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On the Two-Dimensional Generic Rigidity Matroid and Its Dual

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We give a characterization of the dual of the 2-dimensional generic rigidity matroid $R(G)$ of a graph G and derive necessary and sufficient conditions for a connected matroid to be the rigidity matroid of a birigid graph. © 1991 Academic Press, Inc.

I. BASIC DEFINITIONS

Let $G=(V, E)$ be a graph on the edge set E, vertex set V. We define the *support* $\sigma(F)$ of a subset F of E to be the set of endpoints of edges in F.

We define a subset F of E to be *independent* if $|F'| \leqslant 2\,|\sigma(F')| - 3$ holds for all subsets F' of F. It is well known, see [1] or [6], that these independent edge sets are the independent sets of a matroid, the so-called *2-dimensional generic rigidity matroid*, $R(G)$, of the graph G. The closure operator and rank function of this matroid will be denoted by c and r, respectively. The term *circuit* will always refer to a circuit in $R(G)$. Figure 1 provides some examples of circuits on 6 vertices. Examples (a) and (b) are easily generalized to an arbitrary number of vertices. We shall always consider $R(G)$ as a restriction of the rigidity matroid of a complete graph on the same vertex set.

$R(G)$ is usually presented as the row space matroid of the so-called *rigidity matrix*, a matrix with two columns for each vertex representing its coordinates in the plane, and a row for each edge recording the linear equations imposed on the infinitesimal motions by the edges. The column space of the rigidity matrix and its applications to network theory are described in [7].

The degree of freedom f of a graph $G=(V, E)$ is defined to be $f=2n-3-r(E)$, where $n=|V|$. $G=(V, E)$ is called *rigid* if its degree of freedom is 0 or $r(E)=2n-3$. G is called *edge birigid*, if $r(E-e)=2n-3$

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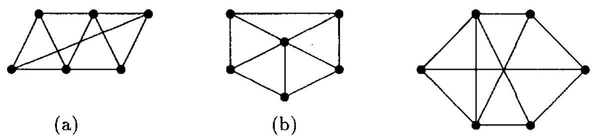

FIGURE 1

for every $e \in E$. G is called *birigid* if G is rigid and $r(E - \text{star}(v)) = 2(n-1) - 3 = 2n - 5$ for every $v \in V$, where $\text{star}(v)$ denotes the set of edges adjacent to v. Examples of graphs with various rigidity properties are depicted in Fig. 2. We will henceforth abbreviate $E - \text{star}(v)$ with $E - v$. To simplify notation and language we will not distinguish between sets of edges and the subgraphs they induce.

The following observations are immediate consequences of the definitions. The union of two graphs G_1 and G_2 having at most one vertex in common is not rigid, and $c(G_1 \cup G_2) = c(G_1) \cup c(G_2)$. If two rigid graphs intersect in two or more vertices, their union is rigid. Rigidity induces an equivalence relation on the edge set of G. The equivalence classes are called r-components. It follows that r-components have at most one vertex in common, we say that rigidity induces a 1-partition on the vertex set, and that birigid graphs are at least (vertex) 3-connected. Moreover, $R(G)$ can be written as the direct sum over the r-components of G.

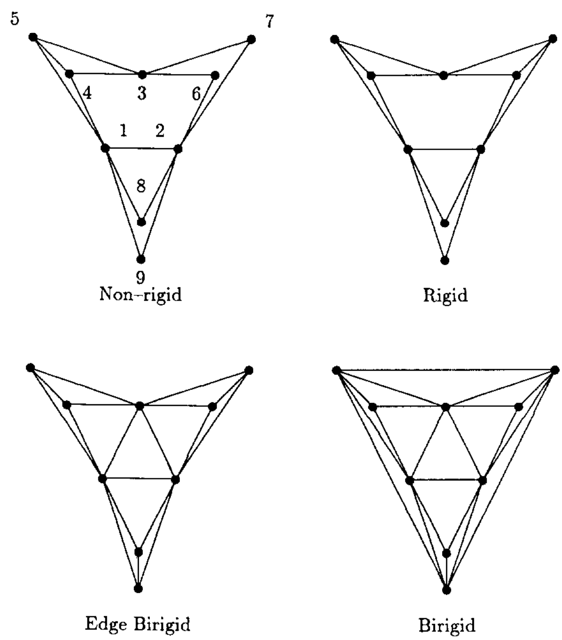

FIGURE 2

We shall use the following property of $R(G)$: Assume the edge set F induces a subgraph of G containing a vertex v of valence 3. Then F is independent if and only if there is an edge e containing neighbors of v such that e is not contained in F and $F-v+e$ is independent. We say $R(G)$ satisfies the 1-extendability property, see [1]. Note that e need not be in G.

Note that an independent rigid graph always contains a vertex of valence at most 3. If v is a vertex of valence 2, $G-v$ is also rigid and independent. If v is a vertex of valence 3, the 1-extendability property ensures the existence of an edge e such that $G-v+e$ is rigid and independent. Therefore any independent rigid graph can be obtained from an edge by successive addition of either a vertex of valence 2, or of a vertex of valence 3 and the removal of an appropriate edge. This technique is known as the *Henneberg replacements* [4] and is used by civil engineers for an inductive analysis of frameworks. Henneberg replacements may also be used to characterize $R(G)$.

II. A CHARACTERIZATION OF $R^*(G)$

Harary [3] calls a set X of edges of a connected graph G a *cutset* of G if the removal of X from G results in a disconnected graph, and then defines a *cocycle* of G to be a minimal cutset of G. We can define an *r-cutset* and a *cocircuit* analogously for a rigid graph. Welsh [10] extends Harary's definition to disconnected graphs by calling a set X of edges a *cutset* of G if its removal from G increases the number of connected components. We cannot simply replace connected components by r-components in this definition to obtain a reasonable definition for an r-cutset of a nonrigid graph, since the number of r-components of a graph may actually decrease with the removal of a set of edges, e.g., if G has n r-components, one of which is an edge e, for example, (3, 6) in the nonrigid graph of Fig. 2, then the removal of e results in a graph with $n-1$ r-components. We know that the rank of $E(G)$ decreases as we remove edges from G, or, equivalently, the degree of freedom of G increases, and we therefore define a *cocircuit* of G to be a set X of edges of G whose removal from G increases its degree of freedom and is minimal with respect to that property. Immediate consequences of this definition are

LEMMA 1. *X is a cocircuit of G if and only if X is a minimal subset of $E(G)$ such that X has nonempty intersection with every base of $R(G)$.*

And

THEOREM 1. *If G is a graph and $C^*(G)$ denotes the set of cocircuits of G, then $C^*(G)$ is the set of circuits of a matroid $R^*(G)$ on $E(G)$ and*

(1) $R^*(G) = (R(G))^*$.

(2) $R(G) = (R^*(G))^*$.

$R(G)^*$ is called the *cocircuit matroid* of G.

Here are some trivial examples of cocircuits:

(a) If the edge set of G is independent, consider for example the nonrigid or the rigid graph in Fig. 2, the collection of cocircuits equals the edge set.

(b) If G is a circuit, consider for example any of the graphs in Fig. 1, the collection of cocircuits is the set of all two element subsets of the edge set.

(c) If G is birigid, and v is a vertex of G, $\text{star}(v) - e$ is a cocircuit for every edge e in $\text{star}(v)$, since in this case every basis of $R(G)$ contains at least two edges of $\text{star}(v)$, and a basis of $R(G-v)$ can be extended to a basis of $R(G)$ by any two element subset of $\text{star}(v)$. Note that $\text{star}(v) - e$ is not necessarily a cocircuit if G is not birigid. For example, removal of the edges (2, 6) and (2, 7) suffices to increase the degree of freedom in the rigid, as well as the edge birigid graph of Fig. 2.

It may be worth noting that these examples have the following analogues in the 1-dimensional generic rigidity matroid, which is well known as the cycle matroid (or connectivity matroid) of a graph G, see [10], where rigidity is replaced by connectivity:

(a) Every edge in a tree is a bridge.

(b) Every two-edge subset of a cycle is a minimal cutset.

(c) If G is 2-connected, $\text{star}(v)$ is a minimal cutset of G for every v. Star(v) is called vertex cocycle. Vertex cocycles span the cocycle space of the cycle matroid of G.

We can characterize the cocircuits of $R(G)$ as follows:

THEOREM 2. *Let $G = (V, E)$ be a rigid graph with $|V| = n$. Let $\{V_1, V_2, \ldots, V_k\}$ be a 1-partition of V such that*

(1) *V_i induces a rigid subgraph G_i in G for all i,*

(2) *for all subsets $\{i_1, \ldots, i_l\}$ of the index set with $l \geq 2$ holds,*

$$\left| \bigcup_{j=1}^{l} V(G_{i_j}) \right| \geq \sum_{j=1}^{l} n_{i_j} - \frac{3l}{2} + 2, \qquad \textit{where} \quad n_{i_j} = |V(G_{i_j})|,$$

and

(3) $\sum_{i=1}^{k} (2n_i - 3) = 2n - 4$.

Then the edges of G connecting different V_i's form a cocircuit of G, and all edge sets obtained by this process form the collection of cocircuits of $R(G)$.

Proof. Consider a graph G' on the vertex set V which is the union of complete graphs on the V_i's. Then

$$r(G') = \sum_{i=1}^{k} (2n_i - 3) = r(G) - 1.$$

Therefore $r(E-X) = r(E) - 1$ where $X = E(G) - E(G')$. So X is an r-cutset. If X is not minimal, then there is an edge e in X such that $r(E - X + e) = r(E) - 1 = r(G')$. The edge e connects different V_i's and is contained in a rigid subgraph G'' of G'. We can consider G'' as a union of some G_i's by enlarging G'' by every G_{i_j} which it intersects. Then

$$r(G'') \leqslant \sum_{j=1}^{l} (2n_{i_j} - 3) < 2\left|\bigcup_{j=1}^{l} V(G_{i_j})\right| - 3 = 2\,|V(G'')| - 3.$$

Therefore G'' cannot be rigid, a contradiction.

Conversely, let C^* be a cocircuit of $R(G)$. Consider the rigid components G_i of $G - C^*$. Then the collection $V(G_i)$ satisfies the hypotheses of the theorem. ∎

For a nonrigid graph G, $R(G)$ can be written as a direct sum of its restrictions to the rigid components of G. The set of cocircuits of $R(G)$ is therefore the union over the sets of cocircuits of the direct summands.

The proof of Theorem 2 only used counting arguments involving the rank function of $R(G)$. For matroids on $G = (V, E)$ defined by $\{F : |F'| \leqslant a\,|\sigma(F')| - b \text{ for all } F' \subseteq F\}$ as collection of independent sets, where a and b are constants such that $a\,|\sigma(F)| - b \geqslant 0$ for all nonempty edge sets, we can formulate a similar theorem since maximal independent sets for which equality holds in the defining relation induce an $\lceil a/b \rceil - 1$ partition on V. For example, if $a = b = 1$ we obtain the cycle matroid of G. In this case, condition (3) of Theorem 2 implies $k = 2$ and the statement of the theorem reduces to: Let V_1, V_2 be a partition of the vertex set such that V_1 and V_2 induce connected graphs, then the edges of G with one endpoint in V_1 and one endpoint in V_2 form a cocycle and conversely.

III. Conditions for a Matroid to be the Rigidity Matroid of a Graph

Graver [2] proved that a matroid is the cycle matroid of a graph if and only if it is binary and has a 2-complete basis of cocircuits. An alternative characterization of graphic matroids is due to Sachs [8]. The key to this result is the equivalence of the biconnectivity of G and the connectivity of the cycle matroid of G, which implies that, given a graphic matroid M, there is a graph G such that $M(G) \cong M$, with the property that star(v) is a cocycle of M for every vertex v in G.

It will in general not be possible to construct a G such that every vertex of G corresponds to a cocircuit of $R(G)$, because birigidity of G implies the connectivity of $R(G)$ and this implication is not an equivalence, see [9].

It is easy to show that $M(G)$ is connected only if G is rigid, in fact edge birigid. There are however edge birigid graphs whose rigidity matroids are disconnected, for example, 3 circuits arranged in a "triangle," see Fig. 2(c). For details and examples see [9]. Thus we have the following sequence of implications:

$$G \text{ birigid} \underset{\not\Leftarrow}{\Rightarrow} R(G) \text{ connected} \underset{\not\Leftarrow}{\Rightarrow} G \text{ edge birigid.} \qquad (*)$$

To find conditions for a connected matroid to be the rigidity matroid of a birigid graph, we first examine the complete graph and its rigidity matroid.

THEOREM 3. *A matroid M on E is isomorphic to $R(K_n)$, the rigidity matroid of the complete graph on n vertices if and only if $|E| = \binom{n}{2}$ and there exists a collection of n subsets E_i of E with the following properties.*

(1) *$E = \bigcup_{i=1}^{n} E_i$ such that $E_i - e$ is a cocircuit of M for all $e \in E_i$ and each e is contained in exactly two of the E_i's.*

(2) *For all $F \subseteq E$ we have $r(F) \leqslant 2n(F) - 3$ where $n(F)$ is the number of E_i's with nonempty intersection with F.*

Proof. For each E_i we draw a vertex v and join two vertices by an edge e if e is contained in the intersection of the corresponding E_i's. Condition (1) implies that we obtain K_n by this process.

Condition (2) implies that a dependent edge set of $R(K_n)$ is also dependent in M. We want to show that every independent subset of $R(K_n)$ is also independent in M.

Consider the set of circuits in M which are independent in $R(K_n)$ and choose an F in that set such that $|\sigma(F)|$ is minimal. F has no vertex of valence $\leqslant 2$, otherwise there would be a cocircuit in M intersecting F in exactly one element by condition (1), which is impossible. Since F is independent in $R(K_n)$, F must contain a vertex v of valence 3. By the 1-extendability property of $R(K_n)$ there is an edge e connecting neighbors of v, such that e is not contained in F and $F - v + e$ is independent in $R(K_n)$. $F + e$ contains a circuit C_e of M with $e \in C_e$. F and C_e must intersect in star(v), otherwise C_e is of smaller support than F and is therefore a circuit of $R(K_n)$ by our minimality assumption, contradicting the fact that $F-v+e$ is independent in $R(K_n)$. But if F and C_e intersect in v, then $F+C_e-f$ must contain a circuit of M for every f in star(v), and since no circuit in M has a vertex of valence less than 3, $F+e-v$ must contain a circuit, again a contradiction. ▮

If M is a matroid satisfying conditions (1) and (2) of the preceding theorem, and M' is a restriction of M, then M' satisfies (2), but not necessarily (1) because the collection $\{E_i|_{M'}\}$ might not be minimal. On the other hand, every matroid satisfying (1) and (2) can be considered as a restriction of a matroid satisfying the hypotheses of the theorem and we have the

COROLLARY. *A matroid satisfying conditions* (1) *and* (2) *is the rigidity matroid* $R(G)$ *of some graph* G.

If G is birigid, $R(G)$ satisfies conditions (1) and (2) since the removal of star(v) decreases the rank of $E(G)$ by 2, hence star(v) $- e$ is a cocircuit for every $e \in$ star(v), and the collection $\{\text{star}(v) \mid v \in G\}$ satisfies (1). Condition (2) is Laman's condition [5].

THEOREM 4. *A matroid* M *is isomorphic to the rigidity matroid* $R(G)$ *of a birigid graph* G *if and only if* M *is connected and satisfies conditions* (1) *and* (2) *of Theorem* 3.

Proof. Since M is connected and isomorphic to the rigidity matroid of a graph G by the corollary, G is rigid by (∗). So $R(G)$ has rank $2n-3$ if G has n vertices. Furthermore, for every vertex v in G and any edge e in star(v), $E-\text{star}(v)+e$ is a hyperplane, hence has rank $2n-4$, so $E-\text{star}(v)$ has rank $2(n-1)-3$, which is to say that G is birigid.

Conversely, if G is birigid, $R(G)$ is connected by (∗) and satisfies conditions (1) and (2).

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