J. Differential Equations 248 (2010) 2608–2627

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Stability of periodic traveling waves for complex modified Korteweg–de Vries equation

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ARTICLE INFO

Article history:
Received 21 November 2009
Revised 28 January 2010
Available online 19 February 2010

Keywords:
Korteweg–de Vries equation
Periodic waves
Stability

ABSTRACT

We study the existence and stability of periodic traveling-wave solutions for complex modified Korteweg–de Vries equation. We also discuss the problem of uniform continuity of the data-solution mapping.

1. Introduction

Consider the complex modified Korteweg–de Vries equation

$$u_t + 6|u|^2 u_x + u_{xxx} = 0, \tag{1.1}$$

where u is a complex-valued function of $(x,t) \in \mathbb{R}^2$. In this paper, we study the orbital stability of the family of periodic traveling-wave solutions

$$u = \varphi(x,t) = e^{i\omega(x+(3a+\omega^2)t)} r\big(x + \big(a + 3\omega^2\big)t\big), \tag{1.2}$$

where $r(y)$ is a real-valued T-periodic function and $a, \omega \in \mathbb{R}$ are parameters. The problem of the stability of solitary waves for nonlinear dispersive equations goes back to the works of Benjamin [5] and Bona [8] (see also [1,18,19]). A general approach for investigating the stability of solitary waves for nonlinear equations having a group of symmetries was proposed in [10]. The existence and stability

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.02.001

of solitary wave solutions for Eq. (1.1) has been studied in [20]. In contrast to solitary waves for which stability is well understood, the stability of periodic traveling waves has received little attention. Recently in [3], the authors developed a complete theory on the stability of cnoidal waves for the KdV equation. Other new explicit formulae for the periodic traveling waves based on the Jacobi elliptic functions, together with their stability, have been obtained in [2,4,11] for the nonlinear Schrödinger equation (NLS), modified KdV equation, and generalized BBM equation. In [12], the stability of periodic traveling-wave solutions of BBM equation which wave profile stays close to the constant state $u = (c-1)^{1/p}$ is considered.

Our purpose here is to study existence and stability of periodic traveling-wave solutions of Eq. (1.1). We base our analysis on the invariants $Q(u)$, $P(u)$, and $F(u)$ (see Section 4). Our approach is to verify that φ is a minimizer of a properly chosen functional $M(u)$ which is conservative with respect to time over the solutions of (1.1). We consider the L^2-space of T-periodic functions in $x \in \mathbb{R}$, with a norm $\|.\|$ and a scalar product $\langle .,. \rangle$. To establish that the orbit

$$\mathcal{O} = \left\{ e^{i\omega\eta} \varphi(\cdot - \xi, t) \colon \omega \in 2\pi\mathbb{Z}/T,\ \xi, \eta \in [0, T] \right\}$$

is stable, we take $u(x,t) = e^{i\omega\eta}\varphi(x-\xi,t) + h(x,t)$, $h = h_1 + ih_2$ and express the leading term of $M(u) - M(\varphi)$ as $\langle L_1 h_1, h_1 \rangle + \langle L_2 h_2, h_2 \rangle$ where L_i are second-order self-adjoint differential operators in $L^2[0,T]$ with potentials depending on r and satisfying $L_1 r' = L_2 r = 0$. The proof of orbital stability requires that zero is the second eigenvalue of L_1 and the first one of L_2. Therefore, we are able to establish stability when $r(y)$ does not oscillate around zero. Sometimes, the waves (1.2) with this property are called "dnoidal waves" because r is expressed by means of the elliptic function $dn(y;k)$.

We would like to mention that (1.1) is the second equation in the NLS hierarchy and it also can be integrated by the inverse spectral transform method. Besides, the complex modified Korteweg–de Vries equation has been obtained from Hasimoto transformation for different vortex Hamiltonians [16]. As seen in Section 3 below, apart of the KdV equation, the complex modified Korteweg–de Vries equation has much more rich family of traveling-wave solutions, including complex-valued ones, obtained when $\omega \neq 0$.

The paper is organized as follows. In Section 2, we discuss in brief the correctness of the Cauchy problem for (1.1) in periodic Sobolev spaces H^s, $s \in \mathbb{R}$ (equipped with a norm $\|.\|_s$). The problem is locally well-posed for $s > \frac{3}{2}$ and ill-posed for $s < \frac{1}{2}$. In Section 3 we outline the existence and the properties of the periodic traveling-wave solutions (1.2) to (1.1), with emphasis on the case when r does not oscillate around zero. In Section 4 we prove our main orbital stability result (Theorem 4.1). In Appendix A, we establish some technical results we need during the proof of our main theorem.

2. Cauchy problem

In this section we discuss the well-posedness of the initial-value problem for the complex modified Korteweg–de Vries equation in the periodic case. We take an initial value $u_0(x)$ in a periodic Sobolev space $H^s = H^s[0,T]$. Local well-posedness means that there exists a unique solution $u(.,t)$ of (1.1) taking values in H^s for a time interval $[0, t_0)$, it defines a continuous curve in H^s and depends continuously on the initial data.

The local well-posedness for (1.1) in the non-periodic case is studied in [20] by applying Kato's theory of abstract quasilinear equations [13,14]. In the periodic case the conditions are verified in the same way as in [20], therefore we present here without proving the following result.

Theorem 2.1. *Let $s > \frac{3}{2}$ and $T > 0$. For each $u_0(x) \in H^s[0,T]$ there exists t_0 depending only on $\|u_0\|_s$ such that* (1.1) *has a unique solution $u(x,t)$, with $u(x,0) = u_0(x)$ and*

$$u \in C\big([0,t_0); H^s\big) \cap C^1\big([0,t_0); H^{s-3}\big).$$

Moreover, the mapping $u_0(x) \to u(x,t)$ is continuous in the $H^s[0,T]$-norm.

Remark. If there exist real numbers $s_1 > s_0 > \frac{3}{2}$ and a nondecreasing function α such that for each $t_0 > 0$ and for any solution $u \in C([0, t_0); H^{s_1})$ of (1.1), one has

$$\|u\|_{s_1} \leqslant \alpha\big(\|u_0\|_{s_0}\big), \quad t \in [0, t_0),$$

then the assertion of Theorem 2.1 holds with $t_0 = \infty$. For example, this is the case for $s_0 = 2$, which follows from the existence of appropriate nonlinear functionals, invariant with respect to time (see [20]).

Sometimes it is more appropriate to consider other version of well-posedness, for example by strengthening our definition, requiring that the mapping data-solution is uniformly continuous, that is: for any ε, there exists $\delta > 0$, such that if $\|u_{01} - u_{02}\|_s < \delta$, then $\|u_1 - u_2\|_s < \varepsilon$, with $\delta = \delta(\varepsilon, M)$, where $\|u_{01}\|_s \leqslant M$ and $\|u_{02}\|_s \leqslant M$. The ill-posedness of some classical nonlinear dispersive equations (KdV, mKdV, NLS) in both periodic and non-periodic cases are studied in [6,7,9,15]. The approach in these papers is based on the existence and good properties of the traveling-wave solutions associated to the respective equations. Below we discuss the problem of the uniform continuity of data-solution mapping for (1.1) in periodic Sobolev spaces with small s.

Theorem 2.2. *The initial-value problem for the complex modified Korteweg–de Vries equation* (1.1) *is locally ill-posed for initial data in the periodic spaces H^s with $s < \frac{1}{2}$.*

Proof. It is easy to see that

$$u_{N,A}(x,t) = A\exp\big(i\big(Nx + \big(N^3 - 6A^2N\big)t\big)\big), \tag{2.1}$$

where A is a real constant and N is a positive integer, solves Eq. (1.1) with initial data $u_0(x) = A\exp(iNx)$. For $A = \alpha N^{-s}$ where α is a real parameter, we have

$$\|u_0\|_s \leqslant C\alpha^2,$$
$$\big\|u_{N,A}(\cdot,t)\big\|_s \leqslant C\alpha^2$$

with $C > 0$. Let $A_1 = \alpha_1 N^{-s}$ and $A_2 = \alpha_2 N^{-s}$. For the Sobolev norm of the difference of two initial data, we have

$$\begin{aligned}\big\|u_{A_1,N}(0) - u_{A_2,N}(0)\big\|_s^2 &= \sum_{\xi\in\mathbb{Z}}\big(1+\xi^2\big)^s\big|\widehat{u_{A_1,N}}(\xi) - \widehat{u_{A_2,N}}(\xi)\big|^2 \\ &\leqslant C|\alpha_1 - \alpha_2|^2 \to 0 \quad \text{as } \alpha_1 \to \alpha_2.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\big\|u_{A_1,N}(\cdot,t) - u_{A_2,N}(\cdot,t)\big\|_s^2 \\ &\quad= \sum_{\xi\in\mathbb{Z}}\big(1+\xi^2\big)^s\big|\widehat{u_{A_1,N}}(\xi) - \widehat{u_{A_2,N}}(\xi)\big|^2 \\ &\quad= \big(1+N^2\big)^s\big|\alpha_1 N^{-s}\exp\big(i\big(N^3 - 6\alpha_1^2 N^{-2s+1}\big)\big) - \alpha_2 N^{-s}\exp\big(i\big(N^3 - 6\alpha_2^2 N^{-2s+1}\big)\big)\big|^2 \\ &\quad\geqslant C\big|\alpha_1 - \alpha_2\exp\big(i6\big(\alpha_1^2 - \alpha_2^2\big)N^{1-2s}\big)\big|^2.\end{aligned}$$

Let $s < \frac{1}{2}$, and α_1, α_2 be chosen so that

$$\left(\alpha_1^2 - \alpha_2^2\right) N^{1-2s} = C N^{2\nu},$$

where $\nu > 0$ and $2\nu + 2s - 1 < 0$. Then for $t = \frac{\pi}{2} C^{-1} N^{-2\nu}$, we have

$$\left\| u_{A_1,N}(x,t) - u_{A_2,N}(x,t) \right\|_s^2 \geqslant C\left(\alpha_1^2 + \alpha_2^2\right).$$

Note that t can be made arbitrary small by choosing N sufficiently large. This completes the proof of the theorem. □

3. Periodic traveling-wave solutions

We are looking for traveling-wave solutions for Eq. (1.1) in the form

$$\varphi(x,t) = e^{i\omega(x+\alpha t)} r(x + \beta t), \tag{3.1}$$

where $\alpha, \beta, \omega \in \mathbb{R}$ and $r(y)$ is a smooth real periodic function with a given period T. Substituting (3.1) into (1.1) and separating real and imaginary parts, we obtain the following equations

$$\beta - 3\omega^2 = \frac{1}{3}\left(\alpha - \omega^2\right) = a \in \mathbb{R},$$
$$r'' + 2r^3 + ar = 0. \tag{3.2}$$

Therefore

$$u = \varphi(x,t) = e^{i\omega(x+(3a+\omega^2)t)} r\left(x + \left(a + 3\omega^2\right)t\right). \tag{3.3}$$

Integrating once again the second equation in (3.2), we obtain

$$r'^2 = c - ar^2 - r^4, \tag{3.4}$$

hence the periodic solutions are given by the periodic trajectories $H(r, r') = c$ of the Hamiltonian vector field $dH = 0$ where

$$H(x,y) = y^2 + x^4 + ax^2.$$

Clearly, two cases appear:

1) *Global center* ($a \geqslant 0$). Then for any $c > 0$ the orbit defined by $H(r, r') = c$ is periodic and oscillates around the center at the origin.
2) *Duffing oscillator* ($a < 0$). Then there are two possibilities:
   2.1) (*outer case*): for any $c > 0$ the orbit defined by $H(r, r') = c$ is periodic and oscillates around the eight-shaped loop $H(r, r') = 0$ through the saddle at the origin.
   2.2) (*left and right cases*): for any $c \in (-\frac{1}{4}a^2, 0)$ there are two periodic orbits defined by $H(r, r') = c$ (the left and right ones). These are located inside the eight-shaped loop and oscillate around the centers at $(\mp\sqrt{-a/2}, 0)$, respectively.

In cases 1) and 2.1) above, $r(x)$ oscillates around zero and for this reason we are unable to study stability properties of the wave (3.3). In the rest of the paper, we will consider the left and right cases of Duffing oscillator.

Remark. One could also consider Eq. (1.1) with a minus sign,

$$u_t - 6|u|^2 u_x + u_{xxx} = 0.$$

It has a traveling-wave solution of the form (3.3) where r is a real-valued periodic function of period T satisfying equation $r'' - 2r^3 + ar = 0$. Taking $H(x, y) = y^2 - x^4 + ax^2$, $a > 0$ (*the truncated pendulum Hamiltonian*), we see that for $c \in (0, \frac{1}{4}a^2)$ the periodic solutions are given by the periodic trajectories $H(r, r') = c$ of the Hamiltonian vector field $dH = 0$ which oscillate around the center at the origin and are bounded by the separatrix contour $H(r, r') = \frac{1}{4}a^2$ connecting the saddles $(\mp\sqrt{a/2}, 0)$. Therefore, we are unable to handle this case, too.

In the left and the right cases, let us denote by $r_0 > r_1 > 0$ the positive roots of $r^4 + ar^2 - c = 0$. Then, up to a translation, we obtain the respective explicit formulas

$$r(z) = \mp r_0 dn(\alpha z; k), \qquad k^2 = \frac{r_0^2 - r_1^2}{r_0^2} = \frac{a + 2r_0^2}{r_0^2}, \qquad \alpha = r_0, \qquad T = \frac{2K(k)}{\alpha}. \tag{3.5}$$

Here and below $K(k)$ and $E(k)$ are, as usual, the complete elliptic integrals of the first and second kind in a Legendre form. By (3.5), one also obtains $a = (k^2 - 2)\alpha^2$ and, finally,

$$T = \frac{2\sqrt{2 - k^2}K(k)}{\sqrt{-a}}, \quad k \in (0, 1),\ T \in I = \left(\frac{2\pi}{\sqrt{-2a}}, \infty\right). \tag{3.6}$$

Lemma 3.1. *For any $a < 0$ and $T \in I$, there is a constant $c = c(a)$ such that the periodic traveling-wave solution* (3.3) *determined by $H(r, r') = c(a)$ has a period T. The function $c(a)$ is differentiable.*

Proof. The statement follows from the implicit function theorem. It is easily seen that the period T is a strictly increasing function of k:

$$\frac{d}{dk}\left(\sqrt{2 - k^2}K(k)\right) = \frac{(2 - k^2)K' - kK}{\sqrt{2 - k^2}} = \frac{K' + E'}{\sqrt{2 - k^2}} > 0.$$

Given a and c in their range, consider the functions $r_0(a, c)$, $k(a, c)$ and $T(a, c)$ given by the formulas we derived above. We obtain

$$\frac{\partial T}{\partial c} = \frac{dT}{dk}\frac{dk}{dc} = \frac{1}{2k}\frac{dT}{dk}\frac{d(k^2)}{dc}.$$

Further, we have in the left and right cases

$$\frac{d(k^2)}{dc} = \frac{d(k^2)}{d(r_0^2)}\frac{dr_0^2}{dc} = -\frac{a}{r_0^4(a + 2r_0^2)}.$$

We see that $\partial T(a, c)/\partial c \neq 0$, therefore the implicit function theorem yields the result. □

4. Stability

In this section we prove our main stability result which concerns the left (right) Duffing oscillator cases. Take $a < 0$, $T > 2\pi/\sqrt{-2a}$ and determine $c = c(a)$ so that the two orbits given by $H(r, r') = c$ have period T. Next, chose $\omega \neq 0$ in (3.3) to satisfy $\omega T/2\pi \in \mathbb{Z}$. Then $\varphi(x, t)$ is a solution of (1.1) having a period T with respect to x.

4.1. Basic statements and reductions

Take a solution $u(x,t)$ of (1.1) of period T in x and introduce the pseudometric

$$d(u,\varphi)=\inf_{(\eta,\xi)\in\mathbb{R}^2}\left\|u(x,t)-e^{i\omega\eta}\varphi(x-\xi,t)\right\|_1. \tag{4.1}$$

Eq. (1.1) possesses the following conservation laws

$$Q(u)=i\int_0^T\bar{u}_x u\,dx,\qquad P(u)=\int_0^T|u|^2\,dx,\qquad F(u)=\int_0^T\left(|u_x|^2-|u|^4\right)dx.$$

Let

$$M(u)=F(u)+\left(\omega^2-a\right)P(u)-2\omega Q(u).$$

For a fixed $q>0$, we denote

$$d_q^2(u,\varphi)=\inf_{(\eta,\xi)\in\mathbb{R}^2}\left(\left\|u_x(x,t)-e^{i\omega\eta}\varphi_x(x-\xi,t)\right\|^2+q\left\|u(x,t)-e^{i\omega\eta}\varphi(x-\xi,t)\right\|^2\right). \tag{4.2}$$

Clearly, the infimum in (4.1) and (4.2) is attained at some point (η,ξ) in the square $[0,T]\times[0,T]$. Moreover, for $q\in[q_1,q_2]\subset(0,\infty)$, (4.2) is a pseudometric equivalent to (4.1). Now, we can formulate our main result in the paper.

Theorem 4.1. *Let φ be given by* (1.2)*, with $r\neq 0$. For each $\varepsilon>0$ there exists $\delta>0$ such that if $u(x,t)$ is a solution of* (1.1) *and $d(u,\varphi)|_{t=0}<\delta$, then $d(u,\varphi)<\varepsilon$ $\forall t\in[0,\infty)$.*

The crucial step in the proof will be to verify the following statement.

Proposition 4.1. *There exist positive constants m,q,δ_0 such that if u is a solution of* (1.1) *such that $P(u)=P(\varphi)$ and $d_q(u,\varphi)<\delta_0$, then*

$$M(u)-M(\varphi)\geqslant m d_q^2(u,\varphi). \tag{4.3}$$

The proof consists of several steps. The first one concerns the metric d_q introduced above.

Lemma 4.1. *The metric $d_q(u,\varphi)$ is a continuous function of $t\in[0,\infty)$.*

Proof. The proof of the lemma is similar to the proof of Lemmas 1, 2 in [8]. □

We fix $t\in[0,\infty)$ and assume that the minimum in (4.1) is attained at the point $(\eta,\xi)=(\eta(t),\xi(t))$. In order to estimate $\Delta M=M(u)-M(\varphi)$, we set

$$u(x,t)=e^{i\omega\eta}\varphi(x-\xi,t)+h(x,t)$$

and integrating by parts in the terms containing h_x and $\bar{h}_x$, we obtain

$$\begin{aligned}\Delta M &= M(u) - M(\varphi) \\ &= 2\,Re\int_0^T e^{i\omega\eta}\big[-\varphi_{xx} + \big(\omega^2 - a - 2|\varphi|^2\big)\varphi + 2i\omega\varphi_x\big]\bar{h}\,dx \\ &\quad + \int_0^T \big[|h_x|^2 + \big(\omega^2 - a - 4|\varphi|^2\big)|h|^2 - 2i\omega h\bar{h}_x - 2\,Re\big(e^{-2i\omega\eta}\overline{\varphi}^2 h^2\big)\big]\,dx \\ &\quad - \int_0^T |h|^2\big(4\,Re\big(e^{i\omega\eta}\varphi\bar{h}\big) + |h|^2\big)\,dx \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Note that the boundary terms annihilate by periodicity. Using that $r(x)$ satisfies Eq. (3.2), we obtain $I_1 = 0$. Let

$$h = (h_1 + ih_2)e^{i\omega(x-\xi+(\omega^2+3a)t+\eta)},$$

where h_1 and h_2 are real periodic functions with period T. Then we have

$$\begin{gathered}|h|^2 = h_1^2 + h_2^2, \\ |h_x|^2 = (h_{1x} - \omega h_2)^2 + (h_{2x} + \omega h_1)^2, \\ \int_0^T \bar{h}_x h\,dx = i\int_0^T \big[h_2 h_{1x} - h_1 h_{2x} - \omega\big(h_1^2 + h_2^2\big)\big]\,dx, \\ Re\big(h^2\overline{\varphi}^2 e^{-2i\omega\eta}\big) = r^2\big(h_1^2 - h_2^2\big).\end{gathered}$$

Thus for I_2 we obtain the expression

$$I_2 = \int_0^T \big[h_{1x}^2 - \big(a + 6r^2\big)h_1^2\big]\,dx + \int_0^T \big[h_{2x}^2 - \big(a + 2r^2\big)h_2^2\big]\,dx = M_1 + M_2.$$

Introduce in $L^2[0,T]$ the self-adjoint operators L_1 and L_2 generated by the differential expressions

$$\begin{aligned}L_1 &= -\partial_x^2 - \big(a + 6r^2\big), \\ L_2 &= -\partial_x^2 - \big(a + 2r^2\big),\end{aligned}$$

with periodic boundary conditions in $[0,T]$.

4.2. Spectral analysis of the operators L_1 and L_2

Consider in $L^2[0,T]$ the following periodic eigenvalue problems

$$\begin{cases} L_1\psi = \lambda\psi & \text{in } [0,T], \\ \psi(0) = \psi(T), & \psi'(0) = \psi'(T), \end{cases} \tag{4.4}$$

$$\begin{cases} L_2\chi = \lambda\chi & \text{in } [0,T], \\ \chi(0) = \chi(T), & \chi'(0) = \chi'(T). \end{cases} \tag{4.5}$$

The problems (4.4) and (4.5) have each a countable infinite set of eigenvalues $\{\lambda_n\}$ with $\lambda_n \to \infty$. We shall denote by ψ_n, respectively by χ_n, the eigenfunction associated to the eigenvalue λ_n. For the periodic eigenvalue problems (4.4) and (4.5) there are associated semi-periodic eigenvalue problems in $[0,T]$, namely (e.g. for (4.5))

$$\begin{cases} L_2\vartheta = \mu\vartheta & \text{in } [0,T], \\ \vartheta(0) = -\vartheta(T), & \vartheta'(0) = -\vartheta'(T). \end{cases} \tag{4.6}$$

As in the periodic case, there is a countable infinity set of eigenvalues $\{\mu_n\}$. Denote by ϑ_n the eigenfunction associated to the eigenvalue μ_n. From the Oscillation Theorem [17] we know that $\lambda_0 < \mu_0 \leqslant \mu_1 < \lambda_1 \leqslant \lambda_2 < \mu_2 \leqslant \mu_3, \ldots, \lambda_0$ is simple and

(a) χ_0 has no zeros on $[0,T)$,
(b) χ_{2n+1} and χ_{2n+2} have exactly $2n+2$ zeros on $[0,T)$,
(c) ϑ_{2n} and ϑ_{2n+1} have exactly $2n+1$ zeros on $[0,T)$.

The intervals $(\lambda_0,\mu_0),(\mu_1,\lambda_1),\ldots$ are called intervals of stability and the intervals $(-\infty,\lambda_0)$, (μ_0,μ_1), $(\lambda_1,\lambda_2),\ldots$ are called intervals of instability.

We use now (3.5) and (3.6) to rewrite operators L_1, L_2 in more appropriate form. From the expression for $r(x)$ from (3.5) and the relations between elliptic functions $sn(x)$, $cn(x)$ and $dn(x)$, we obtain

$$L_1 = \alpha^2\big[-\partial_y^2 + 6k^2 sn^2(y) - 4 - k^2\big],$$

where $y = \alpha x$.

It is well known that the first five eigenvalues of $\Lambda_1 = -\partial_y^2 + 6k^2 sn^2(y,k)$, with periodic boundary conditions on $[0, 4K(k)]$, where $K(k)$ is the complete elliptic integral of the first kind, are simple. These eigenvalues and corresponding eigenfunctions are:

$$\begin{aligned}
&\nu_0 = 2 + 2k^2 - 2\sqrt{1-k^2+k^4}, && \phi_0(y) = 1 - \big(1 + k^2 - \sqrt{1-k^2+k^4}\big)sn^2(y,k), \\
&\nu_1 = 1 + k^2, && \phi_1(y) = cn(y,k)dn(y,k) = sn'(y,k), \\
&\nu_2 = 1 + 4k^2, && \phi_2(y) = sn(y,k)dn(y,k) = -cn'(y,k), \\
&\nu_3 = 4 + k^2, && \phi_3(y) = sn(y,k)cn(y,k) = -k^{-2}dn'(y,k), \\
&\nu_4 = 2 + 2k^2 + 2\sqrt{1-k^2+k^4}, && \phi_4(y) = 1 - \big(1 + k^2 + \sqrt{1-k^2+k^4}\big)sn^2(y,k).
\end{aligned}$$

It follows that the first three eigenvalues of the operator L_1, equipped with periodic boundary condition on $[0, 2K(k)]$ (that is, in the case of left and right family), are simple and $\lambda_0 = \alpha^2(\nu_0 - \nu_3) < 0$, $\lambda_1 = \alpha^2(\nu_3 - \nu_3) = 0$, $\lambda_2 = \alpha^2(\nu_4 - \nu_3) > 0$. The corresponding eigenfunctions are $\psi_0 = \phi_0(\alpha x)$, $\psi_1 = r'(x)$, $\psi_2 = \phi_4(\alpha x)$.

Similarly, for the operator L_2 we have

$$L_2 = \alpha^2\big[-\partial_y^2 + 2k^2 sn^2(y,k) - k^2\big]$$

in the case of left and right family. The spectrum of $\Lambda_2 = -\partial_y^2 + 2k^2 sn^2(y,k)$ is formed by bands $[k^2, 1] \cup [1+k^2, +\infty)$. The first three eigenvalues and the corresponding eigenfunctions with periodic boundary conditions on $[0, 4K(k)]$ are simple and

$$\begin{aligned} \epsilon_0 &= k^2, & \theta_0(y) &= dn(y,k), \\ \epsilon_1 &= 1, & \theta_1(y) &= cn(y,k), \\ \epsilon_2 &= 1+k^2, & \theta_2(y) &= sn(y,k). \end{aligned}$$

From (3.2) it follows that zero is an eigenvalue of L_2 and it is the first eigenvalue in the case of left and right family, with corresponding eigenfunction $r(x)$.

4.3. The estimate for M_2

Below, we will denote by $\langle f, g\rangle = \int_0^T f(x)g(x)\,dx$ and by $\|f\|$ the scalar product and the norm in $L^2[0,T]$. In the formulas that follow, we take $r = r(\bar{x})$ with an argument $\bar{x} = x - \xi + (a+3\omega)t$. From the previous section, we know that when considered in $[0,T]$, the operator L_2 has an eigenfunction r corresponding to zero eigenvalue and the rest of the spectrum is contained in (α^2, ∞).

The derivative of $d_q^2(u,\varphi)$ with respect to η at the point where the minimum is attained is equal to zero. Together with (3.2), this yields

$$\begin{aligned} 0 &= -i\omega \int_0^T \left[e^{i\omega\eta}\varphi_x \bar{h}_x - e^{-i\omega\eta}\overline{\varphi}_x h_x + q\left(e^{i\omega\eta}\varphi\bar{h} - e^{-i\omega\eta}\overline{\varphi}h\right)\right] dx \\ &= 2\omega\, Im \int_0^T (-\varphi_{xx} + q\varphi)e^{i\omega\eta}\bar{h}\,dx \\ &= 2\omega\, Im \int_0^T \left(\left(q+\omega^2+a+2r^2\right)r - 2i\omega r'\right)(h_1 - ih_2)\,dx \\ &= -2\omega \int_0^T \left[\left(q+\omega^2+a+2r^2\right)rh_2 + 2\omega r' h_1\right] dx. \end{aligned} \tag{4.7}$$

We set $h_2 = \beta r(\bar{x}) + \theta$, $\int_0^T \theta r\,dx = 0$. Substituting in (4.7), we obtain

$$0 = \beta\|r\|^2\left(q+\omega^2+a+\frac{2\|r^2\|^2}{\|r\|^2}\right) + 2\int_0^T \left(\theta r^3 + \omega r' h_1\right) dx.$$

Using that $\frac{2\|r^2\|^2}{\|r\|^2} \geqslant -a$ (see estimate A of Appendix A), we obtain the estimate

$$\begin{aligned} |\beta|\|r\| &\leqslant 2\frac{|\int_0^T (\theta r^3 + \omega r' h_1)\,dx|}{(q+\omega^2)\|r\|} \\ &\leqslant \frac{2\|r^3\|\cdot\|\theta\| + 2|\omega|\|r'\|\cdot\|h_1\|}{(q+\omega^2)\|r\|} \\ &\leqslant m_0\left(\|\theta\| + \|h_1\|\right), \end{aligned}$$

where $m_0 = 2m_1(a,\omega)/(q+\omega^2)$ and

$$m_1(a,\omega) = \max_{c\in[-\frac{1}{4}a^2,0]}\left(\frac{\|r^3\|}{\|r\|}, \frac{|\omega|\|r'\|}{\|r\|}, \frac{3\|r^2r'\|}{\|r'\|}, \frac{|\omega|\|ar+2r^3\|}{\|r'\|}\right)$$

(the third and fourth items are included for later use). It is obvious that the first three fractions are bounded. For the last one, see estimate D in Appendix A. We will use below that for a and ω fixed, $m_0 \to 0$ when $q \to \infty$. Further,

$$\|h_2\| \leqslant |\beta|\|r\| + \|\theta\| \leqslant m_0\big(\|\theta\| + \|h_1\|\big) + \|\theta\| = (m_0+1)\|\theta\| + m_0\|h_1\|.$$

Hence, we obtain

$$\|\theta\|^2 \geqslant \frac{\|h_2\|^2}{2(m_0+1)^2} - \left(\frac{m_0}{m_0+1}\right)^2\|h_1\|^2. \tag{4.8}$$

Since $L_2 r = 0$ and $\langle\theta, r\rangle = 0$, then from the spectral properties of the operator L_2, it follows

$$M_2 = \langle L_2 h_2, h_2\rangle = \langle L_2\theta,\theta\rangle \geqslant \alpha^2\langle\theta,\theta\rangle \geqslant -\frac{a}{2}\|\theta\|^2.$$

From here and (4.8), one obtains

$$M_2 \geqslant \frac{|a|}{4(m_0+1)^2}\|h_2\|^2 - \frac{|a|m_0^2}{2(m_0+1)^2}\|h_1\|^2. \tag{4.9}$$

4.4. *The estimate for M_1*

First of all, let us note that the operator L_1 equipped with periodic boundary conditions in $[0,T]$ has the following spectral data:

$$\begin{aligned}
&\lambda_0 = a - 2\sqrt{a^2+3c}, &&\psi_0 = 6r^2 + 3a - \lambda_0,\\
&\lambda_1 = 0, &&\psi_1 = r',\\
&\lambda_2 = a + 2\sqrt{a^2+3c}, &&\psi_2 = 6r^2 + 3a - \lambda_2,
\end{aligned} \tag{4.10}$$

and the rest of the spectrum is contained in (λ_2, ∞).

We set

$$h_1 = \gamma_1\psi_0(\bar{x}) + \gamma_2 r'(\bar{x}) + \theta_1, \qquad r(\bar{x}) = \nu\psi_0(\bar{x}) + \psi, \tag{4.11}$$

where

$$\langle\theta_1,\psi_0\rangle = \langle\theta_1, r'\rangle = \langle\psi,\psi_0\rangle = \langle\psi_0, r'\rangle = \langle\psi, r'\rangle = 0 \tag{4.12}$$

and γ_1, γ_2 and ν are some constants. By (4.12), we have

$$M_1(h_1) = \langle L_1 h_1, h_1\rangle = \gamma_1^2\lambda_0\langle\psi_0,\psi_0\rangle + \langle L_1\theta_1,\theta_1\rangle.$$

Therefore, from spectral properties of the operator L_1 it follows

$$M_1(h_1) \geqslant \gamma_1^2\lambda_0\|\psi_0\|^2 + \lambda_2\|\theta_1\|^2. \tag{4.13}$$

The fundamental difficulty in the estimate of M_1 is the appearance of the negative term $\gamma_1^2\lambda_0\|\psi_0\|^2$. Below, we are going to estimate it. From the condition

$$P(u)=\int_0^T \left|h+e^{i\omega\eta}\varphi(x-\xi,t)\right|^2 dx=P(\varphi)$$

we obtain

$$\|h\|^2=2\,Re\int_0^T e^{i\omega\eta}\varphi(x-\xi,t)\bar{h}\,dx=-2\int_0^T rh_1\,dx.$$

Then using (4.11), we have

$$-\frac{1}{2}\|h\|^2=\nu\gamma_1\|\psi_0\|^2+\int_0^T \psi\theta_1\,dx$$

and therefore

$$\gamma_1^2\|\psi_0\|^2=\frac{1}{\nu^2\|\psi_0\|^2}\left(\frac{1}{2}\|h\|^2+\int_0^T \psi\theta_1\,dx\right)^2. \tag{4.14}$$

From (4.14), we obtain

$$\gamma_1^2\|\psi_0\|^2\leqslant\frac{1}{\nu^2\|\psi_0\|^2}\left(\frac{1+d}{4}\|h\|^4+\frac{d+1}{d}\|\psi\|^2\|\theta_1\|^2\right), \tag{4.15}$$

where d is a positive constant which will be fixed later. Below, we will denote by C_m, D_m positive constants, depending only on d but not on the system parameters a, c, ω. Using (4.14) and (4.13), we derive the inequality

$$\begin{aligned}M_1&\geqslant\left(\lambda_2+\lambda_0\left(1+\frac{1}{d}\right)\frac{\|\psi\|^2}{\nu^2\|\psi_0\|^2}\right)\|\theta_1\|^2+\frac{\lambda_0(1+d)}{4\nu^2\|\psi_0\|^2}\|h\|^4\\&\geqslant C_1\lambda_2\|\theta_1\|^2-D_1|a|^{\frac{1}{2}}\|h\|^4\end{aligned} \tag{4.16}$$

(see the estimates in point C of Appendix A).

We denote $\vartheta=h_1-\gamma_2 r'(\bar{x})=\gamma_1\psi_0(\bar{x})+\theta_1$. Then from (4.12), (4.15) and the inequalities $\lambda_2\leqslant\frac{1}{3}|\lambda_0|\leqslant|a|$, we have

$$\begin{aligned}\|\vartheta\|^2=\gamma_1^2\|\psi_0\|^2+\|\theta_1\|^2&\leqslant\left(1+\frac{(d+1)\|\psi\|^2}{d\nu^2\|\psi_0\|^2}\right)\|\theta_1\|^2+\frac{1+d}{4\nu^2\|\psi_0\|^2}\|h\|^4\\&\leqslant C_2\|\theta_1\|^2+D_2|a|^{-\frac{1}{2}}\|h\|^4.\end{aligned}$$

Then

$$\|\theta_1\|^2\geqslant\frac{\|\vartheta\|^2}{C_2}-\frac{D_2\|h\|^4}{C_2|a|^{\frac{1}{2}}}$$

and hence, by (4.16) and $\lambda_2 \leqslant |a|$,

$$M_1 \geqslant \frac{C_1\lambda_2}{C_2}\|\vartheta\|^2 - \frac{C_1D_2 + C_2D_1}{C_2}|a|^{\frac{1}{2}}\|h\|^4$$
$$= C_3\lambda_2\|\vartheta\|^2 - D_3|a|^{\frac{1}{2}}\|h\|^4. \tag{4.17}$$

After differentiating (4.2) with respect to ξ, we obtain

$$0 = 2\,Re\int_0^T e^{i\omega\eta}(\varphi_{xx}\bar{h}_x + q\varphi_x\bar{h})\,dx = 2\,Re\int_0^T e^{i\omega\eta}\big[\varphi_t + \big(6|\varphi|^2 + q\big)\varphi_x\big]\bar{h}\,dx$$
$$= 2\,Re\int_0^T (h_1 - ih_2)\big[i\omega\big(3a + \omega^2 + 6r^2 + q\big)r + \big(a + 3\omega^2 + 6r^2 + q\big)r'\big]\,dx$$
$$= 2\int_0^T \big[\big(a + 3\omega^2 + 6r^2 + q\big)r'h_1 + \omega\big(3a + \omega^2 + 6r^2 + q\big)rh_2\big]\,dx.$$

From (4.7), we have

$$\int_0^T qrh_2\,dx = -\int_0^T \big[2\omega r'h_1 + \big(a + \omega^2 + 2r^2\big)rh_2\big]\,dx$$

and replacing in the above equality, we obtain

$$\int_0^T \big[\big(a + \omega^2 + 6r^2 + q\big)r'h_1 + \omega\big(2a + 4r^2\big)rh_2\big]\,dx = 0.$$

Substituting $h_1 = \gamma_2 r'(\bar{x}) + \vartheta$ in the above equality and using the orthogonality condition $\langle r', \vartheta\rangle = \langle r', \gamma_1\psi_0 + \theta_1\rangle = 0$, we obtain

$$\gamma_2\|r'\|^2\left(a + \omega^2 + q + \frac{6\|rr'\|^2}{\|r'\|^2}\right) + 2\int_0^T \big[\omega\big(a + 2r^2\big)rh_2 + 3r^2r'\vartheta\big]\,dx = 0.$$

Using that $\frac{6\|rr'\|^2}{\|r'\|^2} \geqslant -a$ (see Appendix A), we further have

$$|\gamma_2|\|r'\| \leqslant \frac{2|\int_0^T [\omega(a + 2r^2)rh_2 + 3r^2r'\vartheta]\,dx|}{(\omega^2 + q)\|r'\|}$$
$$\leqslant 2\frac{|\omega|\|ar + 2r^3\| \cdot \|h_2\| + 3\|r^2r'\| \cdot \|\vartheta\|}{(\omega^2 + q)\|r'\|}$$
$$\leqslant m_0\big(\|\vartheta\| + \|h_2\|\big).$$

Hence

$$\|h_1\| \leqslant |\gamma_2| \|r'\| + \|\vartheta\| \leqslant (m_0+1)\|\vartheta\| + m_0\|h_2\|,$$

which yields

$$\|\vartheta\|^2 \geqslant \frac{\|h_1\|^2}{2(m_0+1)^2} - \left(\frac{m_0}{m_0+1}\right)^2 \|h_2\|^2.$$

Replacing in (4.17), we finally obtain

$$M_1 \geqslant \frac{C_3\lambda_2}{2(m_0+1)^2}\|h_1\|^2 - \frac{C_3\lambda_2 m_0^2}{(m_0+1)^2}\|h_2\|^2 - D_3|a|^{\frac{1}{2}}|h\|^4. \tag{4.18}$$

4.5. The estimate for $\triangle M$

From (4.9) and (4.18), we have

$$M_1 + M_2 \geqslant \frac{C_3\lambda_2 - |a|m_0^2}{2(m_0+1)^2}\|h_1\|^2 + \frac{|a| - 4C_3\lambda_2 m_0^2}{4(m_0+1)^2}\|h_2\|^2 - D_3|a|^{\frac{1}{2}}\|h\|^4.$$

We now fix q so that $|a|m_0^2 \leqslant \frac{1}{2}C_3\lambda_2$ and assuming that $C_3 \leqslant \frac{1}{2}$ (which is no loss of generality), one has also $4C_3\lambda_2 m_0^2 \leqslant \frac{1}{2}|a|$. Therefore we obtain

$$M_1 + M_2 \geqslant C_4\lambda_2\big(\|h_1\|^2 + \|h_2\|^2\big) - D_3|a|^{\frac{1}{2}}\|h\|^4 = C_4\lambda_2\|h\|^2 - D_3|a|^{\frac{1}{2}}\|h\|^4,$$

where C_4 and D_3 are absolute constants independent of the parameters of the system.

On the other hand, estimating directly I_2 from below (for this purpose we use its initial formula), we have

$$\begin{aligned} I_2 &\geqslant \|h_x\|^2 + \int_0^T \big(\omega^2 - a - 4r^2\big)|h|^2\,dx - 2|\omega| \int_0^T |h|\cdot|h_x|\,dx - 2\int_0^T r^2|h|^2\,dx \\ &\geqslant \|h_x\|^2 + \big(\omega^2 - a + 4a\big)\|h\|^2 - 2\omega^2\|h\|^2 - \frac{1}{2}\|h_x\|^2 + 2a\|h\|^2 \\ &= \frac{1}{2}\|h_x\|^2 - \big(\omega^2 - 5a\big)\|h\|^2. \end{aligned}$$

Similarly, $|I_3| \leqslant \max(4|a|^{\frac{1}{2}}|h| + |h|^2)\|h\|^2$. Let $0 < m < \frac{1}{2}$. We have

$$\begin{aligned} \triangle M &= 2mI_2 + (1-2m)(M_1+M_2) + I_3 \\ &\geqslant m\|h_x\|^2 - 2m\big(\omega^2 - 5a\big)\|h\|^2 + (1-2m)\big(C_4\lambda_2\|h\|^2 - D_3|a|^{\frac{1}{2}}\|h\|^4\big) \\ &\quad - \max\big(4|a|^{\frac{1}{2}}|h| + |h|^2\big)\|h\|^2 \\ &= m\|h_x\|^2 + \big[-2m\big(\omega^2 - 5a\big) + (1-2m)C_4\lambda_2\big]\|h\|^2 \\ &\quad - \big[\max\big(4|a|^{\frac{1}{2}}|h| + |h|^2\big) + (1-2m)D_3|a|^{\frac{1}{2}}\|h\|^2\big]\|h\|^2. \end{aligned}$$

We choose m, so that

$$2qm = (1-2m)C_4\lambda_2 - 2m\big(\omega^2 - 5a\big), \quad \text{i.e.} \quad 2m = \frac{C_4\lambda_2}{q+\omega^2+5|a|+C_4\lambda_2} < 1.$$

From the inequality

$$|h|^2 \leqslant \frac{1}{T}\int_0^T |h|^2\,dx + 2\left(\int_0^T |h|^2\,dx \int_0^T |h_x|^2\,dx\right)^{\frac{1}{2}}$$

we obtain

$$|h|^2 \leqslant \frac{1}{T}\int_0^T |h|^2\,dx + \sqrt{q}\int_0^T |h|^2\,dx + \frac{1}{\sqrt{q}}\int_0^T |h_x|^2\,dx.$$

Hence for sufficiently large q, we obtain

$$\max\big|h(x,t)\big|^2 \leqslant \frac{2}{\sqrt{q}} d_q^2(u,\varphi)$$

and moreover $\|h\|^2 \leqslant q^{-1}d_q^2(u,\varphi)$. Consequently we can choose $\delta_0 > 0$, such that for $d_q(u,\varphi) < \delta_0$, we will have $[\max(4|a|^{\frac{1}{2}}|h| + |h|^2) + (1-2m)D_3|a|^{\frac{1}{2}}]\|h\|^2 \leqslant qm$.

Finally, we obtain that if $d_q(u,\varphi) < \delta_0$, then $\Delta M \geqslant m d_q^2(u,\varphi)$. Proposition 4.1 is completely proved.

4.6. *Proof of Theorem 4.1*

We split the proof of our main result into two steps. We begin with the special case $P(u) = P(\varphi)$. Assume that m, q, δ_0 have been selected according to Proposition 4.1. Since ΔM does not depend on t, $t \in [0,\infty)$, there exists a constant l such that $\Delta M \leqslant l d^2(u,\varphi)|_{t=0}$. Below, we shall assume without loss of generality that $l \geqslant 1$, $q \geqslant 1$.

Let

$$\varepsilon > 0, \qquad \delta = \min\left(\left(\frac{m}{lq}\right)\frac{\delta_0}{2}, \left(\frac{m}{l}\right)^{1/2}\varepsilon\right)$$

and $d(u,\varphi)|_{t=0} < \delta$. Then

$$d_q(u,\varphi) \leqslant q^{1/2} d(u,\varphi)|_{t=0} < \frac{\delta_0}{2}$$

and Lemma 4.1 yields that there exists a $t_0 > 0$ such that $d_q(u,\varphi) < \delta_0$ if $t \in [0,t_0)$. Then, by virtue of Proposition 4.1 we have

$$\Delta M \geqslant m d_q^2(u,\varphi), \quad t \in [0,t_0).$$

Let t_{max} be the largest value such that

$$\Delta M \geqslant m d_q^2(u,\varphi), \quad t \in [0,t_{max}).$$

We assume that $t_{max} < \infty$. Then, for $t \in [0, t_{max}]$ we have

$$d_q^2(u,\varphi) \leqslant \frac{\Delta M}{m} \leqslant \frac{l}{m} d^2(u,\varphi)|_{t=0} < \frac{l}{m}\delta^2 \leqslant \frac{\delta_0^2}{4}.$$

Applying once again Lemma 4.1, we obtain that there exists $t_1 > t_{max}$ such that

$$d_q(u,\varphi) < \delta_0, \quad t \in [0, t_1).$$

By virtue of the proposition, this contradicts the assumption $t_{max} < \infty$. Consequently, $t_{max} = \infty$,

$$\Delta M \geqslant m d_q^2(u,\varphi) \geqslant m d^2(u,\varphi), \quad t \in [0,\infty).$$

Therefore,

$$d^2(u,\varphi) \leqslant \frac{\Delta M}{m} \leqslant \frac{l}{m}\delta^2 < \varepsilon^2, \quad t \in [0,\infty),$$

which proves the theorem in the special case.

Now we proceed to remove the restriction $P(u) = \|u\|^2 = \|\varphi\|^2 = P(\varphi)$. We have (see (A.6)) $\|\varphi\| = (2r_0 E(k))^{1/2}$, where r_0 is given by (3.5). Below, we are going to apply a perturbation argument, freezing for a while the period T and the parameters a, c in (3.4). We claim there are respective parameter values a^*, c^*, and corresponding φ^*, r^*, r_0^*, k^*, see (3.3), (3.4) and (3.5), such that φ^* has a period T in x and moreover, $2r_0^* E(k^*) = \|u\|^2$. By (3.5), we obtain the equations

$$\begin{aligned} &\frac{2K(k^*)}{r_0^*} - T = 0, \\ &2r_0^* E\big(k^*\big) - \|u\|^2 = 0. \end{aligned} \tag{4.19}$$

If (4.19) has a solution $k^* = k^*(T, \|u\|)$, $r_0^* = r_0^*(T, \|u\|)$, then the parameter values we need are given by

$$a^* = \big(k^{*2} - 2\big) r_0^{*2}, \qquad c^* = \big(k^{*2} - 1\big) r_0^{*4}.$$

Moreover, one has $\|\varphi^*\| = \|u\|$ and we could use the restricted result we established above. As $k = k^*(T, \|\varphi\|)$, $r_0 = r_0^*(T, \|\varphi\|)$, it remains to apply the implicit function theorem to (4.19). Since the corresponding functional determinant reads

$$\begin{vmatrix} \frac{2K'(k^*)}{r_0^*} & -\frac{2K(k^*)}{r_0^{*2}} \\ 2r_0^* E'(k^*) & 2E(k^*) \end{vmatrix} = \frac{4}{r_0^*}(KE)' = \frac{4}{r_0^*}\left(\frac{1}{2}\pi + KK'\right) > 0$$

(by Legendre's identity), the existence of a^* and c^* with the needed properties is established.

By (4.19) and our assumption, we have

$$\frac{K(k^*)}{r_0^*} = \frac{K(k)}{r_0} = \frac{T}{2}. \tag{4.20}$$

Next, choosing $\eta = 2(a^* - a)t$, $\xi = (a - a^*)t$, by (3.3) and (4.1) one easily obtains the inequality

$$d^2\big(\varphi^*, \varphi\big) \leqslant \big(1 + \omega^2\big)\big\|r^* - r\big\|^2 + \big\|r^{*\prime} - r'\big\|^2.$$

Denote for a while $\Phi(\rho) = \rho\, dn(z\rho; k(\rho)) = \rho\, dn(y; k)$ where $k = k(\rho)$ is determined from $K(k) = \frac{1}{2}\rho T$. Then using (3.5), we have $r^* - r = \Phi(r_0^*) - \Phi(r_0) = (r_0^* - r_0)\Phi'(\rho)$ with some appropriate ρ. Moreover,

$$\Phi'(\rho) = dn(y; k) + \rho\left[z\frac{\partial dn}{\partial y}(y; k) + \frac{T}{2K'(k)}\frac{\partial dn}{\partial k}(y; k)\right]$$

satisfies $|\Phi'(\rho)| \leqslant C_0$ with a constant C_0 independent of the values with $*$ accent. Hence, $|r^* - r| \leqslant C_0|r_0^* - r_0|$. Similarly, $|r^{*\prime} - r'| \leqslant C_1|r_0^* - r_0|$. All this, together with (4.20) yields

$$d\big(\varphi^*, \varphi\big) \leqslant C\big|r_0^* - r_0\big| = \frac{2C}{T}\big|K\big(k^*\big) - K(k)\big| = \frac{2C}{T}\big|K'(\kappa)\big|\,\big|k^* - k\big|. \tag{4.21}$$

From the inequalities

$$\big|\big\|\varphi^*\big\| - \|\varphi\|\big| = \big|\|u\| - \|\varphi\|\big| \leqslant d(u, \varphi)|_{t=0} < \delta$$

it follows that

$$-\big(2r_0E(k)\big)^{-1/2}\delta < \big(\|\varphi\|\big)^{-1}\big\|\varphi^*\big\| - 1 < \big(2r_0E(k)\big)^{-1/2}\delta$$

and, therefore, $1 - \delta_1 < \frac{r_0^*E(k^*)}{r_0E(k)} < 1 + \delta_1$, i.e. $|r_0^*E(k^*) - r_0E(k)| < r_0E(k)\delta_1$, where we have denoted $\delta_1 = (1 + (2r_0E(k))^{-1/2}\delta)^2 - 1$.

On the other hand, we have (using (4.20) again)

$$\big|r_0^*E\big(k^*\big) - r_0E(k)\big| = \frac{2}{T}\big|K\big(k^*\big)E\big(k^*\big) - K(k)E(k)\big| = \frac{2}{T}\big|(KE)'(\kappa)\big|\big|k^* - k\big| \geqslant C_2\big|k^* - k\big|, \tag{4.22}$$

with appropriate $C_2 > 0$ independent of the values bearing $*$ accent. In particular, one has $|k^* - k| \leqslant C_3r_0E(k)\delta_1$. Thus combining (4.21) and (4.22), we get

$$d\big(u, \varphi^*\big)\big|_{t=0} \leqslant d(u, \varphi)|_{t=0} + d\big(\varphi, \varphi^*\big)\big|_{t=0} < \delta + Cr_0E(k)\delta_1 = \delta_0.$$

Let $\varepsilon > 0$. We select δ (and together, δ_0 and δ_1) sufficiently small and apply the part of the theorem which has already been proved to conclude:

$$d\big(u, \varphi^*\big)\big|_{t=0} < \delta_0 \quad \Rightarrow \quad d\big(u, \varphi^*\big) < \frac{\varepsilon}{2}, \quad t \in [0, \infty).$$

Then, choosing an appropriate $\delta > 0$, we obtain that

$$d(u, \varphi) \leqslant d\big(u, \varphi^*\big) + d\big(\varphi, \varphi^*\big) < \frac{\varepsilon}{2} + Cr_0E(k)\delta_1 < \varepsilon$$

for all $t \in [0, \infty)$. The theorem is completely proved.

Acknowledgments

We would like to thank the referee for several relevant suggestions. The first author has been partially supported by Scientific Research Grant RD-05-291/09 of Shumen University. The third author is partially supported by Grant DO 02-360/2008 of the Bulgarian Ministry of Education, Youth and Science.

Appendix A

For $n \in \mathbb{Z}$ and $c \in (-\frac{1}{4}a^2, 0)$, consider the line integrals $I_n(c)$ and their derivatives $I_n'(c)$ given by

$$I_n(c) = \oint_{H=c} x^n y\,dx, \qquad I_n'(c) = \oint_{H=c} \frac{x^n\,dx}{2y}, \tag{A.1}$$

where one can assume for definiteness that the integration is along the right oval contained in the level set $\{H = c\}$. These integrals would be useful because

$$\int_0^T r^n(t)\,dt = 2\int_0^{\frac{1}{2}T} r^n(t)\,dt = 2\int_{r_1}^{r_0} \frac{x^n\,dx}{\sqrt{c - ax^2 - x^4}} = \oint_{H=c} \frac{x^n\,dx}{y} = 2I_n'(c) \tag{A.2}$$

(we applied a change of the variable $r(t) = x$ in the integral and used Eq. (3.4)). The properties of I_n are well known, see, e.g., [11] for a recent treatment. Below, we list some facts we are going to use.

Lemma. (i) *The following identity holds*:

$$(n+6)I_{n+3} + (n+3)aI_{n+1} - ncI_{n-1} = 0, \quad n \in \mathbb{Z},$$

which implies

$$I_3' = -\frac{1}{2}aI_1', \qquad I_4' = \frac{1}{3}cI_0' - \frac{2}{3}aI_2', \qquad I_6' = -\frac{4}{15}acI_0' + \left(\frac{8}{15}a^2 + \frac{3}{5}c\right)I_2'. \tag{A.3}$$

(ii) *The integrals I_0 and I_2 satisfy the system*

$$4cI_0' - 2aI_2' = 3I_0,$$
$$-2acI_0' + \left(12c + 4a^2\right)I_2' = 15I_2.$$

(iii) *The ratio $R(c) = I_2'(c)/I_0'(c)$ satisfies the Riccati equation and related system*

$$\left(8c^2 + 2a^2c\right)R'(c) = ac + 4cR(c) - aR^2(c), \qquad \begin{aligned}\dot{c} &= 8c^2 + 2a^2c,\\ \dot{R} &= ac + 4cR - aR^2,\end{aligned} \tag{A.4}$$

which imply estimates

$$\frac{2c}{a} \leqslant R(c) \leqslant \frac{c}{2a} - \frac{3a}{8}. \tag{A.5}$$

The equations in (i)–(iii) are derived in a standard way, see [11] for more details. The estimates (A.5) follow from the fact that, in the (c, R)-plane, the graph of $R(c)$ coincides with the concave separatrix trajectory of the system (A.4) contained in the triangle with vertices $(0, 0)$, $(-\frac{1}{4}a^2, -\frac{1}{2}a)$ and $(0, -\frac{3}{8}a)$ and connecting the first two of them.

After this preparation, we turn to prove the estimates we used in the preceding sections.

A. The estimate for $A = \frac{2\|r^2\|^2}{\|r\|^2}$. By (A.2), (A.3) and the first inequality in (A.5), we have

$$A = \frac{2\int_0^T r^4\,dt}{\int_0^T r^2\,dt} = \frac{2I_4'}{I_2'} = \frac{2cI_0' - 4aI_2'}{3I_2'} = \frac{2c}{3}\frac{1}{R} - \frac{4a}{3} \geqslant -a.$$

B. The estimate for $B = \frac{6\|rr'\|^2}{\|r'\|^2}$. By (3.4), we have as above

$$\begin{aligned} B &= \frac{6\int_0^T r^2(c - ar^2 - r^4)\,dt}{\int_0^T (c - ar^2 - r^4)\,dt} = \frac{6(cI_2' - aI_4' - I_6')}{cI_0' - aI_2' - I_4'} \\ &= \frac{6}{5}\frac{(2a^2 + 6c)I_2' - acI_0'}{2cI_0' - aI_2'} = \frac{6}{5}\frac{(2a^2 + 6c)R - ac}{2c - aR} \geqslant -\frac{12}{5}a. \end{aligned}$$

To obtain the last inequality, we used that $4c + a^2 \geqslant 0$ and the second estimate in (A.5).

C. The estimate for $C = \lambda_2 + \lambda_0(1 + \frac{1}{d})\frac{\|\psi\|^2}{\nu^2\|\psi_0\|^2}$. By (4.11) and (4.12) we have

$$\|\psi\|^2 = \|r\|^2 - \nu^2\|\psi_0\|^2, \quad \text{where } \nu = \frac{\langle r, \psi_0\rangle}{\|\psi_0\|^2}.$$

Therefore

$$C = \lambda_2 + \lambda_0\left(1 + \frac{1}{d}\right)\left(\frac{\|r\|^2\|\psi_0\|^2}{\langle r, \psi_0\rangle^2} - 1\right).$$

Next,

$$\langle r, \psi_0\rangle = \int_0^T \left[6r^3 + (3a - \lambda_0)r\right]dt = 12I_3' + (6a - 2\lambda_0)I_1' = -2\lambda_0 I_1',$$

$$\begin{aligned} \|r\|^2\|\psi_0\|^2 &= \int_0^T r^2\,dt \int_0^T \left(6r^2 + 3a - \lambda_0\right)^2 dt \\ &= 4I_2'\left[36I_4' + 12(3a - \lambda_0)I_2' + (3a - \lambda_0)^2 I_0'\right] \\ &= 4I_2'\left[\left(12c + (3a - \lambda_0)^2\right)I_0' + (12a - 12\lambda_0)I_2'\right]. \end{aligned}$$

By (3.5), we have

$$I_2'(c) = \frac{1}{2}\int_0^T r^2\,dt = r_0\int_0^{K(k)} dn^2(t)\,dt = r_0 E(k). \tag{A.6}$$

Making use of the identity $E(k) = \frac{1}{2}\pi F(\frac{1}{2}, -\frac{1}{2}, 1, k^2)$ where F is the Gauss hypergeometric function, we obtain an appropriate expansion to estimate E from above

$$E(k) = \frac{\pi}{2}\left(1 - \frac{k^2}{4} - \frac{3k^4}{64} - \frac{5k^6}{512} - \cdots,\right), \qquad E^2(k) \leqslant \frac{\pi^2}{4}\left(1 - \frac{k^2}{2} - \frac{k^4}{32}\right)$$

with all removed terms negative. As $I_1' = \frac{1}{2}\pi$, by (3.5) this implies

$$I_2'^2 \leqslant -I_1'^2 \frac{a^2 + 20ar_0^2 + 4r_0^4}{32r_0^2}.$$

Together with $I_0' I_2' \geqslant I_1'^2$, this yields

$$\begin{aligned}\frac{\|r\|^2\|\psi_0\|^2}{\langle r, \psi_0\rangle^2} - 1 &\leqslant \frac{1}{\lambda_0^2}\left[12c + (3a - \lambda_0)^2 + \frac{3}{8r_0^2}(\lambda_0 - a)\left(a^2 + 20ar_0^2 + 4r_0^4\right)\right] - 1 \\ &= \frac{\lambda_2}{\lambda_0}\left(\frac{\lambda_2 - a}{8r_0^2} - 1\right) \leqslant \frac{\lambda_2}{\lambda_0}\left(\frac{\sqrt{3}}{8} - 1\right),\end{aligned}$$

where the equality is obtained by direct calculations. Therefore,

$$C \geqslant \lambda_2\left(-\frac{1}{d} + \frac{d+1}{d}\frac{\sqrt{3}}{8}\right) = C_1\lambda_2$$

with $C_1 > 0$ an absolute constant when $d \geqslant 4$ is fixed.

As a by-product of our calculations, we easily obtain also the estimate

$$\frac{\lambda_0}{\nu^2\|\psi_0\|^2} = \frac{\lambda_0\|\psi_0\|^2}{\langle r, \psi_0\rangle^2} \geqslant \frac{\lambda_2(\frac{\sqrt{3}}{8} - 1) + \lambda_0}{\|r\|^2} \geqslant -D_1|a|^{\frac{1}{2}}.$$

D. The estimate for $D = \frac{\|ar + 2r^3\|}{\|r'\|}$. Making use of statements (i) and (ii) of the Lemma, we have

$$\begin{aligned}D^2 &= \frac{\int_0^T (a^2r^2 + 4ar^4 + 4r^6)\,dt}{\int_0^T (c - ar^2 - r^4)\,dt} = \frac{a^2 I_2' + 4aI_4' + 4I_6'}{cI_0' - aI_2' - I_4'} \\ &= \frac{4acI_0' + (7a^2 + 36c)I_2'}{5(2cI_0' - aI_2')} = \frac{aI_0 + 6I_2}{I_0} \leqslant -5a.\end{aligned}$$

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