[
x_0 = \mathbf{x} \cdot \mathbf{N}
\]

(3)

In Eqs. (2) and (3), \( \mathbf{x} \) is the position and \( \mathbf{M} \) is the direction of glide, \( \mathbf{N} \) and \( \mathbf{M} \) are supposed to be normalized vectors. The strain associated with the perturbation is:

\[
\Delta\varepsilon = \left\{ \text{grad}(\Delta\mathbf{u}) \right\}_L = \frac{1}{2} \frac{d\Delta\mathbf{u}}{dx_N}(\mathbf{M} \otimes \mathbf{N} + \mathbf{N} \otimes \mathbf{M}).
\]

(4)

In Eq. (4), \( \{a\}_L \) denotes the symmetric part of a tensor \( a \).

The equilibrium equation is satisfied by both the homogeneous and perturbed solution. It gives the following relation for the stress increment \( \Delta\sigma \) associated with the perturbation:

\[
\mathbf{N} \cdot \Delta\sigma = 0.
\]

(5)

The bifurcation condition is fulfilled when such a perturbation may appear from the homogeneous solution at a time which is taken as the origin \( t = 0 \). In other words, there is a loss of uniqueness of the solution of the incremental problem:

\[
\Delta\mathbf{u}(t = 0) = 0 \quad \text{and} \quad \Delta\mathbf{u}(t = 0) \neq 0.
\]

(6)

At \( t = 0 \), the constitutive relations (1) are written in the following form for bifurcation, so long as elastic unloading is excluded for the homogeneous and perturbed solution:

\[
\Delta\sigma = \mathbf{L}(\sigma_1, z_1) : \Delta\varepsilon \quad \text{and} \quad \Delta\sigma = \mathbf{B}(\sigma_1, z_1) : \Delta\varepsilon.
\]

(7)

In these formulas, index 1 applies for the homogeneous solution. According to the equilibrium Eq. (5) and as was established in Rice (1976), the possibility to activate a band of normal \( \mathbf{N} \) is equivalent to the loss of positive-definiteness of the so-called “acoustic” tensor \( (\mathbf{N} \otimes \mathbf{L}(\sigma_1, z_1) \cdot \mathbf{N}) \) and the direction of glide satisfies:

\[
(\mathbf{N} \cdot \mathbf{L}(\sigma_1, z_1) \cdot \mathbf{N}) \cdot \mathbf{M} = 0.
\]

(8)

Unlike the bifurcation condition, the perturbation growth condition assumes that the perturbation pre-exists at \( t = 0 \) and increases with time:

\[
\Delta\mathbf{u}(t = 0) \neq 0 \quad \text{and} \quad \frac{d}{dt} ||\Delta\mathbf{u}|| > 0 \quad \text{for any} \ t > 0.
\]

(9)

The evolution of the perturbations in stress, deformation and internal variable is ruled by the coupled differential equations deduced from the constitutive relations:

\[
\Delta\sigma = \mathbf{L}(\sigma_1, z_1) : \Delta\varepsilon + \frac{\partial \mathbf{L}}{\partial \sigma} : \Delta\sigma + \frac{\partial \mathbf{L}}{\partial z} : \Delta z : \dot{\varepsilon}.
\]
and

\[ \Delta \mathbf{x} = \mathbf{B}[\mathbf{\sigma}_1, \mathbf{\varepsilon}_1] : \Delta \mathbf{\varepsilon} + \left( \frac{\partial \mathbf{B}}{\partial \mathbf{\sigma}} : \Delta \mathbf{\sigma} + \frac{\partial \mathbf{B}}{\partial \mathbf{\varepsilon}} : \Delta \mathbf{\varepsilon} \right) : \mathbf{\varepsilon}_1. \]  

(10)

Let us remark that for perturbations with an infinite growth rate compared to the strain rate of the homogeneous solution, the second terms in the left side of the equality are negligible compared to the first ones: relations (7) and (10) are equivalent and such perturbations obey the bifurcation criterion which is in accordance with the results of Barbier et al. (1998).

Otherwise, relations (10) differ from the same relations obtained in the bifurcation case by terms depending on the effect of the perturbed moduli \( \Delta \mathbf{\varepsilon} \) and \( \Delta \mathbf{\sigma} \) on the homogeneous deformation field \( \mathbf{\varepsilon}_1(t) \).

2.2. Isotropic elasticity and normality of plastic strain rates

In the following, we develop these instability conditions for an isotropic elastic behavior and a plastic flow satisfying the normality rule. For this material, elasticity writes:

\[ \mathbf{\sigma} = \mathbf{A} \mathbf{\varepsilon} - \varphi - \varphi - \mathbf{b}(\mathbf{\sigma}), \]  

(11)

In Eq. (11), \( \mathbf{A} \) and \( \mathbf{G} \) are respectively the Lamé and shear moduli.

In the more general form, plasticity is ruled by a yield surface in the stress space defined by a function \( F \) depending on a single internal state variable \( \varphi \) characterizing the strain hardening:

\[ F(\mathbf{\varphi}, \varphi) = 0. \]  

(12)

The shape of the yield surface is given by the dependence of \( F \) on the different components of \( \mathbf{\varphi} \).

The material is supposed to obey the normality rule:

\[ \varphi = \lambda \frac{\partial F}{\partial \mathbf{n}} \]  

(13)

In Eq. (13), \( \mathbf{n} \) is the normal to the yield surface (we assume that the yield surface is smooth so that the normal is unique). As is usually the case for metals, the yield surface is supposed to depend only on the stress deviator \( \mathbf{s} \) and so \( \mathbf{n} \) is also deviatoric (the plastic deformation develops without volume change).

We suppose that the internal state variable characterizing the strain hardening is the time integral of the plastic multiplier \( \lambda \) and the sensitivity of \( F \) to the internal state variable is noted \( h(\mathbf{\varphi}, \varphi) \):

\[ \lambda = \int_0^t \frac{\partial F}{\partial \mathbf{n}} \]  

(14)

In this case, so long as elastic unloading is excluded, the consistency condition writes:

\[ \mathbf{F} = \mathbf{n} : \mathbf{\varphi} - h \varphi = 0. \]  

(15)

It gives the following relation for the plastic multiplier:

\[ 2 \mathbf{G}(\mathbf{n} : \dot{\mathbf{\varepsilon}}) = \lambda (h + 2 \mathbf{G}(\mathbf{n} : \mathbf{n})). \]  

(16)

So, the incremental relations for the stress tensor and internal state variable are:

\[ \dot{\mathbf{\sigma}} = \mathbf{A} \dot{\mathbf{\varepsilon}} + \frac{4 \mathbf{G}^2}{h + 2 \mathbf{G}(\mathbf{n} : \mathbf{n})} (\mathbf{n} : \dot{\mathbf{\varepsilon}}) \mathbf{n} \]

and:

\[ \dot{\varphi} = \frac{2 \mathbf{G}}{h + 2 \mathbf{G}(\mathbf{n} : \mathbf{n})} (\mathbf{n} : \dot{\mathbf{\varepsilon}}). \]  

(17)

For such materials, the incremental moduli \( \mathbf{L} \) and \( \mathbf{B} \) are functions of \( \mathbf{\sigma} \) and \( \varphi \) through the normal \( \mathbf{n} \) and hardening coefficient \( h \). For a given orientation of band \( \mathbf{N} \), the bifurcation condition is also a function of \( \mathbf{n} \) and \( h \).

The perturbed moduli \( \Delta \mathbf{L} \) and \( \Delta \mathbf{B} \) are functions of the perturbations \( \Delta \mathbf{n} \) and \( \Delta \varphi \). So, the perturbation growth rate depends on the second derivatives \( \partial \mathbf{n}/\partial \mathbf{\sigma}, \partial \mathbf{n}/\partial \mathbf{\varepsilon}, \partial h/\partial \mathbf{\sigma} \) and \( \partial h/\partial \mathbf{\varepsilon} \) of the function \( F \) which characterize the shape of the yield surface around the homogeneous solution (for a given value of \( \varphi \)) and its evolution with strain hardening.

3. Flat plate in proportional loading with a Von Mises yield surface

In this section, the relations of Section 2 are particularized to an isotropic material with a Von Mises yield surface. The material is also supposed to be loaded in such a manner that the homogeneous solution is proportional. The case of the stress encountered in a plate in biaxial tension is developed.

3.1. Material behavior and homogeneous solution

The Von Mises yield surface is defined by a function \( F \) of the following form:

\[ F(\mathbf{\varphi}, \varphi) = (\varphi - \varphi)^2 - Y(\varphi) \]  

(18)

with \( \varphi^2 = \sqrt{s^T \mathbf{s}} \).

In relation (17), the equivalent stress \( \varphi^2 \) is defined in such a way that \( \mathbf{n} \) is normalized:

\[ \mathbf{n} = \frac{1}{\varphi} \mathbf{s} \]  

so \( \mathbf{n} : \mathbf{n} = 1 \).

(19)

For the sake of simplicity, the strain hardening is assumed linear:

\[ Y(\varphi) = Y_0 + H \varphi \]  

so \( h(\mathbf{\varphi}, \varphi) = H \).

(20)

For this behavior, the perturbations on \( \mathbf{n} \) and \( h \) write:

\[ \Delta \mathbf{n} = \frac{1}{\varphi^2} (\Delta \mathbf{s} - (\Delta \mathbf{s} : \mathbf{n}) \mathbf{n}) \]  

and \( \Delta h = 0 \).

(21)

Let us now suppose that the material is loaded in such a way that the homogeneous solution corresponds to \( \varphi^2 = 0 \) (there is no initial plastic deformation) and that the initial stress state \( \sigma_{10} \) and the strain evolution are of the following form:

\[ \sigma_{10} = \sigma_{10}^0 \mathbf{1} + s_{10} \mathbf{n}_1, \]  

(22)

\[ \mathbf{e}_1(t) = (\mathbf{e}_{1}^0 + \mathbf{e}_{1}^0 \mathbf{n}_1) t. \]  

(23)

The strain rates \( \dot{\mathbf{e}}_{1}^0 \) and \( \dot{\mathbf{e}}_{1}^0 \) are supposed to be constants. In this case, the stress deviator \( \mathbf{s} \) and the deviatoric strain \( \mathbf{e}_1 \) remain collinear to the normal tensor \( \mathbf{n}_1 \) (which is also the constant normal to the yield surface). According to relation (17), the stress and internal state variable for the homogeneous solution are:

\[ \sigma_{10} = \sigma_{10}^0 + (3A + 2GH) \dot{\mathbf{e}}_{1}^0 + (\mathbf{s}_{10} + \frac{2GH}{2G} \dot{\mathbf{e}}_{1}^0 \mathbf{n}_1) \mathbf{n}_1, \]  

(24)

\[ \dot{\varphi}_{10} = \frac{2G}{2G + H} \dot{\mathbf{e}}_{1}^0. \]

Let us consider the case of the homogeneous solution for a plate in biaxial tension as presented on Fig. 1, the \( x \) and \( y \) axes being the principal directions of stress in the plane of the plate and the \( z \)-axis.

**Fig. 1.** Plate in biaxial tension and orientation of the first bifurcation.
being normal to the plate. Then, the stress takes the following form in which $\chi$ is the biaxiality rate of the stress tensor:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \chi \sigma_{xx} \end{pmatrix}. \quad (25)$$

In this case, the normal tensor writes:

$$\mathbf{n} = \frac{1}{\sigma_{xy}} \mathbf{S} = \frac{1}{1 + \delta^2 + (1 + \delta)^2} \begin{pmatrix} 1 & \delta \\ \delta & -(1 + \delta) \end{pmatrix}. \quad (26)$$

In this expression, $\delta$ is the biaxiality rate of the stress deviator and is deduced from the biaxiality rate $\chi$:

$$\delta = \frac{2\chi - 1}{2 - \chi}. \quad (27)$$

The biaxiality is supposed to remain constant during loading, so:

$$\sigma_{10}^m = \frac{1 + \delta}{1 + \delta^2 + (1 + \delta)^2} s_{10} \quad \text{and} \quad \dot{\sigma}_{10}^m = \frac{1 + \delta}{1 + \delta^2 + (1 + \delta)^2} \dot{s}_{10}. \quad (28)$$

when different loading paths are considered for $\chi$ varying from 0 to 1, $\delta$ varies at the same time from $-1/2$ to 1. $\delta = 0$ corresponds to the case of plane plastic strain rate:

$$\dot{\varepsilon}^p = \begin{pmatrix} \dot{\varepsilon}_{xx}^p \\ 0 \\ -\dot{\varepsilon}_{xx}^p \end{pmatrix}. \quad (29)$$

### 3.2. Bifurcation condition

In the case of Section 3.1, at each time the bifurcation condition (8) defines a limit hardening coefficient for the activation of a band with orientation $\mathbf{N}$ (the details of the calculation are in Dequiedt, 2010):

$$\bar{H}_{lim}(\mathbf{N}) = \frac{H_{lim}(\mathbf{N})}{2G} = \frac{2((\mathbf{N} \cdot \mathbf{n})^2 - \Gamma(\mathbf{N} \cdot \mathbf{n} \cdot \mathbf{N})^2)}{1 - \delta^2}, \quad (30)$$

In this formula, $\Gamma$ characterizes the compressibility of the material ($\Gamma \leq 1$ and $\Gamma = 1$ for an incompressible material):

$$\Gamma = \frac{A + G}{A + 2G}. \quad (31)$$

The first band to develop when $\bar{H}$ is decreased has the normal $\mathbf{N}$ for which $\bar{H}_{lim}(\mathbf{N})$ is maximum. The associated hardening coefficient $H_{loc}$ and orientation of the bands $\mathbf{N}_{loc}$ are functions of $\Gamma$ and of the normal $\mathbf{n}$.

For the plate in biaxial tension, the first bands have their normal $\mathbf{N}_{loc}$ in the $(x,z)$ plane which contains the minimal and the maximal of the principal stresses. The glide direction $\mathbf{M}_{loc}$ is the symmetric of $\mathbf{N}_{loc}$ to the $x$-axis.

The orientation $\phi_{loc}$ of $\mathbf{N}_{loc}$ with the x-axis (see Fig. 1) and the associated hardening coefficient $H_{loc}$ are functions of the biaxiality rate $\delta$. They are plotted on Fig. 2 for several values of the compressibility $\Gamma$. Except for the case $\delta = 0$, the bifurcation criterion leads to strongly negative values of $\bar{H}_{loc}$. In the small deformation hypotheses, elasticity in shear is a necessity for bifurcation since, for a fixed $H$, $\bar{H}$ tends to 0 when $G$ tends to infinity. Elasticity in compression slightly favors localization.

The angle $\phi_{loc}$ increases with $\delta$ and with the compressibility (i.e., with decreasing $\Gamma$). For $\Gamma = 1, \phi_{loc} = 45^\circ$ for any $\delta$ and the perturbation in strain rate $\Delta \dot{\varepsilon}$ is simple shear. For $\Gamma \leq 1$, the perturbation $\Delta \dot{\varepsilon}$ has a small traction component (or compression depending on the sense of $\mathbf{M}_{loc}$).

### 3.3. Perturbation growth

If we now consider the growth of a perturbation, according to the form of the incremental stress–strain relation (17) and of the homogeneous solution (24), relations (10) for the perturbation become:

$$\Delta \sigma + \frac{1}{4G^2 \delta_{10}} \frac{H}{2G} (\Delta \sigma - (\mathbf{n}_1 : \mathbf{\Delta s}) \mathbf{n}_1) = \lambda |\Delta \varepsilon| \mathbf{1} + 2G \Delta \dot{\varepsilon} - \frac{4C^2}{2G + H} (\mathbf{n}_1 : \Delta \dot{\varepsilon}) \mathbf{n}_1, \quad \Delta \dot{\varepsilon} = \frac{2G}{H + 2G} (\mathbf{n}_1 : \Delta \dot{\varepsilon}). \quad (31)$$

We are now looking for perturbations which grow proportionally:

$$\Delta \sigma(t) = g(t) \Delta \dot{\varepsilon}, \quad \Delta \dot{\varepsilon}(t) = g(t) \Delta \dot{\varepsilon} = g(t) \Delta \dot{\varepsilon} (\mathbf{M} \otimes \mathbf{N}), \quad \Delta \dot{\varepsilon} = g(t) \Delta \dot{\varepsilon}. \quad (32)$$

If perturbations of this form do exist, functions of time and position can be separated in Eq. (31). So, one can find a scalar $\psi$ such that the following relations are satisfied simultaneously:

$$g'(t) = \psi \frac{1}{4G^2 \delta_{10}} \frac{H}{2G} g(t) \quad (33)$$
and
\[ \psi \Delta \epsilon + (\Delta \sigma - (n_1 \cdot \Delta \epsilon) n_1) = \psi \left[ A \text{Tr}(\Delta \epsilon) I + 2G \Delta \epsilon - \frac{4C^2}{2G + H}(n_1 \cdot \Delta \epsilon) n_1 \right]. \]

Relation (33) writes on these components:
\[ \Delta \sigma = \Delta \sigma^m + \Delta \sigma^b n_1 + \Delta \sigma^b. \]
\[ \Delta \epsilon = \Delta \epsilon^m + \Delta \epsilon^b n_1 + \Delta \epsilon^b. \]
\[ \psi = \frac{1}{3}(M \cdot N) I + (M \cdot n_1 - N \cdot n_1) n_1 + (M \otimes N)_3 - (M \cdot n_1 \cdot N \cdot n_1 - \frac{1}{3}(M \cdot N) I). \]

Relation (34) is a second order equation on \( \psi \). The higher of the 2 solutions gives a perturbation growth rate as a function of the compressibility, the hardening modulus, the normal to the yield surface and the band orientation: \( \psi (\Gamma, \bar{H}, n_1, N) \).

In the flat plate case, we study the orientations \( N \) in the \((x,z)\) plane (\( \phi \) is angle made with the \( x \)-axis). \( \psi \) is plotted on Fig. 3 as a function of \( \phi \) for the equi-biaxial stress case, a compressibility parameter \( \Gamma = 0.85 \) and for different values of \( \bar{H} \). It is shown that no positive growth rates can be found when the hardening modulus \( \bar{H} \) is positive.

It is also seen on Fig. 3 that for \( \bar{H} < 0 \) a most favorable orientation \( \phi_{\text{max}} \) is displayed which gives the maximum growth rate \( \psi_{\text{max}} \). For different values of \( \Gamma \) and \( \delta \), \( \psi_{\text{max}} \) and \( \phi_{\text{max}} \) are plotted as functions of \( \bar{H} \) in the interval \([\bar{H}_{\text{loc}}, 0]\) (Figs. 4 and 5).

It is seen on the Fig. 4 that, for any value of \( \Gamma \) and \( \delta \), when \( \bar{H} \) varies from \( \bar{H}_{\text{loc}} \) to 0, \( \psi_{\text{max}} \) decreases from infinity to 0. In other words, for negative hardening moduli which are higher than \( H_{\text{loc}} \), there exist perturbations growing at a finite rate. For a given \( \bar{H} \), a slight compressibility gives a slightly higher value of \( \psi_{\text{max}} \) and so favors localization as has also been exhibited for the bifurcation criterion.

Fig. 5 shows that \( \phi_{\text{max}} \) evolves monotonically from \( \phi_{\text{loc}} \) to 45°; thus, for \( \bar{H} > \bar{H}_{\text{loc}} \), the orientation of the most favorable band is slightly different from the one of the first bifurcation.

Let us now consider the case \( \Gamma = 0.85 \) and the deviatoric stress \( s_{10} = 0.01 \). For the different biaxialities \( \delta \), we can study the growing functions for hardening coefficients which are the half and the tenth of the bifurcation hardening coefficient: \( \Pi_{1/2} = \bar{H}_{\text{loc}}/2 \) and \( \Pi_{1/10} = \bar{H}_{\text{loc}}/10 \) respectively. These values and the ones of the associated growth rate \( (\psi_{1/2} \text{ and } \psi_{1/10}) \) are reported in Table 1.

The growing functions \( g(t) \) are plotted on Fig. 6 for \( \bar{H} \in [0, 0.05] \). In this time interval, the deformations of the homogenous solution remain small and the equivalent stress remains strictly positive in a hypothesis of linear softening:

\[ s_{10} + \frac{\bar{H}}{(1 + \bar{H})} \bar{H} > 0 \quad \text{for} \quad \bar{H} \in [0, 0.05]. \]

Those graphs show high differences between the case \( \bar{H} = \Pi_{1/10} \) and the case \( \bar{H} = \Pi_{1/2} \). In the case \( \bar{H} = \Pi_{1/10} \), the growing functions are nearly linear and grow very slowly. They increase a little more rapidly for the low biaxiality rates. In the case \( \bar{H} = \Pi_{1/2} \), they grow far more rapidly with approximately the same initial slope for the different biaxiality rates. Yet, the slope increases with time for the higher biaxiality rates (which nevertheless also correspond to the lower \( \Pi_{1/2} \)).
The influence of the deviatoric stress $s\_{10}$ is exhibited on Fig. 7 for $\delta = 1$ and $\Pi = \Pi_{1/2}$, a higher stress giving a weaker growing function.

### Table 1

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<th>$\delta$</th>
<th>$\Pi_{1/2}$</th>
<th>$\psi_{1/2}$</th>
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### 4. Evolution of the yield surface shape

In this section, the plasticity law of Section 3 is modified in such a way to take into account the shape of the plasticity yield surface and its evolution with plastic deformation. This new constitutive law is then introduced in the perturbation growth analysis.

Let us assume that this law keeps the homogeneous solution unchanged, i.e. the stress deviator $s$ remains collinear to a constant tensor $n$, which is normal to the yield surface at each time. However, the curvature of the yield surface in these points of the stress space may be different from the one of the Von Mises surface and may evolve during material deformation.
In the neighborhood of the homogeneous solution, the yield surface is approximated by a Hill-ellipsoid the shape of which is a function of the internal state variable \( \alpha \). Function \( F \) then becomes:

\[ F(\sigma, \alpha) = \sigma_{\text{aniso}} - (Y_0 + H\alpha). \tag{40} \]

For the sake of simplicity, it is admitted that all the components of the stress deviator orthogonal to \( \mathbf{n}_1 \) are equivalent. So, the Hill equivalent stress \( \sigma_{\text{aniso}}^g \) is a function of \( s^i \) and \( s^j \) defined with the same notations as in Eqs. (34) and of an evolving anisotropy parameter \( B^i(\alpha) \):

\[ \sigma_{\text{aniso}}^g = \sqrt{(s^i)^2 + B^i(\alpha)(s^i \cdot s^i)}. \]

It is easily checked that the stress–strain relation remains unchanged in the direction of \( \mathbf{n}_1 \) and the homogeneous solution remains the same as in Section 3.

When \( B^i(\alpha) > 1 \), the curvature is higher than in the Von Mises case and when \( B^i(\alpha) < 1 \) it is lower. If \( B^i(\alpha) \) increases with \( \alpha \), the ellipsoid shrinks in the direction orthogonal to \( \mathbf{n}_1 \) and the curvature of the surface increases in the point corresponding to the homogeneous solution. For example, this situation was exhibited by Toth et al. (1996) from poly-crystal plasticity simulations in equi-biaxial loading (\( \beta = 1 \)) for an aluminum sample with an initial cubic texture. The shapes of such an evolving surface and its tangent ellipsoid in the \((s_{xx}, s_{yy})\) plane are displayed in Fig. 8 (this ellipsoid has, of course, no sense for stress states far from the homogeneous solution and the triaxial behavior is not supposed to be ruled by a Hill surface!).

On the contrary, if \( B^i(\alpha) \) decreases with \( \alpha \), the ellipsoid widens in the direction orthogonal to \( \mathbf{n}_1 \) and the curvature of the surface decreases in the point of the homogeneous solution. Such a situation was exhibited for the former aluminum sample in plane strain conditions (\( \beta = 0 \)). The evolving surface and its tangent ellipsoid are displayed in Fig. 9.

The qualitative results exhibited for the aluminum sample may of course be different for a material with a different crystal structure or having a different texture.

Let us now consider a material the behavior of which is ruled by the Hill yield surface given in (40) for both the homogeneous and perturbed solution. The perturbation growth analysis modifies in the following way.

In any point of the surface, the normal is:

\[ \mathbf{n} = \frac{1}{\sigma_{\text{aniso}}} (s^i \mathbf{n}_1 + B^i(\alpha)s^i) \quad \text{and} \quad h = H \frac{d\beta^i}{\sigma_{\text{aniso}}} (s^i \cdot s^i). \tag{41} \]

So, the perturbations on \( \mathbf{n} \) and \( h \) from the homogeneous solution are:

\[ \Delta \mathbf{n} = \frac{1}{\sigma_{\text{aniso}}} B^i(\alpha) \Delta s^i \quad \text{and} \quad \Delta h = 0. \tag{42} \]

The evolution of the perturbation is ruled by the following relations:

\[ \Delta \sigma + \frac{B^i(\alpha)}{2G + H} \Delta s^i = A T r(\Delta \mathbf{e}) \mathbf{1} + 2G \Delta \mathbf{e} - \frac{4G^2}{2G + H} (\mathbf{n}_1 : \Delta \mathbf{e}) \mathbf{n}_1, \]

\[ \Delta \mathbf{e} = \frac{2G}{H + 2G} (\mathbf{n}_1 : \Delta \mathbf{e}). \tag{43} \]

In the following, we suppose that the yield surface shape parameter \( B^i(\alpha) \) is linear in \( \alpha \):

\[ B^i(\alpha) = B_0 + B_1 \alpha. \tag{44} \]

If we still look for perturbations growing proportionally, the growing function now satisfies:

\[ g(t) = \psi \left[ \frac{B_0 + B_1 \frac{t}{1 + H} H}{\frac{4G^2}{2G + H} + \frac{H}{2G}} \right] g(t) \tag{45} \]

or:

\[ \frac{dg}{dt} = \psi \frac{B_0 + B_1 \frac{t}{1 + H} H}{(1 + H) \frac{4G^2}{2G + H} + \frac{H}{2G}} g(t) \]

in non-dimensional variables.

The growing parameter still obeys (38).

Let us now consider the plate in equi-biaxial loading, a deviatoric stress \( s_{10} = 0.01 \) and the hardening parameters \( H = H_{1/2} \) and \( H = H_{1/10} \). The growing function is plotted for different values of \( B_0 \) and \( B_1 \). On Fig. 10, the case of a constant curvature is considered. The effect of the curvature is exhibited varying \( B_0 \) with \( B_1 = 0 \). On Fig. 11, the case of an increasing curvature is considered. The effect of the increasing rate is shown varying \( B_0 \) with \( B_1 = 1 \). (The yield surfaces superimpose the Von Mises surface for \( t = 0 \).) In the first case, the growing function slope increases with \( B_0 \); in the second case, the growing functions have the same slope at the origin but the ones with the higher \( B_1 \) grow faster.

It is confirmed that an initially higher curvature or an increasing curvature of the yield surface both have a destabilizing effect. Nevertheless, high values of \( B_0 \) or \( B_1 \) (in other words a strongly

Fig. 8. Evolving yield surface in the \((s_{xx}, s_{yy})\) plane for equi-biaxial loading – case of a strain-hardening behavior and an increasing curvature.

Fig. 9. Evolving yield surface in the \((s_{xx}, s_{yy})\) plane for plane plastic strain – case of a strain-hardening behavior and a decreasing curvature.
increased curvature) are needed to have a significant effect on the growing function.

Let us remember anyway that, in the former developments, the deviations of the normal \( \mathbf{n} \), which may be induced by texture evolution are neglected. Only curvature changes are taken into account, which is the condition for a proportional loading and for the possibility of quite simple analytic results.

5. Conclusion

In this study, the conditions for localization are analyzed in the simple case of an infinite medium in small deformations and quasi-static loading. The stress state is representative of the one encountered in a flat plate in biaxial tension.

Bands of localization are considered. A bifurcation condition and a condition for growth of a small perturbation of the homogeneous solution are performed and compared. The origins of the differences between both conditions are emphasized. It is proved that, for negative hardening coefficients higher than the one needed for a bifurcation, perturbations may grow with a time scale comparable to the one of the homogeneous solution. The growing function of such a perturbation can be written in an explicit form in the simple case of a homogeneous solution in linear time and keeping a constant direction in the stress space.

This result is coherent with the one established by Dudzinski and Molinari (1991) that perturbation growth conditions are weaker than bifurcation ones. It is also coherent with the results of Jouve (2010) in linear stability analysis in dynamic loading. In such studies, the perturbations are generally assumed to grow at a time scale an order of magnitude shorter than the time scale of the homogeneous solution but Jouve showed that positive growth rates are found in biaxial stretching provided that this hypothesis is partly relaxed.

The evolution of the yield surface shape linked with texture development in the neighborhood of the homogeneous loading path is then introduced in a form as simple as possible. More precisely, the normal \( \mathbf{n} \) remains constant on the homogeneous path but the curvature changes and all the directions orthogonal to \( \mathbf{n} \) in the stress space are equivalent. An increasing curvature seems to be a quite realistic assumption to model the formation of a vertex without losing the regularity and derivability of the yield surface. Unlike the vertex, the curvature has no effect on the bifurcation conditions. However, it changes the perturbation growth conditions which depend on the second derivatives of the yield function. Such perturbations develop more rapidly for a higher curvature justifying the destabilizing effect of this parameter. At the same time, with the former hypotheses, the most favorable orientation of the bands and the glide direction are unchanged with respect to the isotropic case.

Further developments shall be performed on the effect on localization of the shape of yield surfaces and their evolution with plastic deformation. This applies particularly for materials for which instability develops after some amount of plastic deformation or with a low growth rate. In these cases, the changes in the material texture can no longer be ignored. The definition of more realistic evolving yield surfaces shall be deduced from simulations on polycrystals for the materials of interest. We can guess that for such surfaces, perturbation growth conditions as the ones of Sections 3 and 4 may no longer be displayed in an explicit form.

Finally, both finite deformation effects and the influence of the plate geometry shall be taken into account in addition to yield surface shape effects. The modes of instability of a block of finite thickness might be compared to the ones of an infinite medium as was performed in uniaxial strain by Hill and Hutchinson (1975). It is worth evaluating to which extent the addition of
modes with length scales comparable to the block thickness modify the conclusions of the previous study.

References


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