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Isometry theorem for the Segal–Bargmann transform on a noncompact symmetric space of the complex type

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Abstract

We consider the Segal–Bargmann transform on a noncompact symmetric space of the complex type. We establish isometry and surjectivity theorems for the transform, in a form as parallel as possible to the results in the dual compact case. The isometry theorem involves integration over a tube of radius R in the complexification, followed by analytic continuation with respect to R. A cancellation of singularities allows the relevant integral to have a nonsingular extension to large R, even though the function being integrated has singularities.
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1. Introduction

1.1. Euclidean and compact cases

The Segal–Bargmann transform for the Euclidean space $\mathbb{R}^d$, in a form convenient for the present paper, can be expressed as follows. Let t be a fixed positive number and let $e^{t\Delta/2}$ be the time-t forward *heat operator* for $\mathbb{R}^d$. It is not hard to show that for any f in $L^2(\mathbb{R}^d, dx)$,

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$e^{t\Delta/2}f$ admits an entire analytic continuation in the space variable from $\mathbb{R}^d$ to $\mathbb{C}^d$. The Segal–Bargmann transform [2,44] is then the map associating to each $f \in L^2(\mathbb{R}^d)$ the holomorphic function obtained by analytically continuing $e^{t\Delta/2}f$ from $\mathbb{R}^d$ to $\mathbb{C}^d$. Basic properties of this transform are encoded in the following theorem. (See [27] for more information.)

Theorem 1. *The **isometry formula**. Fix f in $L^2(\mathbb{R}^d, dx)$. Then the function $F := e^{t\Delta/2}f$ has an analytic continuation to $\mathbb{C}^d$ satisfying*

$$\int\limits_{\mathbb{R}^d} |f(x)|^2 \, dx = \int\limits_{\mathbb{C}^d} |F(x+iy)|^2 \frac{e^{-|y|^2/t}}{(\pi t)^{d/2}} \, dy \, dx. \tag{1}$$

*The **surjectivity theorem**. Given any holomorphic function F on $\mathbb{C}^d$ for which the right-hand side of (1) is finite, there exists a unique $f \in L^2(\mathbb{R}^d)$ with $F|_{\mathbb{R}^d} = e^{t\Delta/2}f$.*

*The **inversion formula**. If $f \in L^2(\mathbb{R}^d)$ is sufficiently regular and $F := e^{t\Delta/2}f$, then*

$$f(x) = \int\limits_{\mathbb{R}^d} F(x+iy) \frac{e^{-|y|^2/2t}}{(2\pi t)^{d/2}} \, dy.$$

Note that we have $e^{-|y|^2/t}$ in the isometry formula but $e^{-|y|^2/2t}$ in the inversion formula. From the point of view of harmonic analysis, the Segal–Bargmann transform may be thought of as a way of combining information about a function $f(x)$ on $\mathbb{R}^d$ with information about the Fourier transform $\hat{f}(y)$ of f into a single (holomorphic) function $F(x+iy)$ on $\mathbb{C}^d = \mathbb{R}^{2d}$. From the point of view of quantum mechanics, F may be thought of as the phase space wave function corresponding to the position space wave function f. For more information, see [2,13,20,21,24].

Analogous results for compact symmetric spaces have been obtained by Hall [18,19] in the compact group case and by Stenzel [45] in the general case. (See [23,27] for more information. See also [37] for surprising results in the case of the Heisenberg group.) Let X denote a compact symmetric space, assumed for simplicity to be simply connected. Then X can be expressed as $X = U/K$, where U is a simply connected compact Lie group and K is the fixed-point subgroup of an involution. We may define the complexification of U/K to be $U_{\mathbb{C}}/K_{\mathbb{C}}$, where $U_{\mathbb{C}}$ is the unique simply connected Lie group whose Lie algebra is $\mathfrak{u} + i\mathfrak{u}$ and where $K_{\mathbb{C}}$ is the connected Lie subgroup of $U_{\mathbb{C}}$ whose Lie algebra is $\mathfrak{k} + i\mathfrak{k}$. Then $U_{\mathbb{C}}/K_{\mathbb{C}}$ may be identified diffeomorphically with the tangent bundle $T(U/K)$ by means of the map $\Phi : T(U/K) \to U_{\mathbb{C}}/K_{\mathbb{C}}$ given by

$$\Phi(x, Y) \to \exp_x(iY), \tag{2}$$

where Y is a tangent vector to U/K at x and where $\exp_x(iY)$ refers to the analytic continuation of the geometric exponential map for U/K. See [28, Eq. (2)] for a simple explicit formula for $\Phi(x, Y)$ in the case that U/K is a sphere.

If the Lie algebra $\mathfrak{u}$ of U is decomposed in the usual way as $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$, then let G be the connected Lie subgroup of $U_{\mathbb{C}}$ whose Lie algebra is $\mathfrak{g} = \mathfrak{k} + i\mathfrak{p}$. The *dual noncompact symmetric space* to U/K is the manifold G/K, equipped with an appropriate G-invariant Riemannian metric. The identification (2) of $T(U/K)$ with $U_{\mathbb{C}}/K_{\mathbb{C}}$ gives rise to an identification of each fiber in $T(U/K)$ with G/K. Specifically, if x_0 is the identity coset in U/K, then the image of $T_{x_0}(U/K)$ under Φ is precisely the G-orbit of the identity coset in $U_{\mathbb{C}}/K_{\mathbb{C}}$. Furthermore, the stabilizer in G

of the identity coset is precisely K, and so $\Phi(T_{x_0}(U/K)) \cong G/K$. Any other fiber in $T(U/K)$ is then identified with $T_{x_0}(U/K) \cong G/K$ by the action of U. See [27,45] for details.

Having identified each tangent space $T_x(U/K)$ with the noncompact symmetric space G/K, we have on each tangent space the *heat kernel density* ν_t^{nc} (based at the origin) and the *Jacobian* j^{nc} *of the exponential map* with respect to the Riemannian metric for G/K. Here the superscript "nc" indicates a quantity associated to the noncompact symmetric space G/K dual to the original compact symmetric space U/K. The result is then the following. (See [45]; compare [18,19] in the compact group case.)

Theorem 2. *The **isometry formula**. Fix f in $L^2(U/K)$. Then the function $F := e^{t\Delta/2} f$ has an analytic continuation to $U_{\mathbb{C}}/K_{\mathbb{C}}$ satisfying*

$$\int\limits_{U/K} |f(x)|^2 \, dx = \int\limits_{x \in U/K} \int\limits_{Y \in T_x(U/K)} |F(\exp_x(iY))|^2 \nu_{2t}^{nc}(2Y) j^{nc}(2Y) 2^d \, dY \, dx. \qquad (3)$$

Here $d = \dim(U/K)$, dY is the Lebesgue measure on $T_x(U/K)$, and dx is the Riemannian volume measure on U/K.

*The **surjectivity theorem**. Given any holomorphic function F on $U_{\mathbb{C}}/K_{\mathbb{C}}$ for which the right-hand side of (3) is finite, there exists a unique $f \in L^2(U/K)$ with $F|_{U/K} = e^{t\Delta/2} f$.*

*The **inversion formula**. If $f \in L^2(U/K)$ is sufficiently regular and $F := e^{t\Delta/2} f$, then*

$$f(x) = \int\limits_{T_x(U/K)} F(\exp_x(iY)) \nu_t^{nc}(Y) j^{nc}(Y) \, dY. \qquad (4)$$

Note that in the inversion formula we have $\nu_t(Y) j(Y)$ whereas in the isometry formula we have $\nu_{2t}(2Y) j(2Y)$. Note also that the isometry and inversion formulas for Euclidean space are of the same form as Theorem 2, with $\exp_x(iy) = x + iy$, $j(y) \equiv 1$, and $\nu_t(y) = (2\pi t)^{-d/2} e^{-|y|^2/2t}$.

An important special case of Theorem 2 is the compact group case considered in [18,19], i.e., the case in which K is the diagonal subgroup of $U = K \times K$. This case is connected to stochastic analysis and the Gross ergodicity theorem [15,26,30] and to the quantization of Yang–Mills theory on a spacetime cylinder [6,22,51]. Furthermore, in this case the isometry formula can be understood as a unitary pairing map in the context of geometric quantization [11,12,25,49].

In the compact group case, the dual noncompact symmetric space is of the "complex type," and in this case there is a simple explicit formula for the heat kernel ν_t^{nc}, namely,

$$\nu_t^{nc}(Y) = e^{-|\rho|^2 t/2} j^{nc}(Y)^{-1/2} \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}}. \qquad (5)$$

Here ρ is half the sum (with multiplicity) of the positive roots for G/K and there is a simple explicit expression for j^{nc} (change sin to sinh in (32)). Thus, in the compact group case, the isometry formula takes the form

$$\int\limits_{U/K} |f(x)|^2 \, dx = e^{-|\rho|^2 t} \int\limits_{x \in U/K} \int\limits_{Y \in T_x(U/K)} |F(\exp_x(iY))|^2 j^{nc}(2Y)^{1/2} \frac{e^{-|Y|^2/t}}{(\pi t)^{d/2}} \, dY \, dx \qquad (6)$$

and the inversion formula takes the form

$$f(x) = e^{-|\rho|^2 t/2} \int\limits_{T_x(U/K)} F\big(\exp_x(iY)\big) j^{\mathrm{nc}}(Y)^{1/2} \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} \, dY. \tag{7}$$

1.2. The complex case

Since we have nice theories for the Euclidean and compact cases, the natural next step is to consider symmetric spaces of the noncompact type. This would mean applying the heat operator to a function on a symmetric space of the form G/K, where G is a noncompact semisimple Lie group (connected with finite center) and K is a maximal compact subgroup. If we attempt to imitate the constructions in the compact and Euclidean cases, we rapidly encounter difficulties. As in the compact case, we can define a smooth map $\Phi : T(G/K) \to G_{\mathbb{C}}/K_{\mathbb{C}}$ by

$$\Phi(x, Y) = \exp_x(iY).$$

However, in the noncompact case, Φ is *not* a global diffeomorphism; Φ is not globally injective and the differential of Φ becomes degenerate at certain points. The map Φ gives rise to a *local* identification of each fiber in $T(G/K)$ with the dual *compact* symmetric space, but this identification cannot possibly be global, since $T_x(G/K)$ is not compact. In addition to the (global) breakdown of the desired identifications, we have a problem with analytic continuation. For a typical function f in $L^2(G/K)$, the function $e^{t\Delta/2} f$ does not have a global analytic continuation to $G_{\mathbb{C}}/K_{\mathbb{C}}$, but rather becomes both singular and multiple-valued once one moves far enough from G/K.

The paper [29] takes a first step in overcoming these obstacles. (Related but nonoverlapping results were obtained by Krötz, Ólafsson, and Stanton [36]. We discuss [36] in detail in Sections 1.4 and 8. See also [4,5,41] for a different approach, not involving the heat equation.) In [29], we consider the simplest case, that of noncompact symmetric spaces of the "complex type." Here complex type does *not* mean that the symmetric space is a complex manifold, but rather that the group G admits a complex structure, which means that G is the complexification of K. The complex case is nothing but the noncompact dual of the compact group case. The simplest symmetric space of the complex type is hyperbolic 3-space, where $G \cong SO(3, 1)_e \cong PSL(2, \mathbb{C})$.

In the complex case, we develop in [29] (1) an isometry formula for "radial" (i.e., left-K-invariant) functions on G/K and (2) an inversion formula for general functions (sufficiently regular but not necessarily radial). Suppose f is a radial function in $L^2(G/K)$ and let $F = e^{t\Delta/2} f$. Then the isometry formula of [29, Theorem 2] states that the map $Y \to F(\exp_{x_0} Y)$ has a meromorphic extension to $\mathfrak{p}_{\mathbb{C}}$ and that the L^2 norm of F over $\mathfrak{p}_{\mathbb{C}}$ with respect to a certain measure μ is equal to the L^2 norm of f over G/K. See also [42, Theorem 2.8].

The inversion formula of [29], meanwhile, reads

$$f(x) = \text{``} \lim_{R \to \infty} \text{''} \, e^{|\rho|^2 t/2} \int\limits_{\substack{Y \in T_x(G/K) \\ |Y| \leqslant R}} F(\exp_x iY) j^{\mathrm{c}}(Y)^{1/2} \frac{e^{-|Y|^2/2t}}{(2\pi t)^{d/2}} \, dY. \tag{8}$$

(See [29, Theorem 4]. A different approach to inversion formulas is taken in [46].) Here j^{c} is the Jacobian of the exponential mapping for the *compact* symmetric space U/K dual to G/K

and ρ is half the sum (with multiplicities) of the positive roots for G/K. Moreover, "$\lim_{R\to\infty}$" means that the integral on the right-hand side of (8) is well defined for all sufficiently small R and admits a real-analytic continuation in R to $(0, \infty)$. The right-hand side of (8) then is equal to the limit as R tends to infinity of this analytic continuation. That is, a limit with quotation marks means the limit as R tends to infinity of *the real-analytic extension of* the indicated quantity.

It should be noted that although $F(\exp_x iY)$ develops singularities once Y gets sufficiently large, the integral on the right-hand side of (8) does not develop singularities; it has a real-analytic extension to $R \in (0, \infty)$. There is a delicate "cancellation of singularities" going on here, which is explained in [27,29], and the next subsection.

Leaving aside the analytic continuation in R, which is unnecessary in the compact case, (8) is "dual" to the inversion formula (7) for the compact group case. That is, (8) is obtained from (7) by changing j^{nc} to j^c and changing $e^{-|\rho|^2 t/2}$ to $e^{|\rho|^2 t/2}$. (The constant $|\rho|^2$ is related to the scalar curvature, which is positive in the compact case and negative in the noncompact case.)

The main result of the present paper is an isometry formula which bears the same relationship to the inversion formula (8) as (6) bears to (7).

Theorem 3. *For any f in $L^2(G/K)$ (G complex) we have*

$$\int\limits_{G/K} \left| f(x) \right|^2 dx$$

$$= \text{``} \lim_{R\to\infty} \text{''} e^{|\rho|^2 t} \int\limits_{x\in G/K} \int\limits_{\substack{Y\in T_x(G/K) \\ |Y|\leqslant R}} \left| F(\exp_x iY) \right|^2 j^c(2Y)^{1/2} \frac{e^{-|Y|^2/t}}{(\pi t)^{d/2}} \, dY \, dx. \qquad (9)$$

As in the inversion formula, the integral on the right-hand side of (9) is to be taken literally for small R and interpreted by means of analytic continuation in R for large R. See Theorem 7 in Section 6 for a more precise statement. We will also prove a surjectivity theorem (Theorem 8 in Section 7); roughly, if F is any holomorphic function on a G-invariant neighborhood of G/K inside $G_{\mathbb{C}}/K_{\mathbb{C}}$ for which the right-hand side of (9) makes sense and is finite, then there exists a unique $f \in L^2(G/K)$ with $F|_{G/K} = e^{t\Delta/2} f$.

In the case of hyperbolic 3-space, with the usual normalization of the metric, the isometry formula takes the following explicit form (see also [27, Section 5]):

$$\int\limits_{H^3} \left| f(x) \right|^2 dx = \text{``} \lim_{R\to\infty} \text{''} e^t \int\limits_{x\in H^3} \int\limits_{\substack{Y\in T_x(H^3) \\ |Y|\leqslant R}} \left| F(\exp_x iY) \right|^2 \frac{\sin|2Y|}{|2Y|} \frac{e^{-|Y|^2/t}}{(\pi t)^{3/2}} \, dY \, dx. \qquad (10)$$

The isometry formula of Krötz, Ólafsson, and Stanton [36], when specialized to the complex case, is *not* the same as the formula in Theorem 3. We discuss the relationship between the two results in Section 1.4 and in Section 8. If f just happens to be radial, then there is another isometry formula, established in [29, Theorem 2] (see also [42, Theorem 2.8]). For radial functions, it is not immediately obvious how to see directly that the isometry formula in Theorem 3 agrees with the isometry formula of [29].

1.3. Cancellation of singularities

Let $T^R(G/K)$ denote the set of $(x, Y) \in T(G/K)$ with $|Y| < R$. The inversion and isometry formulas assert that for noncompact symmetric spaces of the complex type, certain integrals (those on the right-hand side of (8) and (9)) involving $F(\exp_x iY)$ over $T^R(G/K)$ are "nonsingular," in that they extend in a real analytic way to all positive values of R. On the other hand, $F(\exp_x iY)$ itself does in fact become singular (and multiple-valued) once Y gets sufficiently large, as can be seen, for example, from the formula [14, Proposition 3.2] for the heat kernel on G/K. If $F(\exp_x iY)$ itself becomes singular for large Y but certain integrals involving F remain nonsingular, then some "cancellation of singularities" must be taking place in the process of integration.

In the case of the inversion formula, the cancellation of singularities occurs because the integral on the right-hand side of (8) only "sees" the part of $F(\exp_x iY)$ that is "radial" in Y (i.e., invariant under the adjoint action of K). Meanwhile, the radial part of $F(\exp_x iY)$ can be expanded in terms of analytically continued spherical functions. In the complex case, the analytically continued spherical functions have only a very simple sort of singularity, a certain denominator function (the same one for all spherical functions) that can become zero. (See Section 8 for precise formulas.) The zeros of this denominator function are canceled by the zeros of the function j^c in the integrand of (8).

Meanwhile, in the isometry formula, the integral of $|F|^2$ over $T^R(G/K)$ can be expressed as an integral of $|F|^2$ over G-orbits, followed by an integration over the space of G-orbits in $T^R(G/K)$. Meanwhile, the Gutzmer-type formula of Faraut [8,9] (also used in an important way in [36]) shows that the orbital integrals of $|F|^2$ can again be expressed in terms of the analytically continued spherical functions. As in the case of the inversion formula, the singularities coming from the analytically continued spherical functions are (in the complex case) canceled by the zeros of j^c in the integrand in (9). See (33) and the discussion following it. In the H^3 case, the integral of $|F(\exp_x iY)|^2$ over the set of (x, Y) with $|Y| = R$ blows up at $R = \pi/2$ like $1/\sin 2R$. This blow-up is canceled by the factor of $\sin |2Y|$ in (10).

From a more philosophical point of view, we note work of R. Szőke [48]. Szőke has shown that although the differential of the map $\Phi : T(G/K) \to G_{\mathbb{C}}/K_{\mathbb{C}}$ becomes degenerate at certain points, the pullback of the $(1, 0)$ sub-bundle of $T_{\mathbb{C}}(G_{\mathbb{C}}/K_{\mathbb{C}})$ by means of Φ has a real-analytic extension to the whole of $T(G/K)$. The problem is that this bundle has nonzero intersection with its complex-conjugate at certain points. Nevertheless, Szőke's result suggests that things do not break down entirely when the differential of Φ becomes degenerate.

1.4. The results of Krötz, Ólafsson, and Stanton

We now give a quick comparison of our isometry formula to the one of B. Krötz, G. Ólafsson, and R. Stanton established in [36]; details are provided in Section 8. The paper [36] establishes an isometry formula for the Segal–Bargmann transform on an arbitrary globally symmetric space G/K of the noncompact type, with G not necessarily complex. The authors of [36] consider the integral of $|F|^2$ over G-orbits in a certain open subset Ξ of $G_{\mathbb{C}}/K_{\mathbb{C}}$. These G-orbits are parameterized by points in a certain open subset $2i\Omega$ of $i\mathfrak{a}$, where $\mathfrak{a}$ is a maximal commutative subspace of $\mathfrak{p}$. Thus, we obtain the orbital integral $\mathcal{O}_{|F|^2}(iY)$, denoting the integral of $|F|^2$ over the G-orbit parameterized by $iY \in 2i\Omega \subset i\mathfrak{a}$. Krötz, Ólafsson, and Stanton then show that there is a certain "shift operator" D such that $D\mathcal{O}_{|F|^2}$ has a real-analytic extension from $2i\Omega$ to all

of $i\mathfrak{a}$. The isometry formula, Theorem 3.3 of [36], then asserts that $\int_{G/K} |f(x)|^2 \, dx$ is equal to the integral of $D\mathcal{O}_{|F|^2}$ over $i\mathfrak{a}$ with respect to a certain Gaussian measure.

In the complex case, the isometry formula of [36] *does not* coincide with the one we establish in this paper. Nevertheless, the two isometry formulas are equivalent in a sense that we explain in Section 8. Specifically, in the complex case, D is a differential operator and we will show that an integration by parts can turn the isometry formula of [36] into the one we prove here. (See also the recent preprint [43], which gives a another description of the image of the Segal–Bargmann, different from both [36] and the present paper.)

In the complex case, the form of the isometry formula in (9) seems preferable to the form in [36], simply because (9) is more parallel to what one has in the dual compact case (6). On the other hand, the result of [36] is more general, because it holds for arbitrary symmetric spaces of the noncompact type, not just the complex case. It would be desirable to attempt to carry out this integration by parts in general (not just in the complex case), so as to recast the isometry formula of [36] into a form more parallel to what one has in the general compact case in (3). However, because the singularities in the orbital integral are more complicated once one moves away from the complex case, it remains to be seen whether this integration by parts can be carried out in general.

2. Preliminaries

Although our main result holds only for the complex case, it is instructive to begin in the setting of a general symmetric space of the noncompact type and then specialize when necessary to the complex case. We consider, then, a connected semisimple Lie group G with finite center, together with a fixed maximal compact subgroup K of G. For our purposes, there is no harm in assuming that G is contained in a simply connected complexification $G_{\mathbb{C}}$. There is a unique involution of G whose fixed points are K, and this leads to a decomposition of the Lie algebra $\mathfrak{g}$ of G as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{p}$ is the subspace of $\mathfrak{g}$ on which the associated Lie algebra involution acts as $-I$. The spaces $\mathfrak{k}$ and $\mathfrak{p}$ satisfy $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

We choose on $\mathfrak{p}$ an inner product invariant under the adjoint action of K. We then consider the manifold G/K and we let x_0 denote the identity coset in G/K. We identify the tangent space to G/K at x_0 with $\mathfrak{p}$. The choice of an Ad-K-invariant inner product on $\mathfrak{p}$ gives rise to a Riemannian metric on G/K that is invariant under the left action of G. The manifold G/K, together with a metric of this form, is a symmetric space of the noncompact type, in the terminology of [33].

In the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$, we consider the subalgebra $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p}$. We let U denote the connected Lie subgroup of $G_{\mathbb{C}}$ whose Lie algebra is $\mathfrak{u}$. The inner product on $\mathfrak{p}$ induces an inner product on $i\mathfrak{p}$ in an obvious way. This inner product determines a Riemannian metric on U/K invariant under the left action of U, and U/K with this metric is a Riemannian symmetric space of the compact type, known as the "compact dual" of G/K.

Let $\mathfrak{a}$ be any maximal commutative subspace of $\mathfrak{p}$. Let $\Sigma \subset \mathfrak{a}$ denote the set of (restricted) roots for the pair $(\mathfrak{g}, \mathfrak{k})$, where we use the inner product on $\mathfrak{p}$, restricted to $\mathfrak{a}$, to identify $\mathfrak{a}$ with $\mathfrak{a}^*$. Let Σ^+ denote a set of positive roots. Let W denote the Weyl group, that is, the subgroup of the orthogonal group of $\mathfrak{a}$ generated by the reflections associated to $\alpha \in R$. It is known that any vector in $\mathfrak{p}$ can be moved into $\mathfrak{a}$ by the adjoint action of K, and that the resulting vector in $\mathfrak{a}$ is unique up to the action of W. We let $\mathfrak{a}^+$ denote the closed fundamental Weyl chamber, that is, the set of points Y in $\mathfrak{a}$ with $\alpha(Y) \geqslant 0$ for all $\alpha \in R^+$. Then each Weyl-group orbit contains exactly one point in $\mathfrak{a}^+$.

We let Ω denote the Weyl-invariant domain in $\mathfrak{a}$ given by

$$\Omega = \left\{ Y \in \mathfrak{a} \,\Big|\, |\langle \alpha, Y \rangle| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma \right\}. \tag{11}$$

We may think of Ω as a subset of $\mathfrak{p} = T_{x_0}(G/K)$. We then define a set Λ by

$$\Lambda = G \cdot \Omega \subset T(G/K); \tag{12}$$

that is, Λ is the smallest G-invariant set in $T(G/K)$ containing Ω. Thus, to determine if a point $Y \in T_x(G/K)$ belongs to Λ, we move Y to a vector $Y' \in T_{x_0}(G/K)$ by the action of G and then move Y' to a vector $Y'' \in \mathfrak{a}$ by the action of K. Then $Y \in \Lambda$ if and only if $Y'' \in \Omega$.

We now consider a map $\Phi : T(G/K) \to G_{\mathbb{C}}/K_{\mathbb{C}}$ given by

$$\Phi(x, Y) = \exp_x(iY), \quad x \in G/K, \ Y \in T_x(G/K). \tag{13}$$

Explicitly, we may identify $T(G/K)$ with $(G \times \mathfrak{p})/K$, where K acts on G by right-multiplication and on $\mathfrak{p}$ by $Y \to k^{-1}Yk$. In that case, the geometric exponential map from $T(G/K)$ to G/K is given by $(g, Y) \to ge^Y K_{\mathbb{C}}$ and so Φ may also be expressed as

$$\Phi(g, Y) = ge^{iY} K_{\mathbb{C}}, \quad g \in G, \ Y \in \mathfrak{p}. \tag{14}$$

Here we observe that for $k \in K$, $\Phi(gk, k^{-1}Yk) = \Phi(g, Y)$, so that Φ, written as a map of $G \times \mathfrak{p}$ into $G_{\mathbb{C}}/K_{\mathbb{C}}$ descends to a map of $(G \times \mathfrak{p})/K$ into $G_{\mathbb{C}}/K_{\mathbb{C}}$. From (14) we can see that Φ is a globally defined smooth map of $T(G/K)$ into $G_{\mathbb{C}}/K_{\mathbb{C}}$.

In contrast to the analogous map in the compact case, Φ is not a diffeomorphism of $T(G/K)$ onto $G_{\mathbb{C}}/K_{\mathbb{C}}$. Indeed, Φ is not globally injective and Φ is not even a local diffeomorphism near certain points in $T(G/K)$. Nevertheless, Φ maps Λ diffeomorphically onto its image in $G_{\mathbb{C}}/K_{\mathbb{C}}$. This image, denoted Ξ in [36], is the Akhiezer–Gindikin "crown domain" [1]. That is,

$$\Xi = \left\{ \exp_x(iY) \in G_{\mathbb{C}}/K_{\mathbb{C}} \,\Big|\, (x, Y) \in \Lambda \right\}. \tag{15}$$

We will consistently parameterize points $z \in \Xi$ as $z = \exp_x(iY)$ with $(x, Y) \in \Lambda$. We let

$$T^R(G/K) = \left\{ (x, Y) \,\Big|\, |Y| < R \right\}.$$

Then $T^R(G/K)$ is contained in Λ for all sufficiently small R. We let $R_{\max}$ denote the largest R with this property:

$$R_{\max} = \max\left\{ R \,\Big|\, T^R(G/K) \subset \Lambda \right\}. \tag{16}$$

The complex structure on Ξ (as an open subset of the complex manifold $G_{\mathbb{C}}/K_{\mathbb{C}}$) can be transferred to Λ by the diffeomorphism Φ. This complex structure on $\Lambda \subset T(G/K)$ is in fact the "adapted complex structure" developed in [16,17,40,47]. Indeed, Λ is the maximal connected domain in $T(G/K)$ containing the zero section on which the adapted complex structure is defined. See [1,3,34,35] for more information.

3. Partial isometry for general symmetric spaces of the noncompact type

We continue to work on G/K, with G arbitrary real semisimple (connected with finite center), not necessarily complex.

Given a function $f \in L^2(G/K)$, let $\hat{f}$ denote the Helgason Fourier transform of f, so that $\hat{f}$ is a square-integrable function on $\mathfrak{a}^* \times B$ invariant under the action of the Weyl group on $\mathfrak{a}^*$. Here $B = K/M$, where M is the centralizer of $\mathfrak{a}$ in K. (See Section III.2 of [31].) It is convenient to think of $\hat{f}$ as a function on $\mathfrak{a}^*$ with values in $L^2(B)$. Thus for $\xi \in \mathfrak{a}^*$, we will let $\|\hat{f}(\xi)\|$ be the L^2 norm of the corresponding element of $L^2(B)$; that is,

$$\|\hat{f}(\xi)\|^2 = \int_B |\hat{f}(\xi, b)|^2 \, db.$$

The Plancherel theorem for the Fourier transform states that for $f \in L^2(G/K)$

$$\|f\|^2 = \int_{\mathfrak{a}^*} \|\hat{f}(\xi)\|^2 \frac{d\xi}{|c(\xi)|^2}. \tag{17}$$

Here c is the Harish-Chandra c-function, the norm of f is the L^2 norm with respect to the Riemannian volume measure on G/K, and $d\xi$ denotes the Lebesgue measure on $\mathfrak{a}^*$ (suitably normalized).

Meanwhile, let Δ denote the Laplacian on G/K, and let $e^{t\Delta/2}$ denote the time-t (forward) heat operator. (We take the Laplacian to be a negative operator.) For $f \in L^2(G/K)$, let $F = e^{t\Delta/2}f$. In that case, F is also in $L^2(G/K)$ and the Fourier transform of F is related to the Fourier transform of f by

$$\hat{F}(\xi) = e^{-t(|\xi|^2 + |\rho|^2)/2} \hat{f}(\xi), \tag{18}$$

where ρ is half the sum of the positive roots (with multiplicity).

According to Section 6 of [35], the function F admits an analytic continuation (also denoted F) to the domain $\varXi \subset G_{\mathbb{C}}/K_{\mathbb{C}}$ defined in (15). We now consider the integrals of $|F|^2$ over various G-orbits inside $\varXi$. A Gutzmer-type formula, due to J. Faraut [8,9], tells us that these orbital integrals can computed as follows. Each G-orbit in $\varXi$ contains exactly one point of the form $\exp_{x_0}(iZ)$, where Z belongs to $\Omega^+ := \Omega \cap \mathfrak{a}^+$. Let dg denote the Haar measure on G, normalized so that the push-forward of dg to G/K coincides with the Riemannian volume measure on G/K. Then the Gutzmer formula for F takes the form (in light of (18))

$$\int_G |F(g \cdot \exp_{x_0}(iY/2))|^2 \, dg = \int_{\mathfrak{a}^*} \|\hat{f}(\xi)\|^2 e^{-t(|\xi|^2 + |\rho|^2)} \phi_{\xi}(e^{iY}) \frac{d\xi}{|c(\xi)|^2} \tag{19}$$

for all $Y \in 2\Omega^+$. Here ϕ_{ξ} is the spherical function normalized to equal 1 at $Y = 0$. Note that if $Y = 0$, then (19) simply reduces to (17). Note also that on the left-hand side of (19) we have the G-orbit through the point $\exp_{x_0}(iY/2)$, whereas on the right-hand side we have the spherical function evaluated at $\exp(iY)$. This factor of 2 is the origin of the factors of 2 in the isometry formula relative to the inversion formula. See Appendix A for more details about the Gutzmer formula and the hypotheses under which it holds.

According to Lemma 2.1 of [36], for each $\xi \in \mathfrak{a}^*$, $\phi_\xi(iY)$ is defined and real-analytic for $Y \in 2\Omega$. Furthermore, for a fixed $Y \in 2\Omega$, $\phi_\xi(e^{iY})$ grows at most exponentially with ξ, with bounds that are uniform on each compact subset of 2Ω. Thus, given $f \in L^2(G/K)$, the right-hand side of (19) is a bounded as a function of Y on each compact subset of 2Ω.

We now fix some bounded positive Ad-K-invariant density α on $\mathfrak{p}^{2R_{\max}} \subset T_{x_0}(G/K)$. Using the action of G, we can identify every tangent space $T_x(G/K)$ with $\mathfrak{p}$, and this identification is unique up to the adjoint action of K on $\mathfrak{p}$. Since α is Ad-K-invariant, we may unambiguously think of α as a function on each of the tangent spaces $T_x(G/K)$. We then consider the integral

$$G_F(R) := \int\limits_{x \in G/K} \int\limits_{Y \in T_x^{2R}(G/K)} \left| F\left(\exp_x(iY/2)\right) \right|^2 \alpha(Y)\, dY\, dx, \tag{20}$$

where $T_x^{2R}(G/K)$ denotes the vectors in $T_x(G/K)$ with magnitude less than $2R$. As we shall see shortly, this integral will be well defined and finite for all $R < R_{\max}$.

Now, for each $x \in G/K$, we choose $g_x \in G$ so that $g_x \cdot x_0 = x$, and we arrange for g_x to be a measurable function of x. (We may take, for example, $g_x \in P := \exp \mathfrak{p}$.) Then we obtain a measurable trivialization of the tangent bundle, with each tangent space $T_x(G/K)$ identified with $\mathfrak{p} = T_{x_0}(G/K)$ by means of the action of g_x. The integral in (20) then becomes an integral over $(G/K) \times \mathfrak{p}^{2R}$, where $\mathfrak{p}^{2R}$ denotes the set of points in $\mathfrak{p}$ with magnitude less than $2R$. We now use generalized polar coordinates to change the integration over $\mathfrak{p}^{2R}$ into one over $\mathfrak{a}_{2R}^+ \times K$, where $\mathfrak{a}_{2R}^+ = \mathfrak{a}^+ \cap \mathfrak{p}^{2R}$. This gives, after applying Fubini's Theorem,

$$G_F(R) = \int\limits_{\mathfrak{a}_{2R}^+} \int\limits_{G/K} \int\limits_K \left| F\left(\exp_x(i\mathrm{Ad}_k Y/2)\right) \right|^2 dk\, dx\, \alpha(Y)\mu(Y)\, dY, \tag{21}$$

where μ is the density appearing in the generalized polar coordinates (e.g., [32, Theorem I.5.17]).

Since each coset x in G/K contains a unique element of the form g_x, each element g of G can be decomposed uniquely as $g = g_x k$, where $x = g \cdot x_0 = gK$ and k is an element of K. In this way, we can identify G measurably with $(G/K) \times K$. Let us consider the measure $dx\, dk$ on $(G/K) \times K$, where dx denotes the Riemannian volume measure and dk is the normalized Haar measure on K. If we transfer this measure to G by the above identification, the resulting measure on G is invariant under the left action of G. To see this, note that for $h \in G$ and $x \in G/K$, there exists a unique $k_{h,x} \in K$ such that $hg_x = g_{h \cdot x} k_{h,x}$. Thus, the left action of G on itself, transferred to $(G/K) \times K$, corresponds to the map $(x, k) \to (h \cdot x, k_{h,x} k)$, and this action preserves $dx\, dk$. Thus, $dx\, dk$ corresponds, under our identification, to a Haar measure dg on the (unimodular) group G. Furthermore, by considering the case $Y = 0$ in the Gutzmer formula (19), we can see that this Haar measure is normalized the same way as the one in the Gutzmer formula.

Now, we have identified $T_x(G/K)$ with $\mathfrak{p}$ in such a way that $g_x \cdot (x_0, Y) = (x, Y)$. Since the map Φ in (13) intertwines the action of G on $\Lambda \subset T(G/K)$ with the action of G on $\Xi \subset G_\mathbb{C}/K_\mathbb{C}$, we have that $g_x \cdot \exp_{x_0}(iY/2) = \exp_x(iY/2)$ for all $Y \in \mathfrak{p}$. Thus,

$$(g_x k) \cdot \exp_{x_0}(iY/2) = g_x \cdot \exp_{x_0}(i\mathrm{Ad}_k Y/2) = \exp_x(i\mathrm{Ad}_k Y/2).$$

This means that the integrals over G/K and over K in (21) combine into an integral over a G-orbit, giving

$$G_F(R) = \int_{\mathfrak{a}_{2R}^+} \int_G \left| F\big(g \cdot \exp_{x_0}(iY/2)\big) \right|^2 dg \, \alpha(Y)\mu(Y) \, dY. \tag{22}$$

We may then evaluate the integral over the G-orbits by Faraut's Gutzmer-type formula (19). After another application of Fubini's Theorem, this gives

$$G_F(R) = \int_{\mathfrak{a}} \left\| \hat{f}(\xi) \right\|^2 e^{-t(|\xi|^2+|\rho|^2)} \left[\int_{\mathfrak{a}_{2R}^+} \phi_\xi\big(e^{iY}\big)\mu(Y)\alpha(Y) \, dY \right] \frac{d\xi}{|c(\xi)|^2}. \tag{23}$$

We now use polar coordinates in the opposite direction to turn the integral in square brackets back into an integral over $\mathfrak{p}^{2R}$:

$$\int_{\mathfrak{a}_{2R}^+} \phi_\xi\big(e^{iY}\big)\mu(Y)\alpha(Y) \, dY = \int_{\mathfrak{p}^{2R}} \phi_\xi\big(e^{iY}\big)\alpha(Y) \, dY.$$

Since, as we have noted earlier, $\phi_\xi(iY)$ grow at most exponentially as a function of ξ with Y fixed, with estimates that are locally uniform in Y (Lemma 2.1 of [36]), it follows that $G_F(R)$ is finite for all $R < R_{max}$. (The growth of the quantity in square brackets on the right-hand side of (23) is less rapid than the decay of $\exp[-t(|\xi|^2 + |\rho|^2)]$.)

We have established, then, the following result.

Proposition 4. *For $f \in L^2(G/K)$ (G not necessarily complex), let $F = e^{t\Delta/2}f$ and let α be a bounded,* Ad-K-invariant, positive density on $\mathfrak{p}^{2R_{max}}$. *Then for all $R < R_{max}$ the function*

$$G_F(R) := \int_{x \in G/K} \int_{Y \in T_x^{2R}(G/K)} \left| F\big(\exp_x(iY/2)\big) \right|^2 \alpha(Y) \, dY \, dx$$

is well defined and finite and given by

$$G_F(R) = \int_{\mathfrak{a}} \left\| \hat{f}(\xi) \right\|^2 e^{-t(|\xi|^2+|\rho|^2)} \left[\int_{\mathfrak{p}^{2R}} \phi_\xi\big(e^{iY}\big)\alpha(Y) \, dY \right] \frac{d\xi}{|c(\xi)|^2}. \tag{24}$$

Clearly, the quantity in square brackets on the right-hand side of (24),

$$\int_{\mathfrak{p}^{2R}} \phi_\xi\big(e^{iY}\big)\alpha(Y) \, dY, \tag{25}$$

is of vital importance in understanding Proposition 4. We call this result a "partial" isometry formula, in that it involves integration of $|F(\exp_x(iY)|^2$ only over a tube of finite radius in

$T(G/K)$. The "global" isometry formula, established in Section 6 in the complex case, will involve a (suitably interpreted) limit of such partial isometries as the radius R goes to infinity.

To close this section, we wish to discuss why it is necessary to let the radius tend to infinity. (Compare Section 4 of [36].) The goal, in the end, is to have the right-hand side of (24) be equal to $\|f\|^2$. To achieve greater flexibility in obtaining this goal, we could replace $\mathfrak{p}^{2R}$ by any convex K-invariant set in $\mathfrak{p}$ whose intersection with $\mathfrak{a}$ is contained in the domain 2Ω. The largest such domain is $\Gamma := \mathrm{Ad}_K(2\Omega)$. Even if we replace $\mathfrak{p}^{2R}$ by Γ, the evidence strongly suggests that there does not exist any Ad-K-invariant density α on Γ for which the right-hand side of (24) is equal to $\|f\|^2$.

In order to have (24) equal to $\|f\|^2$ for all f, α would have to satisfy

$$\int_{\Gamma} \phi_{\xi}\big(e^{iY}\big)\alpha(Y)\,dY = e^{t(|\xi|^2+|\rho|^2)} \tag{26}$$

for almost every ξ. (Essentially the same condition was obtained in a slightly different way by Krötz, Ólafsson, and Stanton in [36, Eq. (4.29)].) At least in the complex case (but almost certainly also in general), a weight satisfying (26) does not exist, as demonstrated in Section 4 of [36].

Let us consider, for example, the case of hyperbolic 3-space. Then Γ is just a ball of radius π and the explicit formulas for the spherical functions turns (26) into

$$\int_{\substack{Y\in\mathbb{R}^3 \\ |Y|\leqslant\pi}} \frac{\sinh(\xi|Y|)}{\xi\sin|Y|}\alpha(Y)\,dY = e^{t(|\xi|^2+|\rho|^2)}, \quad \xi\in\mathbb{R}. \tag{27}$$

Suppose α is any non-negative, rotationally invariant density for which the left-hand side of (27) is finite for almost all ξ. Then it is not hard to see that the left-hand side of (27) grows at most like $e^{\pi|\xi|}$, and thus cannot equal the right-hand side of (27). A similar argument applies to all symmetric spaces of the complex type, as explained in [36, Section 4].

This argument shows that (at least in the complex case), it is not possible to express $\|f\|^2$ as a G-invariant integral of $|F|^2$ over the domain Ξ. Thus, to obtain our isometry formula in the complex case, we extend the integration beyond Ξ, using analytic continuation and a cancellation of singularities, as explained in Section 6.

4. Strategy for a global isometry formula

If we work by analogy to the results of Hall [18,19] and Stenzel [45] in the compact case (see Theorem 2 in the introduction), then we want to take α to be something related to the heat kernel for the *compact* symmetric space U/K dual to G/K. Specifically, according to [39,45], there is a natural local identification of the fibers in $T(G/K)$ with the dual compact symmetric space U/K. We would like, if possible, to choose α so that $\alpha(Y)\,dY$ is the heat kernel *measure* on U/K, based at the identity coset and evaluated at time $2t$. More precisely, the results of [29] indicate that one should take $\alpha(Y)\,dY$ to be a sort of "unwrapped" version of this heat kernel measure. (See Theorem 5 of [29] and Section 5 below for further discussion of the unwrapping concept.) This means that we would like to take

$$\alpha(Y) = \nu_{2t}^{c}(Y)\,j^{c}(Y), \tag{28}$$

where ν_t^c is the unwrapped heat kernel *density* for U/K and j^c is the Jacobian of the exponential mapping for U/K.

With α as given above, the quantity in (25) is given by

$$\int_{\mathfrak{p}^{2R}} \phi_\xi\big(e^{iY}\big)\alpha(Y)\,dY = \int_{\mathfrak{p}^{2R}} \phi_\xi\big(e^{iY}\big)\nu_{2t}^c(Y)j^c(Y)\,dY. \tag{29}$$

Now, ϕ_ξ is an eigenfunction of the Laplacian on G/K with eigenvalue $-(|\xi|^2 + |\rho|^2)$. It then follows that the (locally defined) function on U/K given by $f(e^Y) = \phi_\xi(e^{iY})$ is an eigenfunction of the Laplacian for U/K with eigenvalue $|\xi|^2 + |\rho|^2$. (This assertion can be verified by direct computation but also follows from Theorem 1.16, Propositions 1.17, 1.19 and Theorem 8.5 of [39].) If, by letting R tend to infinity, we could somehow make Proposition 4 into a global result (with α given by (28)), then we would be integrating an eigenfunction of the Laplacian for U/K against the heat kernel for U/K. Thus, the limit as R tends to infinity of (29) "ought" to be $e^{t(|\xi|^2 + |\rho|^2)}\phi_\xi(x_0)$. Since the spherical functions are normalized so that $\phi_\xi(x_0) = 1$, we would get that the right-hand side of (24) tends to $\|f\|^2$ as R tends to infinity.

If we could actually implement this program, we would then obtain an isometry formula analogous to the one in the compact case: $\|f\|^2$ would be equal to the integral of $|F|^2$ first over the fibers with respect to the heat kernel measure for the dual symmetric space and then over the base with respect to the Riemannian volume measure. Unfortunately, because of the singularities that occur in the analytically continued spherical functions and because the identification of $\mathfrak{p}$ with U/K is only local, we do not know how to carry out the above strategy in general.

By contrast, J. Faraut has shown, using a Gutzmer-type formula due to Lassalle [38], that one can carry out a similar line of reasoning if one *starts* on a compact symmetric space. This leads [10] to a new proof of Stenzel's isometry formula for compact symmetric spaces.

In the noncompact case, the case in which G is complex is the most tractable one and we now specialize to this case. We will first work out very explicitly the partial isometry formula in this case, by evaluating the quantity in square brackets in (24), with α given by (28). Then we let the radius tend to infinity, using an appropriate cancellation of singularities.

5. Partial isometry in the complex case

We now assume that G is a connected *complex* semisimple group and K a maximal compact subgroup. The assumption that G is complex is equivalent to the assumption that the (restricted) roots for (G, K) form a reduced root system with all roots having multiplicity 2. The complex case is nothing but the noncompact dual to the compact group case studied in [18,19].

We make use of several (closely related) results that are specific to the complex case and do not hold for general symmetric spaces of the noncompact type. First, in the complex case, the dual compact symmetric space U/K is isometric to a compact group with a bi-invariant metric. There is, as a result, a particular simple formula for the heat kernel on U/K, due to Èskin [7]. (See also [50].) We use an "unwrapped" version of the heat kernel density on U/K, given by

$$\nu_{2t}^c(Y) = e^{t|\rho|^2} j^c(Y)^{-1/2} \frac{e^{-|Y|^2/4t}}{(4\pi t)^{d/2}}. \tag{30}$$

This means that we want to take α in Proposition 4 to be (as in (28))

$$\alpha(Y) = \nu_{2t}^{\mathrm{c}}(Y) j^{\mathrm{c}}(Y) = e^{t|\rho|^2} j^{\mathrm{c}}(Y)^{1/2} \frac{e^{-|Y|^2/4t}}{(4\pi t)^{d/2}}, \tag{31}$$

where on $\mathfrak{a}$ we have, explicitly,

$$j^{\mathrm{c}}(Y)^{1/2} = \prod_{\alpha \in R^+} \frac{\sin\alpha(Y)}{\alpha(Y)}. \tag{32}$$

As shown in [29, Theorem 5], the signed measure $\nu_{2t}^{\mathrm{c}}(Y) j^{\mathrm{c}}(Y) \, dY$ on $\mathfrak{p}$ is an "unwrapped" version of the heat kernel measure for U/K. This means that the push-forward of this measure by $\exp: \mathfrak{p} \to U/K$ is precisely the heat kernel measure on U/K at time $2t$, based at the identity coset.

With α given by (31), the expression in (25) is given by

$$e^{t|\rho|^2} \int_{\mathfrak{p}^{2R}} \phi_\xi\big(e^{iY}\big) j^{\mathrm{c}}(Y)^{1/2} \frac{e^{-|Y|^2/4t}}{(4\pi t)^{d/2}} \, dY. \tag{33}$$

Our next task is to compute (33) as explicitly as possible. Although there is an explicit formula for ϕ_ξ in the complex case (see (52) in Section 8), it is not quite straightforward to compute (33) using that formula. We use instead a more geometric argument, which will also be useful in studying the Segal–Bargmann transform on compact quotients of symmetric spaces of the complex type.

It is known that the function ϕ_ξ is an eigenfunction for the (non-Euclidean) Laplacian on G/K with eigenvalue $-(|\xi|^2 + |\rho|^2)$. In the complex case, we have special "intertwining formulas" for the Laplacian; see Proposition V.5.1 in [31] and the calculations for the complex case on p. 484. These formulas tell us that the function $Y \to \phi_\xi(e^Y) j^{\mathrm{nc}}(Y)^{1/2}$ is an eigenfunction of the *Euclidean* Laplacian for $\mathfrak{p}$ with eigenvalue $-|\xi|^2$. (Here j^{nc} is the Jacobian of the exponential mapping for the noncompact symmetric space G/K.) Since $j^{\mathrm{nc}}(iY) = j^{\mathrm{c}}(Y)$ (as is easily verified from the formulas for these Jacobians) we see that the function

$$\Psi_\xi(Y) := \phi_\xi\big(e^{iY}\big) j^{\mathrm{c}}(Y)^{1/2} \tag{34}$$

is an eigenfunction of the Euclidean Laplacian on $\mathfrak{p}^R$ with eigenvalue $|\xi|^2$.

Lemma 5. *Let Ψ be a smooth function on the ball $B(2R_0, 0)$ in $\mathbb{R}^d$ satisfying $\Delta\Psi = \sigma\Psi$ for some constant $\sigma \in \mathbb{R}$, where Δ is the Euclidean Laplacian. Let β be a non-negative, bounded, measurable, rotationally invariant function on $B(2R_0, 0)$. Then for all $R < R_0$ we have*

$$\int_{|Y| \leqslant 2R} \Psi(Y)\beta(Y) \, dY = \Psi(0) \int_{|Y| \leqslant 2R} e^{\sqrt{\sigma}\, y_1} \beta(Y) \, dY. \tag{35}$$

Here $Y = (y_1, \ldots, y_d)$ and $\sqrt{\sigma}$ is either of the two square roots of σ.

Proof. We let $\tilde{\Psi}$ denote the radialization of Ψ in the Euclidean sense, that is, the average of Ψ with respect to the action of the rotation group. Then $\tilde{\Psi}$ is also an eigenfunction of the Euclidean Laplacian with the same eigenvalue σ, and $\tilde{\Psi}(0) = \Psi(0)$. Since β is rotationally invariant, replacing Ψ with $\tilde{\Psi}$ does not change the value of the integral. But since $\tilde{\Psi}$ is radial, it satisfies differential equation

$$\frac{d^2\tilde{\Psi}}{dr^2} + \frac{(d-1)}{r}\frac{d\tilde{\Psi}}{dr} = \sigma\tilde{\Psi}, \tag{36}$$

with $\tilde{\Psi}(0)$ finite and $d\tilde{\Psi}/dr|_{r=0} = 0$.

When $d = 1$, Eq. (36) is nonsingular at the origin and standard uniqueness results show that $\tilde{\Psi}$ is determined by $\tilde{\Psi}(0)$. When $d \geqslant 2$, (36) is a second-order, linear, nonconstant-coefficient equation, with a regular singular point at $r = 0$. A simple calculation with the theory of regular singular points shows that there is, up to a constant, only one solution of this equation that is nonsingular at the origin.

Now let $\gamma(Y) = e^{\sqrt{\sigma}y_1}$, which is also an eigenfunction of the Laplacian with eigenvalue σ. If $\tilde{\gamma}$ denotes the Euclidean radialization of γ, then $\tilde{\gamma}(0) = 1$ and $\tilde{\gamma}$ also solves Eq. (36) above. Thus we must have $\tilde{\Psi} = \tilde{\Psi}(0)\tilde{\gamma} = \Psi(0)\tilde{\gamma}$. So in the integral on the left-hand side of (35) we may replace Ψ by $\tilde{\Psi}$ and then by $\tilde{\Psi}(0)\tilde{\gamma}$ and finally by $\Psi(0)\gamma$, which establishes the lemma. $\quad\square$

We are now ready to put everything together. We apply Proposition 4 with α as given in (31). We make use of Lemma 5 with $\beta(Y)$ equal to the quantity $(4\pi t)^{-d/2}e^{t|\rho|^2}\exp(-|Y|^2/4t)$, Ψ equal to the function Ψ_ξ in (34), and σ equal to $|\xi|^2$. We also make the change of variable $Y \to 2Y$ (for cosmetic reasons) in the integral that defines $G_F(R)$. The result is the following.

Theorem 6 (*Partial Isometry Formula*). *Let f be in $L^2(G/K)$ (G complex) and let $F = e^{t\Delta/2}f$. Then for all $R < R_{\max}$ the function $G_F(R)$ defined by*

$$G_F(R) = \int\limits_{x \in G/K} \int\limits_{Y \in T_x^R(G/K)} \left|F\big(\exp_x(iY)\big)\right|^2 v_{2t}^{\mathfrak{c}}(2Y)j^{\mathfrak{c}}(2Y)2^d \, dY \, dx$$

may be computed as

$$G_F(R) = \int\limits_{\mathfrak{a}} \|\hat{f}(\xi)\|^2 e^{-t(|\xi|^2+|\rho|^2)}\left[e^{t|\rho|^2} \int\limits_{\substack{Y \in \mathbb{R}^d \\ |Y| \leqslant 2R}} e^{|\xi|y_1}\frac{e^{-|Y|^2/4t}}{(4\pi t)^{d/2}} \, dy\right]\frac{d\xi}{|c(\xi)|^2}, \tag{37}$$

where $Y = (y_1, \ldots, y_d)$. Here $T_x^R(G/K)$ is the set of vectors in $T_x(G/K)$ with magnitude less than R and $v_{2t}^{\mathfrak{c}}$ and $j^{\mathfrak{c}}$ are as in (30) and (32).

Note that for a given R, the expression in square brackets on the right-hand side of (37) depends only on $|\xi|$. Since the effect of the Laplacian on the Fourier transform of f is to multiply $\hat{f}(\xi)$ by $-(|\xi|^2 + |\rho|^2)$, we can rewrite (37) as

$$G_F(R) = \big\langle f, \beta_R(-\Delta)f\big\rangle_{L^2(G/K)}, \tag{38}$$

where β_R is the function given by

$$\beta_R(\lambda) = e^{-t\lambda} e^{t|\rho|^2} \int_{\substack{y \in \mathbb{R}^d \\ |y| \leqslant 2R}} \exp\left(\sqrt{\lambda - |\rho|^2}\, y_1\right) \frac{e^{-|y|^2/4t}}{(4\pi t)^{d/2}}\, dy. \tag{39}$$

Note that the L^2 spectrum of $-\Delta$ is $[|\rho|^2, \infty)$, so that the argument of the square root on the right-hand side of (39) is always non-negative.

6. Global isometry in the complex case

Our goal now is to "let R tend to infinity" in our partial isometry formula for the complex case (Theorem 6). That it is possible to do so reflects a *cancellation of singularities*. The function $F(\exp_x iY)$ becomes singular (and multiple-valued) for large Y. Reflecting this, the orbital integrals of $|F|^2$ become unbounded as the orbits approach the boundary of the domain Ξ. However, Faraut's Gutzmer-type formula tells us that the singularities in the orbital integrals are controlled by the singularities in the analytically continued spherical functions. In the complex case, the singularities of the analytically continued spherical functions are of a particularly simple sort (see (52) in Section 8). These singularities are canceled by the zeros in the density against which we are integrating the orbital integrals, namely, the function α given in (31). (Compare (32) to (52).) This cancellation of singularities allows $G_F(R)$ to be nonsingular, even though both F itself and the orbital integrals of $|F|^2$ are singular.

In Theorem 6, the above-described cancellation of singularities is reflected in the fact that the expression in square brackets on the right-hand side of (37) is well defined and finite for all R. It is not hard, then, to show that $G_F(R)$ admits a real-analytic extension to the whole positive half-line. Furthermore, the limit as R tends to infinity of this analytic extension is easily evaluated by setting $R = \infty$ on the right-hand side of (37) and evaluating a standard Gaussian integral. This will lead to the following result.

Theorem 7 (*Global Isometry Formula*). *Let f be in $L^2(G/K)$, with G complex, and let $F = e^{t\Delta/2} f$. Then for all $R < R_{\max}$, the quantity*

$$G_F(R) := \int_{x \in G/K} \int_{Y \in T_x^R(G/K)} \left|F\left(\exp_x(iY)\right)\right|^2 \nu_{2t}^c(2Y) j^c(2Y) 2^d \, dY \, dx$$

is defined and finite. Furthermore, the function G_F has a real-analytic extension from $(0, R_{\max})$ to $(0, \infty)$ and this extension (also denoted G_F) satisfies

$$\lim_{R \to \infty} G_F(R) = \|f\|_{L^2(G/K)}^2.$$

Proof. We consider the right-hand side of (37) and wish to show that this expression is finite for all $R \in (0, \infty)$ and that it is real-analytic in R. The quantity in square brackets in (37) is bounded

by its limit as R tends to infinity, which is equal to $e^{t(|\xi|^2+|\rho|^2)}$. (This is a simple Gaussian integral.) Thus the right-hand side of (37) is bounded by

$$\int_{\mathfrak{a}} \left\| \hat{f}(\xi) \right\|^2 \frac{d\xi}{|c(\xi)|^2} = \|f\|^2 < \infty.$$

To see that the right-hand side of (37) is real-analytic as a function of R, we reverse the order of integration (since everything is positive) and write it as

$$\int_{\substack{y \in \mathbb{R}^d \\ |y| \leqslant 2R}} \left[\int_{\mathfrak{a}} \left\| \hat{f}(\xi) \right\|^2 e^{-t(|\xi|^2+|\rho|^2)} e^{t|\rho|^2} e^{|\xi||y_1|} \frac{d\xi}{|c(\xi)|^2} \right] \frac{e^{-|y|^2/4t}}{(4\pi t)^{d/2}} \, dy. \tag{40}$$

Now, for any complex number y_1, the quantity $e^{-t|\xi|^2} e^{|\xi||y_1|}$ is a bounded function of ξ. It is therefore not hard to see (using Morera's Theorem) that the expression in square brackets in (40) admits an extension (given by the same formula) to an entire function of y_1. It follows that the whole integrand in (40) is a real-analytic function of y. It is then a straightforward exercise to verify that the integral of a real-analytic function over a ball of radius R is a real-analytic function of R.

To evaluate the limit as R tends to infinity of the right-hand side of (37), we use monotone convergence to put the limit inside. The quantity in square brackets then becomes an easily evaluated Gaussian integral:

$$e^{t|\rho|^2} \int_{y \in \mathbb{R}^d} e^{|\xi||y_1|} \frac{e^{-|y|^2/4t}}{(4\pi t)^{d/2}} \, dy = e^{t|\rho|^2} e^{t|\xi|^2}. \tag{41}$$

Thus, the right-hand side of (37) converges as R tends to infinity to

$$\int_{\mathfrak{a}} \left\| \hat{f}(\xi) \right\|^2 d\xi / \left| c(\xi) \right|^2 = \|f\|_{L^2(G/K)}^2,$$

which is what we want. $\square$

7. Surjectivity theorem in the complex case

Our goal is to show that if F is any holomorphic function for which the isometry formula makes sense and is finite, then F is the analytic continuation of $e^{t\Delta/2} f$, for some unique $f \in L^2(G/K)$. In contrast to the surjectivity result in [36], we do not assume that the restriction of F to G/K is in $L^2(G/K)$ with rapidly decaying Fourier transform. Rather, this property of F holds automatically, in light of the strong form of the Gutzmer formula established in [9]. (See also Appendix A.)

Theorem 8. *Suppose F is a holomorphic function on a domain of the form*

$$\left\{ \exp_x(iY) \in \varXi \mid (x, Y) \in T^{R_0}(G/K) \right\} \tag{42}$$

for some $R_0 \leqslant R_{\max}$ and suppose that the function

$$G_F(R) := \int\limits_{x \in G/K} \int\limits_{Y \in T_x^R(G/K)} \left| F\left(\exp_x(iY)\right) \right|^2 \nu_{2t}^c(2Y) j^c(2Y) 2^d \, dY \, dx \tag{43}$$

is finite for all sufficiently small R. Suppose further that G_F has a real-analytic extension to $(0, \infty)$ and that

$$\lim_{R \to \infty} G_F(R)$$

exists and is finite. Then there exists a unique $f \in L^2(G/K)$ with $F|_{G/K} = e^{t\Delta/2} f$.

Although we initially assume that F is holomorphic only on a domain of the form (42), after the fact we see that the function F, being the analytic continuation of a function of the form $e^{t\Delta/2} f$, can be extended holomorphically to all of Ξ. Furthermore, once $F = e^{t\Delta/2} f$, the isometry theorem tells us that the limit as $R \to \infty$ of $G_F(R)$ is $\|f\|^2$.

Proof of Theorem 8. The uniqueness of f follows from the injectivity of the heat operator $e^{t\Delta/2}$, which in turn follows from the spectral theorem or from the Fourier transform or from the isometry formula.

We turn now to proving the existence of f. According to results of Faraut [9], the assumption that F is square-integrable over the domain in (42) implies that the restriction of F to G/K is in $L^2(G/K)$, that the orbital integrals of $|F|^2$ inside this domain are finite, and that these orbital integrals are given by the Gutzmer formula (19). Thus, if we compute the right-hand side of (43) by the method of the previous section (as in the proof of (40)), we conclude that

$$G_F(R) = \int\limits_{\substack{y \in \mathbb{R}^d \\ |y| \leqslant 2R}} \left[\int\limits_{\mathfrak{a}} \|\widehat{F|_{G/K}}\|^2 e^{t|\rho|^2} e^{|\xi||y_1|} \frac{d\xi}{|c(\xi)|^2} \right] \frac{e^{-|y|^2/4t}}{(4\pi t)^{d/2}} \, dy \tag{44}$$

for $R < R_0$.

We now wish to show that the analytic continuation of G_F must be given (for all $R \in (0, \infty)$) by the expression on the right-hand side of (44). Since the Gaussian factor on the right-hand side of (44) is rotationally invariant, the whole integral is unchanged if we replace the quantity in square brackets (viewed a function of y) by its average over the action of the rotation group. This averaging can be put inside the integral over $\mathfrak{a}$, at which point it affects only $e^{|\xi||y_1|}$. Averaging this function gives (after interchanging an integral with a uniformly convergent sum)

$$\sum_{n \text{ even}} \frac{1}{n!} |\xi|^n |y|^n \int\limits_{S^d} (u \cdot e_1)^n \, du, \tag{45}$$

where du is the normalized volume measure on S^d and where the terms for n odd are zero.

If we replace $e^{|\xi||y_1|}$ by (45) on the right-hand side of (44), all quantities involved will be positive, so by Fubini's Theorem we may freely rearrange the sums and integrals. Rearranging and using polar coordinates on the integral over $\mathbb{R}^d$ gives

$$G_F(R) = \int\limits_0^{2R} \left[\sum_{n \text{ even}} r^n \int\limits_{S^d} (u \cdot e_1)^n \, du \int\limits_{\mathfrak{a}} \| \widehat{F|_{G/K}} \|^2 e^{t|\rho|^2} |\xi|^n \frac{d\xi}{|c(\xi)|^2} \right]$$

$$\times \frac{e^{-r^2/4t}}{(4\pi t)^{d/2}} c_d r^{d-1} \, dr, \tag{46}$$

where c_d is the volume of the unit sphere in $\mathbb{R}^d$. Differentiating with respect to R and moving some factors to the other side gives

$$G_F'(R)(4\pi t)^{d/2} e^{R^2/4t} c_d^{-1} (2R)^{1-d}$$

$$= 2 \sum_{n \text{ even}} (2R)^n \int\limits_{S^d} (u \cdot e_1)^n \, du \int\limits_{\mathfrak{a}} \| \widehat{F|_{G/K}} \|^2 e^{t|\rho|^2} |\xi|^n \frac{d\xi}{|c(\xi)|^2} \tag{47}$$

for $R < R_0$.

Now, since $G_F(R)$ admits a real-analytic extension to all of $(0, \infty)$, so does the right-hand side of (47). Since the coefficient of R^n in (47) is non-negative for all n, it follows (see Lemma 9 below) that the series on the right-hand side of (47) must have infinite radius of convergence. Then both sides of (47) are defined and real-analytic for all positive R; since they are equal for small R, they must be equal for all R. It then follows that (46) also holds for all R. Undoing the reasoning that led to (46), we conclude that (44) also holds for all R.

Now that we know that the analytic continuation of $G_F(R)$ is given by (44) for all R, the Monotone Convergence Theorem tells us that

$$\lim_{R \to \infty} G_F(R) = \int\limits_{\mathbb{R}^d} \left[\int\limits_{\mathfrak{a}} \| \widehat{F|_{G/K}} (\xi) \|^2 e^{t|\rho|^2} e^{|\xi| y_1} \frac{d\xi}{|c(\xi)|^2} \right] \frac{e^{-|y|^2/4t}}{(4\pi t)^{d/2}} \, dy.$$

Reversing the order of integration and using again the Gaussian integral (41) will then give

$$\int\limits_{\mathfrak{a}} \| \widehat{F|_{G/K}} (\xi) \|^2 e^{t|\rho|^2} e^{t|\xi|^2} \frac{d\xi}{|c(\xi)|^2} = \lim_{R \to \infty} G(R) < \infty. \tag{48}$$

We may then conclude that $F|_{G/K}$ is of the form $e^{t\Delta/2} f$, where f is the function whose Fourier transform is given by

$$\hat{f}(\xi) = \widehat{F|_{G/K}}(\xi) e^{t(|\rho|^2 + |\xi|^2)/2}.$$

(That there really is an L^2 function f with this Fourier transform follows from (48).)

This concludes the proof of surjectivity, except for the following elementary lemma about power series with non-negative terms. □

Lemma 9. *Suppose H is a real-analytic function on $(0, \infty)$ such that on $(0, \varepsilon)$, H is given by a convergent power series $H(R) = \sum_{n=0}^{\infty} a_n R^n$. Suppose also the coefficients a_n are non-negative. Then the series $\sum_{n=0}^{\infty} a_n R^n$ has infinite radius of convergence.*

Proof. Assume, to the contrary, that the series $\sum a_n x^n$ has radius of convergence $S < \infty$. Since both $H(R)$ and $\sum a_n R^n$ are real analytic on $(0, S)$ and they are equal on $(0, \varepsilon)$, they are equal on $(0, S)$. We may then differentiate $H(R)$ term by term for $R < S$. Since $H^{(k)}$ is continuous on $(0, \infty)$, letting R approach S gives, by monotone convergence,

$$\frac{H^{(k)}(S)}{k!} = \sum_{n=0}^{\infty} a_n \binom{n}{k} S^{n-k},$$

where $\binom{n}{k}$ is defined to be 0 for $k > n$.

Using Fubini's Theorem (since all terms are non-negative) and the binomial theorem, we have for any $\delta > 0$,

$$\sum_{k=0}^{\infty} \frac{H^{(k)}(S)}{k!} \delta^k = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} \binom{n}{k} S^{n-k} \delta^k = \sum_{n=0}^{\infty} a_n (S + \delta)^n = \infty,$$

because $\sum a_n R^n$ has radius of convergence S. This shows that the Taylor series of H at S has radius of convergence zero, contradicting the assumption that H is real-analytic on $(0, \infty)$. $\quad\square$

8. Comparison with the results of Krötz, Ólafsson, and Stanton

As we have already pointed out in Section 1.4, the isometry formula of Krötz, Ólafsson, and Stanton (Theorem 3.3 of [36]), when specialized to the complex case, *does not* reduce to our isometry formula. We now explain the relationship between the two formulas. Since both formulas already have complete proofs, we will not attempt to give a completely rigorous reduction of one formula to the other. Rather, we will show formally how the isometry formula in [36] can be reduced to the one we prove here, by means of an integration by parts.

Let us begin in the setting of [36], which means that we consider a symmetric space of the form G/K, where G is a real connected semisimple group with finite center and K is a maximal compact subgroup. At the moment, we do not assume that G is complex. After adjusting for differences of normalization of the heat operator ($e^{t\Delta}$ in [36] versus $e^{t\Delta/2}$ here), the isometry formula of [36, Theorem 3.3] can be written as

$$\|f\|^2 = \frac{e^{t|\rho|^2}}{|W|(4\pi t)^{\frac{n}{2}}} \int\limits_{\mathfrak{a}} D\big(\mathcal{O}_{|F|^2}(iY)\big) e^{-|Y|^2/4t} \, dY, \tag{49}$$

where $n = \dim \mathfrak{a}$ is the rank of G/K. Here $\mathcal{O}_{|F|^2}(iY)$ denotes the "orbital integral" of $|F|^2$ appearing in (19), namely,

$$\mathcal{O}_{|F|^2}(iY) = \int\limits_{G} \big|F\big(g \cdot \exp_{x_0}(iY/2)\big)\big|^2 \, dg \tag{50}$$

and D is a pseudodifferential "shift operator" that takes the spherical functions to their Euclidean counterparts. Although $\mathcal{O}_{|F|^2}(iY)$ itself is defined only for small Y, the shift operator D cancels out all the singularities and produces a function that is defined real-analytically on all of $\mathfrak{a}$.

(There appears to be a slight inconsistency in the way the orbital integral is defined in [36], as in (50) in the original definition, but with $Y/2$ replaced by Y in Eq. (3.19) in the proof of the isometry formula. We have maintained the original definition (Eq. (1.2) of [36]) of the orbital integral and adjusted the isometry formula accordingly. This adjustment along with the difference in normalization of the heat equation account for the differences between (49) and Theorem 3.3 of [36].)

If we ignored the singularities in $\mathcal{O}_{|F|^2}$, we could formally move the operator D off of the orbital integral, at the expense of applying the adjoint operator D^* to the Gaussian factor. We could then use Weyl invariance to reduce the domain of integration from $\mathfrak{a}$ to $\mathfrak{a}^+$, giving the nonrigorous expression

$$\|f\|^2 \overset{?}{=} \frac{e^{t|\rho|^2}}{(4\pi t)^{\frac{n}{2}}} \int_{\mathfrak{a}^+} \mathcal{O}_{|F|^2}(iY) D^* \big(e^{-|Y|^2/4t}\big)\, dY. \tag{51}$$

The idea is that D^* is also a sort of shift operator (or Abel transform) and should have the effect of turning the Euclidean heat kernel $\exp(-|Y|^2/4t)$ into the non-Euclidean heat kernel for the *compact* symmetric space dual to G/K. If (51) were really correct it would express $\|f\|^2$ as an integral of $|F|^2$ as an integral over G-orbits followed by an integral over the space of G-orbits, which is just the sort of thing we have in this paper.

In general, it is not at all clear that the right-hand side of (51) makes sense. Even assuming that $D^*(\exp(-|Y|^2/4t))$ is well defined, there will be singularities in the orbital integral $\mathcal{O}_{|F|^2}(iY)$, which are related to the singularities in the analytically continued spherical functions that appear in the Gutzmer formula. Examples show that in general, the singularities in the orbital integral will *not* be canceled by zeros in $D^*(\exp(-|Y|^2/4t))$ and so the right-hand side of (51) will not be well defined without some further "interpretation."

In the complex case, however, D is a simple differential operator and taking its adjoint amounts to integrating by parts. We will now compute D^* explicitly and see that, in this case, $D^*(\exp(-|Y|^2/4t))$ has zeros in all the places that the orbital integral is singular, so that (51) is actually nonsingular. Indeed, in the complex case, (51) is essentially just our isometry formula (Theorem 7).

In this calculation, there are various constants, depending only on the choice of symmetric space, whose values are not worth keeping track of. In the remainder of this section, C will denote such a constant whose value changes from line to line.

In the complex case, the explicit formula for the spherical function (e.g., Theorem 5.7, p. 432, of [31]) implies that

$$\phi_\xi\big(e^{iY}\big) = \frac{C}{\pi(\xi)} \cdot \frac{\sum_{w \in W}(\det w)e^{-\langle w \cdot \xi, Y\rangle}}{\prod_{\alpha \in \Sigma^+}\sin\langle \alpha, Y\rangle}, \tag{52}$$

where π is the Weyl-alternating polynomial given by $\pi(Y) = \prod_{\alpha \in \Sigma^+}\langle \alpha, Y\rangle$. Meanwhile, D is supposed to be the operator that takes the spherical functions to their Euclidean counterparts ψ_ξ, which satisfy

$$\psi_\xi(iY) = \sum_{w \in W} e^{-\langle w \cdot \xi, Y\rangle}.$$

(Note that, following [36], we normalize the Euclidean spherical functions to have the value $|W|$ at the origin.)

Let D_α denote the directional derivative in the direction of α and observe that

$$\left(\prod_{\alpha \in \Sigma^+}(-D_\alpha)\right)e^{-\langle\xi,w\cdot Y\rangle} = \left(\prod_{\alpha\in\Sigma^+}\langle w\cdot\xi,\alpha\rangle\right)e^{-\langle\xi,w\cdot Y\rangle} = (\det w)\pi(\xi)e^{-\langle\xi,w\cdot Y\rangle}$$

because the polynomial π is alternating. Thus, we can see that

$$D = C\left(\prod_{\alpha\in\Sigma^+}(-D_\alpha)\right)\left(\prod_{\alpha\in\Sigma^+}\sin\langle\alpha,Y\rangle\right). \tag{53}$$

(To be precise, the operator that we are here calling D is the operator that takes the *analytic continuation* of the spherical function ϕ_ξ for G/K to the *analytic continuation* of the Euclidean spherical function ψ_ξ. The operator that takes ϕ_ξ itself to ψ_ξ would involve hyperbolic sines instead of ordinary sines. With our definition of D, it is correct to write $D(\mathcal{O}_{|F|^2}(iY))$ rather than $(D\mathcal{O}_{|F|^2})(iY)$ as in [36].)

Taking the adjoint of (53), we obtain

$$D^* = C\left(\prod_{\alpha\in\Sigma^+}\sin\langle\alpha,Y\rangle\right)\left(\prod_{\alpha\in\Sigma^+}D_\alpha\right).$$

We now claim that

$$\left(\prod_{\alpha\in\Sigma^+}D_\alpha\right)e^{-|Y|^2/4t} = \left(\prod_{\alpha\in\Sigma^+}\frac{-\langle\alpha,Y\rangle}{2t}\right)e^{-|Y|^2/4t}. \tag{54}$$

To see this, we first observe that the *Fourier transform* of the left-hand side of (54) is a constant times a Gaussian times the polynomial π. Since π is alternating with respect to the action of the Weyl group and since the Fourier transform commutes with the action of the Weyl group, it follows that the left-hand side of (54) is also alternating. The left-hand side of (54) is a polynomial $h(Y)$ times $e^{-|Y|^2/4t}$, and the polynomial h must be alternating. Furthermore, the leading order term in h is easily seen to be the polynomial appearing on the right-hand side of (54). The lower-order terms in h are also alternating, and an alternating polynomial whose degree is less than the number of positive roots must be identically zero. (Compare Lemma 4 of [50].)

In the complex case, then, (51) takes the form

$$\|f\|^2 = C\frac{e^{t|\rho|^2}}{t^{d/2}}\int_{\mathfrak{a}^+}\mathcal{O}_{|F|^2}(iY)\left(\prod_{\alpha\in\Sigma^+}\frac{\sin\alpha(Y)}{\alpha(Y)}\right)e^{-|Y|^2/4t}\left(\prod_{\alpha\in\Sigma^+}\alpha(Y)\right)^2 dY, \tag{55}$$

where we have rearranged the polynomial factors in a convenient way and where $d = \dim(G/K)$. (In the complex case, $\dim(G/K) = \dim\mathfrak{a} + 2|\Sigma^+|$.) We claim that in this case, (55) actually makes sense. Specifically, $\mathcal{O}_{|F|^2}(iY)$ may be computed by the Gutzmer formula (19) and the explicit formula (52) for the spherical functions then indicates that sine factors on the right-hand side of (55) cancel all the singularities in $\mathcal{O}_{|F|^2}$.

Meanwhile, in the complex case the density for generalized polar coordinates (integration of Ad-K-invariant functions on $\mathfrak{p}$) is given by

$$\mu(Y) = C\left(\prod_{\alpha \in \Sigma^+} \alpha(Y)\right)^2.$$

(This is Theorem I.5.17 of [32] in the case where each m_α is equal to 2.) Also, the product over Σ^+ of $\sin\alpha(Y)/\alpha(Y)$ is just the Jacobian factor $j^c(Y)^{1/2}$ of (32). Thus, if we rewrite (55) as a limit of integrals over $\mathfrak{a}_R^+$ and use the equality of (20) and (21) we see that (51) becomes

$$\lim_{R \to \infty} C \frac{e^{t|\rho|^2}}{t^{d/2}} \int\limits_{x \in G/K} \int\limits_{Y \in T_x^{2R}(G/K)} \left|F\left(\exp_x(iY/2)\right)\right|^2 j^c(Y)^{1/2} e^{-|Y|^2/4t} \, dY \, dx.$$

This is nothing but the isometry formula established in Theorem 7, disguised by the change of variable $Y \to Y/2$.

Presumably, this line of reasoning could be used to give a rigorous reduction of our isometry formula to that of [36]. However, some care would have to be given to the boundary terms in the integration by parts.

Appendix A. The Gutzmer-type formula of Faraut

In this appendix, we discuss Faraut's Gutzmer-type formula, established in [8] and then in a stronger form in [9]. We are particularly concerned with the conditions under which this formula can be applied. In [8], Faraut established the Gutzmer formula under the assumption that the Fourier transform of the restriction of F to G/K has compact support. We will show that this result can easily be extended to any F of the form $F = e^{t\Delta/2} f$, with $f \in L^2(G/K)$, something we require in the proof of the isometry formula. Meanwhile, in [9], Faraut established the Gutzmer formula under the assumption that F is square-integrable over (a domain in) Ξ with respect to a nice G-invariant measure. We require the result of [9] in the proof of the surjectivity theorem.

First, fix $f \in L^2(G/K)$ and let $F := e^{t\Delta/2} f$. If the Fourier transform of f is $\hat{f}$ (in the notation established in Section 3), then the Fourier transform of F is given by $\hat{F}(\xi) = \hat{f}(\xi)e^{-t(|\xi|^2+|\rho|^2)/2}$. Let F_n be the function whose Fourier transform is given by

$$\hat{F}_n(\xi) = \hat{f}(\xi)e^{-t(|\xi|^2+|\rho|^2)/2}\chi_n(\xi), \tag{A.1}$$

where χ_n is the indicator function of the ball of radius n in $\mathfrak{a}^*$. Since the Fourier transform of F_n has compact support, the hypotheses of the Gutzmer formula in [8] hold. Thus, F_n has a holomorphic extension to Ξ and the Gutzmer formula in (19) holds.

Meanwhile, according to Lemma 2.1 and Remark 2.2 of [36], for each Y in Ω, there exists C_Y such that

$$\phi_\xi\left(e^{iY}\right) \leqslant C_Y e^{|\xi||Y|}$$

for all $\xi \in \mathfrak{a}$, and C_Y may be taken to be a locally bounded function of Y. Then, using the Gutzmer formula and (A.1), we see that F_n converges in L^2 on each G-orbit $G \cdot e^{iY}$, and the L^2 convergence is locally uniform as a function of Y. This means that the F_n's are converging in

L^2_{loc}, which then implies that the limiting function Φ is holomorphic. Since also the restriction of Φ to G/K is the L^2 limit of the F_n's, namely, F, we conclude that $\Phi = F$. By the continuity of the L^2 norm, then, we conclude that the Gutzmer formula holds for F.

Meanwhile, the paper [9] establishes the Gutzmer formula for weighted Bergman spaces. This means that we assume F is holomorphic on a G-invariant domain $\mathcal{D} \subset \Xi$ that contains G/K and with the property that the intersection of Γ with each $T_x(G/K)$ is convex. We then assume that F is square-integrable over $\mathcal{D}$ with respect to a G-invariant measure p that has a positive density that is locally bounded away from zero. We let $\mathcal{B}^2(\mathcal{D}, p)$ denote (in Faraut's notation) the space of holomorphic functions on $\mathcal{D}$ that are square-integrable with respect to p. Faraut proves that if $F \in \mathcal{B}^2(\mathcal{D}, p)$, then: (1) the restriction of F to G/K is square-integrable, (2) the restriction of F to each G-orbit inside $\mathcal{D}$ is square-integrable, and (3) the Gutzmer formula holds.

Let us elaborate briefly on one point that is used in the proof of this form of the Gutzmer formula. The result is that given $F \in \mathcal{B}^2(\mathcal{D}, p)$, there exists a sequence $F_n \in \mathcal{B}^2(\mathcal{D}, p)$ converging to F in the norm topology of $\mathcal{B}^2(\mathcal{D}, p)$ such that the Fourier transform of $F_n|_{G/K}$ has compact support. The argument for the existence of such a sequence is implicit in [9], but we felt it might be helpful to spell it out explicitly, since this is the key to extending the Gutzmer formula to functions in $\mathcal{B}^2(\mathcal{D}, p)$.

Faraut shows (Proposition 3.1) that the restriction map $R : \mathcal{B}^2(\mathcal{D}, p) \to L^2(G/K)$ is bounded and injective. It follows that the adjoint map $R^* : L^2(G/K) \to \mathcal{B}^2(\mathcal{D}, p)$ is bounded with dense image. Thus, since functions whose Fourier transform has compact support are dense in $L^2(G/K)$, given $F \in \mathcal{B}^2(\mathcal{D}, p)$, we can choose $g_n \in L^2(G/K)$ with compactly supported Fourier transform such that $R^* g_n \to F$ in $\mathcal{B}^2(\mathcal{D}, p)$. Meanwhile, the operator RR^* is a convolution operator on $L^2(G/K)$ (see p. 104 in [9]), which preserves the space of functions with compactly supported Fourier transform. Thus the Fourier transform of $RR^* g_n$ has compact support, which means that $F_n := R^* g_n$ is the desired sequence in $\mathcal{B}^2(\mathcal{D}, p)$.

In the surjectivity theorem, we wish to apply the Gutzmer formula in the case $\mathcal{D} = T^{R_0}(G/K)$ and p is the measure associated to the density $\alpha(Y) = \nu^c_{2t}(Y) j^c(Y)$ as in Proposition 4.

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