A uniqueness theorem for Dirichlet series satisfying a Riemann type functional equation

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Abstract

We will prove a uniqueness theorem for L-functions in terms of the pre-images of two values in the complex plane.
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1. Introduction and the result

L-functions are Dirichlet series with the Riemann zeta function \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) as the prototype, which are important objects in number theory and have been studied extensively (cf. the recent monograph [7] and various references therein). Throughout the paper, an L-function always means a Dirichlet series \( \mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \) of a complex variable \( s = \sigma + it \), satisfying the following axioms (see e.g. [7, p. 111]):

(i) Ramanujan hypothesis. \( a(n) \ll n^\varepsilon \) for every \( \varepsilon > 0 \);
(ii) Analytic continuation. There is a non-negative integer \( k \) such that \( (s-1)^k \mathcal{L}(s) \) is an entire function of finite order;

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(iii) Functional equation. \( L \) satisfies a functional equation of type
\[ A_L(s) = \omega L(1 - s), \]
where
\[ A_L(s) = L(s) Q^s \prod_{j=1}^K \Gamma(\lambda_j s + \mu_j) \]
with positive real numbers \( Q, \lambda_j, \) and complex numbers \( \mu_j, \omega \) with \( \text{Re} \mu_j \geq 0 \) and \( |\omega| = 1. \)

The degree \( d_L \) of an L-function \( L \) is defined to be \( d_L = 2 \sum_{j=1}^K \lambda_j, \) where \( K, \lambda_j \) are the numbers in the axiom (iii).

Note that the above does not assume the Euler product hypothesis: \( a(1) = 1, \) and \( \log L(s) = \sum_{n=1}^{\infty} b(n) n^{-s}, \) where \( b(n) = 0 \) unless \( n \) is a positive integer power of a prime and \( b(n) \ll n^{\theta} \) for some \( \theta < \frac{1}{2}. \) An L-function satisfying (i)–(iii) and also the Euler product hypothesis is called an L-function in the Selberg class (see e.g. [6,7]), which includes the Riemann zeta-function \( \zeta \) and essentially those Dirichlet series where one might expect a Riemann hypothesis. At the same time, there are a whole host of interesting Dirichlet series not possessing Euler product (cf. \( \zeta \)). The class \( S^2 \) of L-functions satisfying only (ii) and (iii) was introduced in [4], where an L-function satisfying (ii) and (iii) was shown to already satisfy (i) when the degree is between 0 and 1, and the structure of such L-functions was given. In the present paper, we will show how an L-functions \( L \) satisfying (i)–(iii) are uniquely determined by the zeros of \( L - c \) for two distinct complex numbers \( c. \) The result obtained particularly applies to the Selberg class.

By the analytic continuation axiom, an L-function can be analytically continued as a meromorphic function in the complex plane \( \mathbb{C}. \) The zero set of a meromorphic function \( f, \) or more generally, the zero set \( f^{-1}(c) := \{ s \in \mathbb{C}: f(s) = c \} \) of \( f - c \) for a complex value \( c, \) i.e., the set of the pre-image of \( c \) under \( f \) or the \( c \)-values of \( f, \) is one of the main objects in the value distribution theory of meromorphic functions. It is a famous theorem of Nevanlinna, often referred to as Nevanlinna’s uniqueness or unicity theorem, that no two nonconstant meromorphic functions \( f, g \) in \( \mathbb{C} \) must be identically equal if \( f^{-1}(c_j) = g^{-1}(c_j), \) i.e., if \( f - c_j \) and \( g - c_j \) have the same zeros (ignoring multiplicities) for five distinct values \( c_j \in \mathbb{C} \cup \{\infty\} \) (see e.g. [3] or [7]).

We refer the reader to the monograph [7] for a detailed discussion on the topic and related works. In particular, two L-functions in the Selberg class must be identically equal if they have the same zeros with counting multiplicities, which follows from a result in [5] on difference of multiplicities of zeros of two L-functions, see also [2] for a related result. In fact, it was shown in [7, p. 152] that two L-functions (not necessarily in the Selberg class) with \( a(1) = 1 \) must be identically equal if \( L_1 - c \) and \( L_2 - c \) have the same zeros with counting multiplicities for a complex number \( c. \) When ignoring multiplicities, the problem becomes more subtle. The simple example \( \zeta \) and \( \zeta^2, \) which have the same zeros (ignoring multiplicities), shows that the above result is no longer true when ignoring multiplicities; and the following theorem was proved by Steuding (see [7, p. 152]):

**Theorem A.** If two L-functions \( L_1 \) and \( L_2 \) satisfy the same functional equation with \( a(1) = 1 \) and \( L_1^{-1}(c_j) = L_2^{-1}(c_j) \) for two distinct complex numbers \( c_1 \) and \( c_2 \) such that

\[
\liminf_{T \to \infty} \frac{\tilde{N}^{c_1}_{L_j}(T) + \tilde{N}^{c_2}_{L_j}(T)}{\tilde{N}^{c_1}_{L_j}(T) + N^{c_2}_{L_j}(T)} > \frac{1}{2} + \epsilon
\]

for some positive \( \epsilon \) with either \( j = 1 \) or \( j = 2, \) then \( L_1 \equiv L_2. \)
On the above, \( N_c^L(T) \) denotes the number of the zeros of \( L(\sigma + it) - c \) in the rectangle \( 0 \leq \sigma \leq 1, |t| \leq T \) (counting multiplicities), and \( \tilde{N}_c^L(T) \) the number of the same zeros but ignoring multiplicities.

Condition (1.1) reflects that more than 50% of the \( c_1 \) and \( c_2 \) values of \( L_j \) are supposed to be distinct. It was noted by Steuding that such a condition is very difficult to verify; for instance, it is known that more than 63% of the zeros of the Riemann zeta-function are distinct; however, any extension to \( L \)-functions of large degree seems to be hard to realize, as pointed out in [7, p. 152].

The purpose of this paper is to completely remove this condition (1.1). That is, we prove the following

**Theorem 1.** If two \( L \)-functions \( L_1 \) and \( L_2 \) satisfy the same functional equation with \( a(1) = 1 \) and \( L_1^{-1}(c_j) = L_2^{-1}(c_j) \) for two distinct complex numbers \( c_1 \) and \( c_2 \), then \( L_1 \equiv L_2 \).

Unlike the proof of Theorem A in [7], we will employ Nevanlinna theory combined with other analytic tools in our proof, with a careful analysis on the growth and distribution of zeros of the involved functions. Since \( L \)-functions are meromorphic functions and Nevanlinna theory is known as an important tool in studying meromorphic functions, it would be profitable to explore its further applications to the theory of \( L \)-functions.

2. Proof of Theorem 1

For convenience of the reader who might not be familiar with Nevanlinna theory, we list here the notations and results from Nevanlinna theory, which will be used in the proof (see e.g. [3]). Let \( f \) be a meromorphic function in \( \mathbb{C} \). Then the Nevanlinna characteristic \( T(r, f) \) is defined as

\[
T(r, f) = m(r, f) + N(r, f),
\]

where

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta;
\]
\[
\log^+ |x| = \max(0, \log |x|),
\]

and

\[
N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r,
\]

where \( n(t, f) \) denotes the number of poles of \( f \) (counting multiplicities) in \( |s| < t \). Recall the following known result (see e.g. [3, pp. 5, 18, 40]).

(i) The arithmetic properties of \( T(r, f) \) and \( m(r, f) \):

\[
T(r, fg) \leq T(r, f) + T(r, g), \quad T(r, f + g) \leq T(r, f) + T(r, g) + O(1).
\]

The same inequalities hold for \( m(r, f) \).
(ii) \( \log \max_{|s| = r} |f(s)| \leq \frac{R + r}{R - r} T(R, f) \) for \( R > r > 0 \), if \( f \) is entire.

(iii) The Nevanlinna first fundamental theorem: \( T(r, f) = T(r, \frac{1}{f}) + O(1) \).

(iv) The logarithmic derivative lemma: \( m(r, f') = O(\log r) \), if the order \( \rho(f) := \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \) of \( f \) is finite.

**Proof.** We first look at the simple case that one of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), say \( \mathcal{L}_1 \), is constant. Then \( \mathcal{L}_1 \equiv 1 \) by the assumption that \( a(1) = 1 \). Since \( \mathcal{L}_2 - c_j \) and \( \mathcal{L}_1 - c_j \) have the same zeros by the assumption, it is easy to see that \( \mathcal{L}_2 \equiv 1 \) (when \( c_1 \) or \( c_2 \) is 1), or \( \mathcal{L}_2 \neq c_1, c_2 \) in \( \mathbb{C} \) (when \( c_1, c_2 \neq 1 \)). In the latter case, noting that an \( L \)-function has at most one pole, \( \mathcal{L}_2 \) must be constant and thus \( \mathcal{L}_2 \equiv 1 \) since \( a(1) = 1 \), by the classic Picard theorem (see e.g. [1, p. 186]) that a nonconstant meromorphic function in \( \mathbb{C} \) assumes each value in \( \mathbb{C} \cup \{ \infty \} \) infinitely many times with at most two exceptions (or by the “less known” fact that a nonconstant \( L \)-function assumes each complex number, cf. the Riemann–von Mangoldt formula mentioned below). Therefore, \( \mathcal{L}_1 \equiv \mathcal{L}_2(\equiv 1) \).

We thus assume, in the following, that \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are nonconstant. Suppose, to the contrary, that \( \mathcal{L}_1 \neq \mathcal{L}_2 \). Our aim below is to derive a contradiction. To this end, consider the following auxiliary function

\[
F(s) = \frac{(s - 1)^q \mathcal{L}_1' \mathcal{L}_2' (\mathcal{L}_1 - \mathcal{L}_2)^2}{(\mathcal{L}_1 - c_1)(\mathcal{L}_1 - c_2)(\mathcal{L}_2 - c_1)(\mathcal{L}_2 - c_2)}. \tag{2.1}
\]

where \( q \) is an integer chosen so that the function \( F \) does not have a pole or zero at \( s = 1 \). Clearly, \( F \) is not identically zero by the above assumptions. The proof below is to show that \( F \) is identically zero, which serves our purpose.

We claim that \( F \) is an entire function. In fact, the functions \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) both have only one possible pole at \( s = 1 \), which cannot be a pole of \( F \) due to the use of the factor \( (s - 1)^q \). Thus, possible poles of \( F \) may only come from the zeros of \( \mathcal{L}_1 - c_j \) or \( \mathcal{L}_2 - c_j \), \( j = 1, 2 \). If \( w \) is a zero of \( \mathcal{L}_1 - c_j \) of order \( m \geq 1 \) and thus a zero of \( \mathcal{L}_2 - c_j \) of order \( n \geq 1 \) since \( \mathcal{L}_1 - c_j \) and \( \mathcal{L}_2 - c_j \) have the same zeros (\( m \) might be different from \( n \)), it is then a zero of \( (\mathcal{L}_1 - \mathcal{L}_2)^2 \) of multiplicity at least two. Note also that \( w \) is a zero of \( \mathcal{L}_1' \) of order \( m - 1 \) and a zero of \( \mathcal{L}_2' \) of order \( n - 1 \). Thus, the numerator in (2.1) vanishes at \( w \) with a multiplicity at least equal to \( (m - 1) + (n - 1) + 2 = m + n \), which is the order of the zero \( w \) of the denominator in (2.1). Hence, \( w \) is not a pole of \( F \). This shows that \( F \) does not have any poles and thus is an entire function in \( \mathbb{C} \).

We next control the number of zeros of \( F \) in the disc \( |s| < r \) by estimating the counting function \( N(r, \frac{1}{F}) \). We deduce from (2.1) that

\[
N(r, \frac{1}{\mathcal{L}_1 - c_1}) + N(r, \frac{1}{\mathcal{L}_1 - c_2}) + N(r, \frac{1}{\mathcal{L}_2 - c_1}) + N(r, \frac{1}{\mathcal{L}_2 - c_2})
\]

\[
= N(r, \frac{1}{(\mathcal{L}_1 - c_1)(\mathcal{L}_1 - c_2)(\mathcal{L}_2 - c_1)(\mathcal{L}_2 - c_2)})
\]

\[
= N(r, F). \tag{2.2}
\]
Since \( F \) is entire, a possible pole of \( \frac{F}{(s - 1)^q \mathcal{L}_1' \mathcal{L}_2' (\mathcal{L}_1 - \mathcal{L}_2)^2} \) may only come from a zero of \((s - 1)^q \mathcal{L}_1' \mathcal{L}_2' (\mathcal{L}_1 - \mathcal{L}_2)^2\). Recall that \( F \) does not have a zero at \( s = 1 \), the only possible pole of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). Thus, if \( F(w) = 0 \) at a point \( w \), then the numerator \((s - 1)^q \mathcal{L}_1' \mathcal{L}_2' (\mathcal{L}_1 - \mathcal{L}_2)^2\) in (2.1) must vanish also at \( w \) with an equal or higher order. With this information, we deduce that

\[
N\left(r, \frac{F}{(s - 1)^q \mathcal{L}_1' \mathcal{L}_2' (\mathcal{L}_1 - \mathcal{L}_2)^2}\right) \\
\leq N\left(r, \frac{1}{(s - 1)^q \mathcal{L}_1' \mathcal{L}_2' (\mathcal{L}_1 - \mathcal{L}_2)^2}\right) - N\left(r, \frac{1}{F}\right) \\
\leq N\left(r, \frac{1}{(s - 1)^q}\right) + N\left(r, \frac{1}{\mathcal{L}_1}\right) + N\left(r, \frac{1}{\mathcal{L}_2}\right) + 2N\left(r, \frac{1}{\mathcal{L}_1 - \mathcal{L}_2}\right) - N\left(r, \frac{1}{F}\right) \\
= N\left(r, \frac{1}{\mathcal{L}_1}\right) + N\left(r, \frac{1}{\mathcal{L}_2}\right) + 2N\left(r, \frac{1}{\mathcal{L}_1 - \mathcal{L}_2}\right) - N\left(r, \frac{1}{F}\right) + O(\log r), \quad (2.3)
\]

in view of the fact that \( N(r, \frac{1}{(s - 1)^q}) = O(\log r) \). Also, by the first fundamental theorem, we deduce that

\[
N\left(r, \frac{1}{\mathcal{L}_1'}\right) = T\left(r, \frac{1}{\mathcal{L}_1}\right) - m\left(r, \frac{1}{\mathcal{L}_1'}\right) \\
= m(r, \mathcal{L}_1) + N(r, \mathcal{L}_1') + O(1) - m\left(r, \frac{1}{\mathcal{L}_1'}\right) \\
= m\left(r, \frac{\mathcal{L}_1'}{\mathcal{L}_1}\right) + N(r, \mathcal{L}_1') + O(1) - m\left(r, \frac{1}{\mathcal{L}_1'}\right) \\
\leq m\left(r, \frac{\mathcal{L}_1'}{\mathcal{L}_1}\right) + m(r, \mathcal{L}_1) + N(r, \mathcal{L}_1') + O(1) - m\left(r, \frac{1}{\mathcal{L}_1'}\right).
\]

Note that \( N(r, \mathcal{L}_1') = O(\log r) \) since \( \mathcal{L}_1' \) has at most one pole \( s = 1 \), and \( m(r, \frac{\mathcal{L}_1'}{\mathcal{L}_1}) = O(\log r) \) by the logarithmic derivative lemma. We obtain that

\[
N\left(r, \frac{1}{\mathcal{L}_1'}\right) \leq m(r, \mathcal{L}_1') - m\left(r, \frac{1}{\mathcal{L}_1'}\right) + O(\log r).
\]

We then use the first fundamental theorem and logarithmic derivative lemma again to deduce that

\[
N\left(r, \frac{1}{\mathcal{L}_1'}\right) \leq T\left(r, \mathcal{L}_1\right) - m\left(r, \frac{1}{\mathcal{L}_1'}\right) + O(\log r) \\
\leq T\left(r, \mathcal{L}_1 - c_j\right) - m\left(r, \frac{1}{\mathcal{L}_1'}\right) + O(\log r) \\
= T\left(r, \frac{1}{\mathcal{L}_1 - c_j}\right) - m\left(r, \frac{1}{\mathcal{L}_1'}\right) + O(\log r)
\]
These last two inequalities together with (2.3) and (2.2) yield that

\[ N\left(r, \frac{1}{\mathcal{L}_1 - c_j}\right) + m\left(r, \frac{1}{\mathcal{L}_1 - c_j}\right) - m\left(r, \frac{1}{\mathcal{L}'_1}\right) + O(\log r) \]

\[ = N\left(r, \frac{1}{\mathcal{L}_1 - c_j}\right) + m\left(r, \frac{\mathcal{L}'_1}{\mathcal{L}_1 - c_j \mathcal{L}'_1}\right) - m\left(r, \frac{1}{\mathcal{L}'_1}\right) + O(\log r) \]

\[ \leq N\left(r, \frac{1}{\mathcal{L}_1 - c_j}\right) + m\left(r, \frac{\mathcal{L}'_1}{\mathcal{L}_1 - c_j \mathcal{L}'_1}\right) + m\left(r, \frac{1}{\mathcal{L}'_1}\right) - m\left(r, \frac{1}{\mathcal{L}'_1}\right) + O(\log r) \]

\[ = N\left(r, \frac{1}{\mathcal{L}_1 - c_j}\right) + O(\log r). \]

In the exactly same way, we have that

\[ N\left(r, \frac{1}{\mathcal{L}_2}\right) \leq N\left(r, \frac{1}{\mathcal{L}_2 - c_j}\right) + O(\log r). \]

These last two inequalities together with (2.3) and (2.2) yield that

\[ N\left(r, \frac{1}{\mathcal{L}_1 - c_1}\right) + N\left(r, \frac{1}{\mathcal{L}_1 - c_2}\right) + N\left(r, \frac{1}{\mathcal{L}_2 - c_1}\right) + N\left(r, \frac{1}{\mathcal{L}_2 - c_2}\right) \]

\[ \leq N\left(r, \frac{1}{\mathcal{L}_1 - c_2}\right) + N\left(r, \frac{1}{\mathcal{L}_2 - c_2}\right) + 2N\left(r, \frac{1}{\mathcal{L}_1 - \mathcal{L}_2}\right) - N\left(r, \frac{1}{F}\right) + O(\log r) \]

or

\[ N\left(r, \frac{1}{F}\right) \leq 2N\left(r, \frac{1}{\mathcal{L}_1 - \mathcal{L}_2}\right) - N\left(r, \frac{1}{\mathcal{L}_1 - c_1}\right) - N\left(r, \frac{1}{\mathcal{L}_2 - c_1}\right) + O(\log r) \]

\[ = \int_0^r \left\{ 2n\left(t, \frac{1}{\mathcal{L}_1 - \mathcal{L}_2}\right) - n\left(t, \frac{1}{\mathcal{L}_1 - c_1}\right) - n\left(t, \frac{1}{\mathcal{L}_2 - c_1}\right) \right. \]

\[ - 2n\left(0, \frac{1}{\mathcal{L}_1 - \mathcal{L}_2}\right) + n\left(0, \frac{1}{\mathcal{L}_1 - c_1}\right) + n\left(0, \frac{1}{\mathcal{L}_2 - c_1}\right) \frac{dt}{t} \]

\[ + O(\log r) \]  

(2.4)

by the definition of the counting function \(N(r, \cdot)\). We then turn to estimate the un-integrated counting functions \(n(t, \cdot)\) on the right-hand side of (2.4). By the assumption of the theorem, \(\mathcal{L}_1\) and \(\mathcal{L}_2\) satisfy the same functional equation and thus have the same degree. Also, \(\mathcal{L}_1 - \mathcal{L}_2\) clearly satisfies the axioms (i)–(ii), and also (iii) with the same functional equation and thus the same degree. For convenience, for an \(L\)-function \(\mathcal{L}\) and a complex number \(c\) we denote by \(n_-(t, \frac{1}{\mathcal{L} - c})\) the number of zeros (counting multiplicities) of \(\mathcal{L} - c\) within \(|s| < t\) and on the left half-plane \(\{\sigma \leq 0\}\). It is known that the zeros of \(\mathcal{L} - c\) on the left half-plane \(\{\sigma \leq 0\}\) have bounded imaginary parts and the number of these zeros having real part in \([-t, 0]\) is \(\frac{1}{2}d_{\mathcal{L}}t + O(1)\) as \(t \to +\infty\), where \(d_{\mathcal{L}}\) is the degree of \(\mathcal{L}\) (see [7, p. 145]). Thus,

\[ n_-(t, \frac{1}{\mathcal{L} - c}) = \frac{1}{2}d_{\mathcal{L}}t + O(1). \]
We then have that
\[
2n_-(t, \frac{1}{L_1 - L_2}) - n_-(t, \frac{1}{L_1 - c_1}) - n_-(t, \frac{1}{L_2 - c_1}) = O(1) \tag{2.5}
\]
as \(t \to \infty\). On the other hand, by the Riemann–von Mangoldt formula for L-functions (see [7, pp. 145 and 147]), the number of zeros (counting multiplicities) of \(L(s) - c\) in the region \(\text{Re } s > 0, |\text{Im } s| \leq T\), denoted by \(N_{L}^c(T)\), is given by
\[
N_{L}^c(T) = \frac{dL}{\pi} T \log \frac{T}{e} + \frac{T}{\pi} \log(\lambda Q^2) + O(\log T),
\]
where \(\lambda = \prod_{j=1}^{K} \lambda_j^{2\lambda_j}\), and \(\lambda_j, K, Q\) are the numbers in the axiom (iii). (This was proved for \(c \neq 1\). We may assume \(c_1 \neq 1\). Otherwise, replace \(c_1\) by \(c_2\) in (2.4).) Also, notice that \(L - c\) does not have any zeros when \(\sigma = \text{Re } s\) is large, say \(\sigma \geq \alpha > 0\), which follows easily from the Dirichlet series of \(L\). [In fact, this is clear if \(c \neq 1\), since \(L(s) = 1 + O(2^{-s}) \to 1\) as \(\sigma \to +\infty\). If \(c = 1\), then \(|L - c| = |L - 1| \geq \frac{C_1}{n_1^2}\) for an integer \(n_1 \geq 2\) and a constant \(C_1 > 0\), cf. (2.12) below.] Thus, if \(n_+(t, \frac{1}{L-c})\) denotes the number of zeros of \(L - c\) within \(|s| < t\) but on the right half-plane \(\{\sigma > 0\}\), then
\[
n_+(t, \frac{1}{L-c}) \geq N_{L}^c(\sqrt{t^2 - \alpha^2})
\]
for large \(t\), which implies that
\[
n_+(t, \frac{1}{L_1 - c_1}) + n_+(t, \frac{1}{L_2 - c_1}) \geq \frac{d}{\pi} \sqrt{t^2 - \alpha^2} \log \frac{\sqrt{t^2 - \alpha^2}}{e} + \frac{\sqrt{t^2 - \alpha^2}}{\pi} \log(\lambda Q^2) + O(\log \sqrt{t^2 - \alpha^2}), \tag{2.6}
\]
where \(d = d_{L_1} = d_{L_2}\) is the degree of \(L_1\) and \(L_2\). Recalling that \(L_1 - L_2\) has the same degree \(d\), we have that
\[
2n_+(t, \frac{1}{L_1 - L_2}) \leq 2N_{L_1-L_2}^0(t) = 2 \left(\frac{d}{\pi} t \log \frac{t}{e} + \frac{t}{\pi} \log(\lambda Q^2) + O(\log t)\right). \tag{2.7}
\]
It then follows from (2.6) and (2.7) that
\[
2n_+(t, \frac{1}{L_1 - L_2}) - n_+(t, \frac{1}{L_1 - c_1}) - n_+(t, \frac{1}{L_2 - c_1}) \leq 2 \left(\frac{d}{\pi} t \log \frac{t}{e} + \frac{t}{\pi} \log(\lambda Q^2) + O(\log t)\right)
- 2 \left(\frac{d}{\pi} \sqrt{t^2 - \alpha^2} \log \frac{\sqrt{t^2 - \alpha^2}}{e} + \frac{\sqrt{t^2 - \alpha^2}}{\pi} \log(\lambda Q^2) + O(\log \sqrt{t^2 - \alpha^2})\right)
\]
\[
2 \frac{d}{\pi} \log \frac{t}{e} - 2 \frac{d}{\pi} \sqrt{t^2 - \alpha^2} \log \frac{\sqrt{t^2 - \alpha^2}}{e} + O(\log t)
\]
\[
= 2 \frac{d}{\pi} \log t - 2 \frac{d}{\pi} \sqrt{t^2 - \alpha^2} \log \sqrt{t^2 - \alpha^2} + O(\log t)
\]
\[
= 2d \frac{t}{\pi} \log t - 2d \frac{t}{\pi} \sqrt{1 - \frac{\alpha^2}{t^2}} \left( \log t + \log \sqrt{1 - \frac{\alpha^2}{t^2}} \right) + O(\log t)
\]
\[
= 2d \frac{t}{\pi} \log t - 2d \frac{t}{\pi} \left( 1 - O(1) \right) \left( \log t + O(1) \right) + O(\log t)
\]
\[
= O(\log t)
\]
as \( t \to \infty \). Combining this with (2.5) yields that

\[
2n \left( t, \frac{1}{L_1 - L_2} \right) - n \left( t, \frac{1}{L_1 - c_1} \right) - n \left( t, \frac{1}{L_2 - c_1} \right)
\]
\[
= 2n_+ \left( t, \frac{1}{L_1 - L_2} \right) - n_+ \left( t, \frac{1}{L_1 - c_1} \right) - n_+ \left( t, \frac{1}{L_2 - c_1} \right)
\]
\[
+ 2n_- \left( t, \frac{1}{L_1 - L_2} \right) - n_- \left( t, \frac{1}{L_1 - c_1} \right) - n_- \left( t, \frac{1}{L_2 - c_1} \right)
\]
\[
= O(\log t) \leq C \log t
\]

for all \( t \geq r_0 \), where \( C, r_0 \) are two positive numbers. We then obtain, from (2.4), the following control on the zeros of \( F \):

\[
N \left( r, \frac{1}{F} \right) \leq \int_{r_0}^{r} \left\{ 2n \left( t, \frac{1}{L_1 - L_2} \right) - n \left( t, \frac{1}{L_1 - c_1} \right) - n \left( t, \frac{1}{L_2 - c_1} \right) - 2n \left( 0, \frac{1}{L_1 - L_2} \right) + n \left( 0, \frac{1}{L_1 - c_1} \right) + n \left( 0, \frac{1}{L_2 - c_1} \right) \right\} \frac{dt}{t} + O(1) + O(\log r)
\]
\[
\leq \int_{r_0}^{r} \frac{C \log t}{t} \frac{dt}{t} + O(\log r) = O(\log^2 r).
\]

We next give a tight estimate on the modulus of \( F \), which is needed later in the proof, using the above estimate (2.8). In order to do this, suppose that the nonzero zeros of \( F \) are \( a_1, a_2, a_3, \ldots \), arranged in the order of increasing moduli and repeated according to multiplicities, and that \( s = 0 \) is a zero of \( F \) of order \( l \geq 0 \). Then the function \( \frac{F(s)}{s^l} \) does not vanish at \( s = 0 \) and its zeros are exactly \( a_1, a_2, a_3, \ldots \). Then it holds (for any nonzero meromorphic function \( F \), see e.g. [3, p. 25]) that

\[
\sum_{k=1}^{\infty} |a_k|^{-1} \leq \int_{0}^{\infty} \frac{N(t, \frac{F}{t^l})}{t^2} \, dt.
\]
Note, by (2.8), that
\[ N\left( t, \frac{s}{F} \right) \leq N\left( t, s^I \right) + N\left( t, \frac{1}{F} \right) \]
\[ \leq O(\log t) + O(\log^2 t) = O(\log^2 t). \] (2.9)

Thus, \( \sum_{k=1}^{\infty} |a_k|^{-1} \) is convergent. This convergence guarantees (see [3, p. 27]) that the canonical product \( P(s) := \prod_{\rho} (1 - \frac{s}{\rho_1}) \) is an entire function of order \( \rho(P) \) equal to \( \lim sup_{r \to \infty} \frac{\log N(r, s^I)}{\log r} \), which is zero by (2.9). Thus, by the definition of order, \( T(r, P) = O(r^\varepsilon) \) for any \( 0 < \varepsilon < 1 \), which implies, by the inequality (ii) mentioned in the beginning of Section 2 with \( R = 2r \) and \( r = |s| \), that
\[ \log |P(s)| \leq 3T(2|s|, P) = O(|s|^\varepsilon). \] (2.10)

It is known (see [7, p. 150]) that for an L-function \( L \),
\[ T(r, L) = \frac{dL}{\pi} r \log r + O(r). \]

Also,
\[ T\left( r, L' \right) = m\left( r, L' \right) + N\left( r, L' \right) \]
\[ = m\left( r, \frac{L'}{L} \right) + O(\log r) \]
\[ \leq m\left( r, \frac{L'}{L} \right) + m(r, L) + O(\log r) \leq T\left( r, L \right) + O(\log r) \]
by the logarithmic derivative lemma. Hence, by the arithmetic properties of the characteristic function and the first fundamental theorem, we deduce, in view of (2.1), that
\[ T\left( r, F \right) \leq T\left( r, (s - 1)^9 \right) + T\left( r, L'_1 \right) + T\left( r, L'_2 \right) + 4T\left( r, L_1 \right) + 4T\left( r, L_2 \right) + O(1) \]
\[ \leq 10 \frac{d}{\pi} r \log r + O(r). \]

This clearly implies, by the definition of order, that \( F \) is of order at most 1. We then have, by the classic Hadamard factorization theorem (see e.g. [1, p. 384]), that \( F(s) = s^I P(s)e^{As+B} \), where \( A, B \) are two complex numbers. We thus obtained, in view of (2.10), the following estimate on the modulus of \( F \):
\[ F(s) = |s^I P(s)e^{As+B}| = O(e^{\alpha|s|}) \] (2.11)
for some \( \alpha > 0 \) and all large \( |s| \). (It is worth mentioning here that a rougher estimate on \( |F| \) may be obtained directly using the above estimate on \( T(r, F) \), which is however not good enough for our use below.)
Our next step is to examine the end behavior of $F(s)$ as $\sigma \to +\infty$ and $\sigma \to -\infty$, where and in the sequel $s = \sigma + it$. Since $L_1(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^\sigma}$ with $a(1) = 1$, we clearly have that

$$\frac{C_1}{n_1^\sigma} \leq |L_1 - 1| \leq \frac{C_2}{n_1^\sigma}, \quad \text{and} \quad L_1' = O \left( \frac{1}{n_1^\sigma} \right), \quad \text{as} \quad \sigma \to +\infty, \quad (2.12)$$

for some positive constants $C_1, C_2$, where $n_1$ is the smallest integer $n \geq 2$ such that $a(n) \neq 0$.

Similarly, there is an integer $n_2 \geq 2$ such that

$$\frac{C_3}{n_2^\sigma} \leq |L_2 - 1| \leq \frac{C_4}{n_2^\sigma}, \quad \text{and} \quad L_2' = O \left( \frac{1}{n_2^\sigma} \right), \quad \text{as} \quad \sigma \to +\infty, \quad (2.13)$$

for some $C_3, C_4 > 0$. We also have that

$$L_1 - L_2 = O \left( \frac{1}{2\sigma} \right). \quad (2.14)$$

It is then clear from (2.1) that if $c_1, c_2 \neq 1$,

$$F(s) = \frac{(s - 1)^q O \left( \frac{1}{n_1^\sigma} \right) O \left( \frac{1}{n_2^\sigma} \right) (O \left( \frac{1}{2\sigma} \right))^2}{(1 - c_2)^2(1 - c_2)^2}$$

as $\sigma \to +\infty$ for any fixed $t$. If one of $c_1, c_2$, say $c_1$, is 1, then by (2.12) and (2.13), we have that

$$|F(s)| = \frac{\frac{C_1}{n_1^\sigma} \frac{C_3}{n_2^\sigma} (1 - c_2)(1 - c_2)}{C_1 n_1^\sigma C_3 n_2^\sigma (1 - c_2)^2 (1 - c_2)^2}$$

as $\sigma \to +\infty$. Thus, in any case, we always have that

$$|F(s)| = O \left( \frac{1}{4\sigma} \right) = O \left( \frac{1}{e^{q|t|}} \right) \quad (2.15)$$

as $\sigma \to +\infty$ (for any fixed $r$).

We will prove that (2.15) holds also as $\sigma \to -\infty$. However, in this case, the Dirichlet series does not give us (2.12) and (2.13) any more. We turn to use the following result for a nonconstant $L$-function $L$: For any $\sigma$ and large $t$, say $t \geq t_0 > 1$,

$$\epsilon_0 (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{(\frac{1}{2} - \sigma) d_L} |L(1 - \sigma + it)| \leq |L(\sigma + it)| \leq C_0 (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{(\frac{1}{2} - \sigma) d_L} |L(1 - \sigma + it)|$$

for two positive constants $\epsilon_0, C_0$ independent of $\sigma, t$, where $\lambda = \sum_{j=1}^{K} \lambda_j^{\frac{1}{2} - \sigma}$, and $\lambda_j, Q, K$ are the numbers in the axiom (iii) (see Lemma 6.7 in [7, p. 125]). (Note that Lemma 6.7 in [7, p. 125] was stated for $L$ functions in the Selberg class, but the proof does not use the Euler product hypothesis.) As $\sigma \to -\infty$, we have that $1 - \sigma \to +\infty$. Hence, $L(1 - \sigma + it) \to 1$ (cf.
(2.12) or (2.13)) and then $\frac{1}{2} \leq |L(1 - \sigma + it)| \leq 2$ as $\sigma \to -\infty$, which implies that for $t \geq t_0$ and as $\sigma \to -\infty$,

$$
\epsilon (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{\frac{1}{2} - \sigma} L \leq |L(\sigma + it)| \leq C (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{\frac{1}{2} - \sigma} L
$$

(2.16)

for two positive constants $\epsilon, C$. By the Cauchy formula and (2.16), we have for $t \geq t_0$ and $\sigma \to -\infty$,

$$
|L'(\sigma + it)| = \left| \frac{1}{2\pi i} \int_{|w - (\sigma + it)| = 1} \frac{L(w)}{w - (\sigma + it)} \, dw \right| \leq C_1 (\lambda Q^2)^{\frac{3}{2} - \sigma} t^{(1 - \frac{3}{2} - \sigma) L}.
$$

(2.17)

noting that when $|w - (\sigma + it)| = 1$, $\sigma - 1 < \text{Re} \, w < \sigma + 1$ and $t - 1 < \text{Im} \, w < t + 1$, where $C_1 = C$ if $\lambda Q^2 > 1$ and $C_1 = (\lambda Q^2)^{-2} C$ if $\lambda Q^2 < 1$. The inequalities (2.16) and (2.17) hold for $\mathcal{L}_1$ and $\mathcal{L}_2$ with $d_L = d$. Applying the above stated result to $\mathcal{L}_1 - \mathcal{L}_2$, we also have that for $t \geq t_0$ and as $\sigma \to -\infty$,

$$
|L_1 - L_2)(s) \leq C_0 (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{\frac{1}{2} - \sigma} |L_1 - L_2)(1 - \sigma + it)|
\leq C_0 (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{\frac{1}{2} - \sigma} O\left(\frac{1}{2^{1 - \sigma}}\right).
$$

(2.18)

If $d \neq 0$, then for fixed $t \geq t_0 > 1$ and as $\sigma \to -\infty$,

$$
(\lambda Q^2)^{\frac{1}{2} - \sigma} t^{(1/2 - \sigma) d} \to +\infty.
$$

This also holds when $d = 0$ and $Q > 1$. (When $d = 0$, the functional equation in the axiom (iii) does not contain Gamma factors and $\lambda$ above is 1, cf. (2.20) below.) Thus, in these two cases, we have, by (2.1), (2.18), (2.16), and (2.17), that for fixed large $t \geq t_0$ and $\sigma \to -\infty$,

$$
|F(s)| \leq \frac{\epsilon (s - 1)^d |C_1 (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{\frac{1}{2} - \sigma} L|^2 |C_0 (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{\frac{1}{2} - \sigma} L| \cdot O\left(1\right)}{\left|\epsilon (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{(1/2 - \sigma) d} - |c_1| \right|^2 \left|\epsilon (\lambda Q^2)^{\frac{1}{2} - \sigma} t^{(1/2 - \sigma) d} - |c_2| \right|^2}
\leq A_1 |s - 1|^d (t + 1)^{2d} \left(1 + \frac{1}{t} \right)^{(1 - 2\sigma) d} O\left(\frac{1}{4^{1 - \sigma}}\right)
\leq A_1 |s - 1|^d (t + 1)^{2d} \left(1 + 2(1 - 2\sigma) d \frac{1}{t} \right) O\left(\frac{1}{4^{1 - \sigma}}\right)
= O\left(\frac{1}{e^{\sigma}}\right) = O\left(\frac{1}{e^{\sigma_0}}\right).
$$

(2.19)

for some constant $A_1 > 0$, in view of the simple inequality that $(1 + x)^m \leq 1 + 2mx$ for any positive $m$ and small positive $x$. Thus (2.15) holds for large $t \geq t_0$ and as $\sigma \to -\infty$, when $d \neq 0$ and when $d = 0$, $Q > 1$. 


Consider now the cases \( d = 0, Q \leq 1 \). The first inequality in (2.19) may not hold any more. But, when \( d = 0 \), the functional equation in the axiom (iii) does not have any Gamma factors and becomes

\[
\mathcal{L}(s) = \omega \frac{Q^{1-2s}}{L(1-s)}.
\]

When \( \sigma \to -\infty, 1 - \sigma \to +\infty \). Thus for such \( \sigma , \mathcal{L}(1-\bar{s}) = \mathcal{L}(1-\sigma + it) \) has Dirichlet series representation with \( a(1) = 1 \). Hence, \( \mathcal{L} \) is of the form

\[
\mathcal{L}(s) = \omega \frac{Q^{1-2s}}{L(1-s)}
\]

(2.20)

If \( Q = 1 \), then \( \lim_{\sigma \to -\infty} \mathcal{L}(s) = \omega \) for \( \mathcal{L} = \mathcal{L}_1, \mathcal{L}_2 \), and \( \mathcal{L}_1 - \mathcal{L}_2 = O\left(\frac{1}{2^{1-\sigma}}\right) \). Taking derivative in (2.21), \( \mathcal{L}'(s) = O\left(\frac{1}{2^{1-\sigma}}\right) \) for \( \mathcal{L} = \mathcal{L}_1, \mathcal{L}_2 \). Thus, we deduce by (2.1) that

\[
|F(s)| \leq \frac{|s - 1|^q \left(O\left(\frac{1}{2^{1-\sigma}}\right)\right)^2 \left(O\left(\frac{1}{2^{1-\sigma}}\right)\right)^2}{|\omega - c_1|^2 \left|\omega - c_2\right|^2} = O\left(\frac{1}{e^{\sigma}}\right)
\]

as \( \sigma \to -\infty \), provided that \( c_1, c_2 \neq \omega \). If one of \( c_1, c_2 \) is \( \omega \), say \( c_1 = \omega \), then by (2.21) and as in (2.12),

\[
|\mathcal{L}_1 - c_1| = |\mathcal{L}_1 - \omega| \geq \frac{A_2}{n_1^{1-\sigma}}, \quad \mathcal{L}_1' = O\left(\frac{1}{2^{1-\sigma}}\right)
\]

as \( \sigma \to -\infty \) for some \( A_2 > 0 \), where \( n_1 \geq 2 \) is the smallest integer \( n \) such that \( a(n) \neq 0 \) in (2.21) with \( \mathcal{L} = \mathcal{L}_1 \). Similarly,

\[
|\mathcal{L}_2 - c_1| \geq \frac{A_3}{n_2^{1-\sigma}}, \quad \mathcal{L}_2' = O\left(\frac{1}{2^{1-\sigma}}\right)
\]

for some \( A_3 > 0 \) and an integer \( n_2 \geq 2 \); and

\[
(\mathcal{L}_1 - \mathcal{L}_2)(s) = O\left(\frac{1}{2^{1-\sigma}}\right)
\]

as \( \sigma \to -\infty \). Thus,

\[
|F(s)| \leq \frac{|s - 1|^q \left(O\left(\frac{1}{n_1^{1-\sigma}}\right)\right)^2 \left(O\left(\frac{1}{n_2^{1-\sigma}}\right)\right)^2 \left(O\left(\frac{1}{2^{1-\sigma}}\right)\right)^2}{\left|\frac{A_2}{n_1^{1-\sigma}}\frac{A_3}{n_2^{1-\sigma}}(\omega - c_2)^2\right|} = O\left(\frac{1}{e^{\sigma}}\right).
\]

That is, (15) holds as \( \sigma \to -\infty \), when \( d = 0 \) and \( Q = 1 \).

If \( d = 0, Q < 1 \), then by (2.21), \( \lim_{\sigma \to -\infty} \mathcal{L}(s) = 0 \) for \( \mathcal{L} = \mathcal{L}_1, \mathcal{L}_2 \), and \( \mathcal{L}_1 - \mathcal{L}_2 = O\left(\frac{1}{2^{1-\sigma}}\right) \). Taking derivative in (2.21), \( \mathcal{L}'(s) = O\left(Q^{1-2\sigma}\right) \). We then have that
\[ |F(s)| \leq \frac{|s-1|^q (O(Q^{1-2\sigma}))^2 (O(\frac{1}{21-\sigma}))^2}{|c_1^2 c_2^2|} = O\left( \frac{1}{e^{|\sigma|}} \right) \]
as \( \sigma \to -\infty \) provided that \( c_1 c_2 \neq 0 \). If one of \( c_1, c_2 \) is 0, say \( c_1 = 0 \), then by (2.21),

\[ L_1 - c_1 = \omega Q^{1-2\sigma} \left( 1 + O\left( \frac{1}{21-\sigma} \right) \right) \]
for \( \mathcal{L} = L_1, L_2 \). We then have that

\[ |F(s)| \leq \frac{|s-1|^q (O(Q^{1-2\sigma}))^2 (O(\frac{1}{21-\sigma}))^2}{|\omega Q^{1-2\sigma} (1 + O(\frac{1}{21-\sigma})))^2 c_2^2|} = O\left( \frac{1}{e^{|\sigma|}} \right) \]
as \( \sigma \to -\infty \). Hence, (2.15) holds also in this case.

We have thus showed that (2.15) always holds for fixed large \( t \geq t_0 \) and both \( \sigma \to +\infty \) and \( \sigma \to -\infty \). Fix such a large \( t = t_1 \). We have that

\[ |F(\sigma + it_1)| = O\left( \frac{1}{e^{|\sigma|}} \right) \quad (2.22) \]
as \( \sigma \to -\infty \) and as \( \sigma \to +\infty \). We are now ready to finish up the proof by appealing to the following Carlson theorem (see [8, p. 185]): Let \( f \) be holomorphic and of the form \( O(e^{k|z|}) \) for \( \text{Re}(s) > 0 \) (\( k \) is a real number) and let \( f(s) = O(e^{-\alpha|s|}) \) for a constant \( \alpha > 0 \) on the imaginary axis. Then \( f(s) = 0 \) identically. To apply this result to our situation, we make the variable transformation:

\[ z = x + iy = is + t_1 = t_1 - t + i\sigma, \]
or \( s = \frac{z - t_1}{i} \). Set

\[ G(z) = F(s) = F\left( \frac{z - t_1}{i} \right). \]

Then by (2.11), we have

\[ |G(z)| = \left| F\left( \frac{z - t_1}{i} \right) \right| = O\left( e^{\alpha|\frac{z - t_1}{i}|} \right) = O\left( e^{\alpha|z|} \right) \]
for all \( z \). Also, on the imaginary axis of the \( z \)-plane, \( x = 0 \) and \( z = iy \), which correspond to \( t = t_1 \) and \( y = \sigma \), or \( s = \sigma + it_1 \). Thus we have, by (2.22), that when \( z = iy \),

\[ |G(z)| = |F(\sigma + it_1)| = O\left( \frac{1}{e^{|\sigma|}} \right) = O\left( \frac{1}{e^{|z|}} \right). \]

Therefore, the function \( G \) satisfies the conditions of the above result and thus \( G \) is identically zero, from which it follows that \( F \) is identically zero. This finally completes the proof of the theorem. \( \Box \)
References