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Reliable compartmental models for double-pipe heat exchangers: An analytical study

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Abstract

In this work, the analytical properties of the heat exchanger infinite-dimensional dynamic model are discussed. More importantly, those of a 2nd-order lumped-parameter model using the *logarithmic mean temperature difference* (LMTD) as *driving force* are derived and shown to agree with those of the former. Three essential aspects are focused: existence and uniqueness of solutions, equilibrium states, and stability properties. The results developed in this work are intended to supply a solid support for the reliability on the use of the kind of simple compartmental model that is treated. This is specially addressed to works where it is not the quantitative solutions but the qualitative behavior that is important, like modelling and simulation of heat exchanger networks and complex industrial processes where heat exchangers are involved.

Keywords: Heat exchangers; Distributed/lumped-parameter model; Dynamic/analytical properties; Stability; Logarithmic mean temperature difference

1. Introduction

Heat exchangers are widely used in industrial processes such as power plants [1], gas turbines [2], air conditioning [3], refrigeration [4], (domestic, urban, or central) heating [5], and cryogenic systems [6], among many others. Their *universal* application has conducted to the research for a better comprehension of their dynamic behavior, modelling, simulation, identification, and control since the 1940s (with a boom in the 50s and 60s) [7,8]. Nevertheless, given their extremely complex dynamics[1] and the increasing demands imposed to the operation requirements of current industrial processes, they are still the subject of many studies under the above mentioned frames. However, since they are generally part of a complex system or (heat exchanger) network, dynamic analysis based on simplifying but acceptable assumptions and/or simple but suitable models are desirable [10–14].

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[1] See for instance [9, Section 1], where a large list of phenomena involved in their dynamic behavior (rendering it complex) is given.

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The dynamics of heat exchangers is represented in two main ways: through *distributed*- and *lumped-parameter models* [8,15–17]. Since variations of the states that are concerned take place not only in time but also in space, distributed-parameter models are those that best fit to the nature of heat exchangers [7]. Such models are represented by a set of partial differential equations. Since these are in general difficult to analyze, complicated for simulation, and complex for control synthesis, approximations through lumped-parameter models are generally preferable (at least for such purposes) [14].

Lumped-parameter models have been extensively used for dynamic simulation [16,18–21], control design [22–25], network modelling [12,15,26,27], or parameter identification [28–30], for instance. They are constructed considering the division of the whole exchanger in a finite number of elements (lumps) or cells, permitting its dynamic representation through a set of ordinary differential equations. Such a lumping procedure generally assumes that every element behaves like a perfectly stirred tank (or well-mixed compartment) [8,10,11,16,18,26]. Consequently, the fluid temperature is considered to hold throughout each of the cells. Its spatial distribution at every lump is therefore neglected. Only jumps in its values are considered at points delimiting the elements (when more than one are considered). As a consequence, a large number of cells is in general required for an acceptable modelling (approaching the distributed characteristic) [14]. This gives rise to high order models that complicate the dynamic analysis and control design [21,25]. However, there is a special low-order model that has been concluded to be—and actually used as—a reliable representation of the dynamics of double-pipe heat exchangers [10,11,22]. But an analytical study supporting such an assertion is still missing in the literature. This is what constitutes the main motivation of the present work. Our goal is to develop a formal study that contributes concrete analytical results that justify the above mentioned reliability on the use of such a special low-order model.

1.1. Review of previous works

Reduction of the lumped-parameter model order with diminutive accuracy loss may be contemplated by taking into account the temperature distribution at each lump instead of assuming a perfect mix of the fluid. This idea was explored for instance in [10,11], where a comparative numerical study among three different one-cell bi-compartmental models (one element per fluid covering the whole tube length) is proposed. Steady-state characteristics as well as outlet-temperature step and frequency responses (of one of the fluids) to inlet-temperature and flow disturbances (at the other fluid) were compared to those of the distributed dynamics of counterflow exchangers. Each of the considered compartmental models included two first-order (coupled) ordinary differential equations, one per compartment (fluid), which is the simplest modelling case. The difference among them consisted on the respective consideration of a uniform (perfect mix), linear, and exponential temperature distribution for the selection of the heat exchange *driving force* expression. For each of those cases, such a term was respectively expressed in terms of the outlet temperatures difference, the *arithmetic mean temperature difference* (AMTD), and the *logarithmic mean temperature difference* (LMTD). The results showed that the model using the LMTD as driving force is the one that best approaches the distributed dynamics, while that assuming perfect mix (uniform distribution) is the worst. Moreover, it is concluded that a one-cell bi-compartmental (2nd-order) model with *LMTD-driving-force* keeps the dynamic properties of the distributed one. The consideration of more than one bi-compartmental cells was though suggested for stringent quantitative modelling requirements.

Lumped-parameter models with LMTD-driving-force have been shown to be appropriate in several other works. In [19], for instance, such dynamic representations were tested through simulation considering 2–10 compartments. It was concluded that a two-compartment model indeed keeps the qualitative behavior of the distributed dynamics. The consideration of more than two compartments was though reported to be convenient to improve accuracy in steady-state characteristics and achieve sufficient transport time lag. Similar tests were done and conclusions drawn in [20], where quantitative adjustments from two to five bi-compartmental cells were even found to be slight. Superiority of low-order compartmental models using the LMTD as driving force (compared to those using the AMTD or the outlet temperature difference) was also corroborated (through similar tests) in [27], where they were concluded to be advantageous for simulation of heat exchanger networks.

Lumped-parameter models with LMTD-driving-force have also been considered by Alsop and Edgar in [22] where a 2nd-order model was used for control synthesis on a counterflow exchanger. The closed-loop scheme was tested through simulation using a 20 bi-compartmental cell (40th-order) model. Temperature responses to flow step changes were shown using both (the 2nd- and the 40th-order) models; small differences were observed among them. It is worth noting that Alsop and Edgar went further in the low-order modelling by taking the bulk temperature to be the average among the inlet and outlet temperatures at each fluid (for the accumulation term), instead of just the outlet temperature. Since constant inlet temperatures were considered, the time derivative of the bulk (average) temperature ended up in half the time derivative of the outlet one. Consequently, the model used in [22] keeps the same traditional structure of the previous works but doubles the convective and conduction heat transfer rates (*i.e.*, a 2 appears multiplying every right-hand-side term of the dynamic equations). This, on the one hand, has no effect in the steady-state values and, on the other, speeds up the system dynamics improving the response time (getting closer to that of the distributed-parameter model trajectories).

Based on the above mentioned studies, low-order lumped-parameter models with LMTD-driving-force have been used by several authors as reliable dynamic representations of heat exchangers. For instance, they were used in [23] and [24] for control design [31] and [32] for stability limit closed-loop analysis [19] and [20] for the development of dynamic simulators, and [12] for heat exchanger network modelling and simulation. However, the fact that such representations keep the dynamic properties of the distributed-parameter models was concluded in the previous works [10,11,19,20,22,27], from numerical results through simulation. An exhaustive study showing such a fact under an analytical framework is still missing in the literature.

1.2. The contribution of this work

In this work, we aim at filling in the gap that was just mentioned above: under specific assumptions generally made for the derivation of the heat exchanger infinite-dimensional dynamic equations, analytical proofs are developed showing that a 2nd-order compartmental model with the LMTD as driving force keeps the main dynamic properties of the distributed-parameter model it is approached from. Three essential aspects are focused. First, existence and uniqueness of solutions are shown to be a common characteristic of both dynamic models. Then, equilibrium solutions of the finite-dimensional dynamics are obtained and shown to accurately agree with the outlet values of the temperature equilibrium profiles of the distributed model. Furthermore, such equilibrium states are concluded to be exponentially stable and globally attractive on the system state-space domain in both dynamic representations. These results formally state the qualitative behavior of heat exchangers under the stated assumptions. From such a point of view, they further bring to the fore the exact analogy of the simple compartmental model (considered in this work) with the original distributed dynamics it is approached from.

Other studies on the dynamic behavior of double-pipe heat exchangers that are found in the literature propose a different perspective. For instance, frequency analysis through transfer functions were developed in [33–35]. But such approaches implicitly assume a linear behavior which is suitable only locally. On the contrary, the analysis developed in this work is based on a non-linear model that captures the whole global essential phenomena. Other works derived exact (space-time) solutions for special cases [36] or characterized them *via* transient response analysis [37]. But these approaches focus on quantitative system responses. Fundamental properties (like existence and uniqueness of solutions) are bypassed and qualitative aspects (like equilibrium stability and convergence of solutions) are omitted. This work, on the contrary, focuses on fundamental and qualitative aspects rather than quantitative ones. In [38], on the other hand, the effect of data uncertainty on the performance of a concentric tube heat exchanger was studied. But this study was carried out through stochastic analysis and only considered the steady-state response of the distributed-parameter model. This work, on the contrary, focuses on the system dynamics studied within a deterministic model analysis framework. Furthermore, qualitative analyses of heat exchangers may be found in [39–41]. But they are developed within an infinite-dimensional dynamic systems context, since it is the distributed-parameter model that is analyzed in those works. In this work, on the contrary, it is a simple 2nd-order compartmental model that is proved to accurately capture the qualitative behavior of the former. This was intended to be done in [10,11], but such studies were carried out through numerical simulations.

On the contrary, this work develops the formal proofs within a framework suitable for the qualitative analysis of dynamic systems.

The work is organized as follows: Section 2 states the notation used throughout the paper. In Section 3, the distributed-parameter model is presented and its analytical properties are discussed. Section 4 presents the compartmental model, proves its dynamic properties, and brings to the fore their accurate agreement with those of the distributed dynamics. Conclusions are given in Section 5.

2. Nomenclature and notation

The following nomenclature is defined for its use throughout this work:

F	mass flow rate
C_p	specific heat
M	total mass inside the tube
υ	fluid velocity
l	total exchanger length
U	overall heat transfer coefficient
A	heat transfer surface area
T	temperature
t	time
x	position on the exchanger
ΔT	temperature difference
$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	set of n-tuples $(x_j)_{j=1,\ldots,n}$ with $x_j \in \mathbb{R}$
$\mathbb{R}_+$	set of positive real numbers
$\mathbb{R}_+^n$	set of n-tuples $(x_j)_{j=1,\ldots,n}$ with $x_j \in \mathbb{R}_+$

Subscripts

c	cold
h	hot
i	inlet
o	outlet

Let $T_\mathrm{h} = T_\mathrm{h}(t, x)$ and $T_\mathrm{c} = T_\mathrm{c}(t, x)$, respectively denote the temperature of the hot and cold fluids at time t and position $x \in [0, l]$. Furthermore, let ΔT_1 and ΔT_2 stand for the temperature difference at each terminal side of the heat exchanger, *i.e.*, (see Figs. 1 and 2)

$$\Delta T_1 = \begin{cases} T_\mathrm{hi} - T_\mathrm{co} & \text{if counterflow,} \\ T_\mathrm{hi} - T_\mathrm{ci} & \text{if parallel flow,} \end{cases} \tag{1}$$

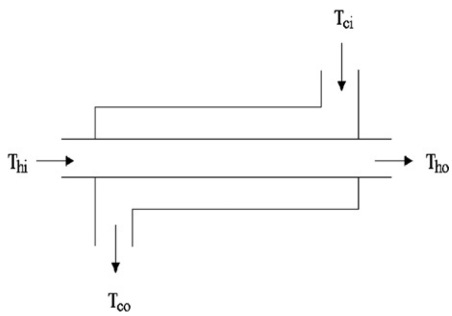

Fig. 1. Counterflow heat exchanger.

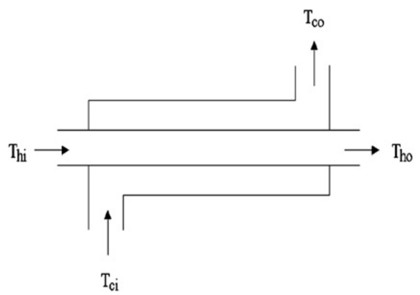

Fig. 2. Parallel flow heat exchanger.

and

$$\Delta T_2 = \begin{cases} T_{\mathrm{ho}} - T_{\mathrm{ci}} & \text{if counter flow,} \\ T_{\mathrm{ho}} - T_{\mathrm{co}} & \text{if parallel flow.} \end{cases} \tag{2}$$

Consider the sets B, C, and D with $B \subset C$, and a mapping $f: C \to D$. We denote $f|_B$ the restriction of f to B, *i.e.*, $f|_B : B \to D : y \mapsto f|_B(y) = f(y)$, $\forall y \in B$. The boundary of a subset, say B, is represented as ∂B. Within the distributed dynamics framework, the system state space will be considered to be the Hilbert space $H = \mathscr{L}_2(0,l) \times \mathscr{L}_2(0,l)$ (with the standard inner product $\langle f, g\rangle = \int_0^l [f_1(x)g_1(x) + f_2(x)g_2(x)]\,\mathrm{d}x$), where $\mathscr{L}_2(0,l)$ denotes the space of continuous functions that are square-integrable on $[0,l]$.

3. The distributed-parameter model

Let us consider the following assumptions:

A1. The fluid temperatures and velocities are radially uniform.
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A3. The thermophysical properties of the fluids are constant (in time and space).
A4. There is no heat transfer with the surroundings (perfectly insulated external tube).
A5. The heat transfer coefficient is axially uniform and is flow, temperature, and time invariant.
A6. The fluids are incompressible and single phase.
A7. Heat conduction along the flow axis is negligible.
A8. There is no energy storage in the walls.
A9. Inlet temperatures, *i.e.*, T_{ci} and T_{hi}, are constant.

Under Assumptions 1–6, a distributed-parameter dynamic representation of heat exchangers may be derived (see for instance [7]). With them in mind and the consideration of Assumption 7, such a model is given by (see for instance [25,41])

$$\begin{aligned} \frac{\partial T_{\mathrm{c}}}{\partial t} &= \alpha v_{\mathrm{c}} \frac{\partial T_{\mathrm{c}}}{\partial x} + \frac{UA}{M_{\mathrm{c}} C_{p\mathrm{c}}}(T_{\mathrm{h}} - T_{\mathrm{c}}), \\ \frac{\partial T_{\mathrm{h}}}{\partial t} &= -v_{\mathrm{h}} \frac{\partial T_{\mathrm{h}}}{\partial x} - \frac{UA}{M_{\mathrm{h}} C_{p\mathrm{h}}}(T_{\mathrm{h}} - T_{\mathrm{c}}), \end{aligned} \tag{3}$$

where

$$\alpha = \begin{cases} 1 & \text{for countercurrent flow,} \\ -1 & \text{for parallel flow.} \end{cases} \tag{4}$$

Furthermore, Assumption 8 guarantees the existence of an equilibrium solution of system (3) (as will be corroborated later in the present section). In vector notation, such a model may be expressed as

$$\frac{\partial T}{\partial t} = A_1 \frac{\partial T}{\partial x} + A_2 T, \tag{5}$$

where

$$T = \begin{pmatrix} T_c \\ T_h \end{pmatrix}, \quad A_1 = \begin{pmatrix} \alpha v_c & 0 \\ 0 & -v_h \end{pmatrix}, \quad \text{and} \quad A_2 = \begin{pmatrix} -\frac{UA}{M_c C_{pc}} & \frac{UA}{M_c C_{pc}} \\ \frac{UA}{M_h C_{ph}} & -\frac{UA}{M_h C_{ph}} \end{pmatrix},$$

or more compactly through the *abstract differential equation*

$$\dot{T} = A_0 T, \tag{6}$$

where A_0 is the unbounded linear operator $A_0 : \mathscr{D}(A_0) \subset H \to H : f \mapsto A_0 f = A_1 \frac{df}{dx} + A_2 f$ with $\mathscr{D}(A_0) = \{f \in H \mid \frac{df}{dx} \in H, f_2(0) = T_{hi}, f_1(\frac{1+\alpha}{2} l) = T_{ci}\}$. Let us note that, letting T^* stand for the equilibrium solution of the system, such that $A_1 \frac{\partial T^*}{\partial x} + A_2 T^* = 0$, (due to linearity of A_0) the model may be taken to represent the dynamic behavior of the temperatures of the fluids, or that of their deviation with respect to T^*. The former case is generally considered to get the temperature equilibrium profile of the system (as will be seen later in the present section). In the latter case, the boundary conditions (inlet temperatures) shall be taken zero in the definition of $\mathscr{D}(A_0)$, *i.e.*, $\mathscr{D}(A_0) = \{f \in H \mid \frac{df}{dx} \in H, f_2(0) = f_1(\frac{1+\alpha}{2} l) = 0\}$. Such a way of considering T (*i.e.*, as the temperature deviation with respect to T^*) in the distributed-parameter model is standard in stability analysis of the system equilibrium profile (see for instance [39–41]). Subsequently, the distributed-parameter model will be indistinctly referred as (3), (5), or (6).

Existence and uniqueness of solutions: Existence of solutions of the distributed-parameter model is guaranteed. This is corroborated through Theorem 91 in [42, Chapter V]. Furthermore, the unbounded linear operator A_0 in (6) is the *infinitesimal generator* of a *C_0-semigroup*, $\Phi(t)$, on H (this was proved in [40]). Then, solutions of (6), $T: H \to H: T_0 \mapsto T(t)$, are uniquely defined for each $T_0 \in H$ as $T(t) = \Phi(t)T_0 \in H$ (see for instance Theorem 2.1.10 in [43]).

Equilibrium profiles: In equilibrium $\frac{\partial T_h^*}{\partial t} = \frac{\partial T_c^*}{\partial t} = 0$. Then, from (5), the temperature equilibrium profile is determined by

$$\frac{dT^*}{dx} = -A_1^{-1} A_2 T^*, \tag{7}$$

whose unique solution is given in Appendix A.

Stability: Stability of the equilibrium profile is treated in [39–41]. In [39], asymptotic stability is proved through a Lyapunov functional. Strong stability in the counterflow case is proved in [40] within an infinite-dimensional linear systems context, while later on, such stability property is proved to be exponential in [41].

4. The lumped-parameter model

Application of the lumping procedure to the distributed dynamics, considering the whole exchanger as a unique bi-compartmental cell, results in a 2nd order lumped-parameter model given by (see for instance [10,11,22])

$$\begin{aligned} \dot{T}_{co} &= \frac{a}{M_c}\left[F_c(T_{ci} - T_{co}) + \frac{UA}{C_{pc}} \Delta T(T_{co}, T_{ho})\right], \\ \dot{T}_{ho} &= \frac{a}{M_h}\left[F_h(T_{hi} - T_{ho}) - \frac{UA}{C_{ph}} \Delta T(T_{co}, T_{ho})\right], \end{aligned} \tag{8}$$

where $a = 1$ if the bulk temperature is taken to be that at the outlet of the tube, as in [10,11], or $a = 2$ if it is approached as the average of those at the inlet and outlet, as in [22]. $\Delta T(\cdot,\cdot)$ is the mean temperature difference throughout the exchanger, modelled in this work through the LMTD. It is actually this term that shall be adjusted for (8) to express the lumped dynamics of the countercurrent or the parallel flow exchanger. The typical expression is given as (see for instance [44–47])

$$\Delta T = \Delta T_\ell \triangleq \frac{\Delta T_2 - \Delta T_1}{\ln \frac{\Delta T_2}{\Delta T_1}},$$

with ΔT_1 and ΔT_2 as defined in (1) and (2), which may be rewritten as

$$\Delta T_\ell = \frac{T_{\mathrm{ho}} - T_{\mathrm{hi}} + \alpha(T_{\mathrm{co}} - T_{\mathrm{ci}})}{\ln \frac{\Delta T_2}{\Delta T_1}}, \tag{9}$$

(see (4)). It is worth noting that this expression reduces to an indeterminate form when $\Delta T_1 = \Delta T_2$, which is specially problematic in the counterflow case. However, taking

$$\Delta T = \Delta T_L \triangleq \begin{cases} \Delta T_\ell & \text{if } \Delta T_2 \neq \Delta T_1, \\ \Delta T_0 & \text{if } \Delta T_2 = \Delta T_1 = \Delta T_0, \end{cases} \tag{10}$$

results in a well-defined continuously differentiable LMTD for every positive ΔT_1 and ΔT_2 (see [48]). This and other useful analytical properties of ΔT_L in (10) are proved in Appendix B.

Subsequently, system (8) will be generically represented as $\dot{T}_{\mathrm{o}} = f(T_{\mathrm{o}})$, with $T_{\mathrm{o}} = (T_{\mathrm{o}1}, T_{\mathrm{o}2})^{\mathrm{T}} \triangleq (T_{\mathrm{co}}, T_{\mathrm{ho}})^{\mathrm{T}}$ and

$$f(T_{\mathrm{o}}) = \begin{pmatrix} f_1(T_{\mathrm{o}}) \\ f_2(T_{\mathrm{o}}) \end{pmatrix} \triangleq \begin{pmatrix} \frac{a}{M_{\mathrm{c}}}\left[F_{\mathrm{c}}(T_{\mathrm{ci}} - T_{\mathrm{co}}) + \frac{UA}{C_{pc}}\Delta T(T_{\mathrm{co}}, T_{\mathrm{ho}})\right] \\ \frac{a}{M_{\mathrm{h}}}\left[F_{\mathrm{h}}(T_{\mathrm{hi}} - T_{\mathrm{ho}}) - \frac{UA}{C_{ph}}\Delta T(T_{\mathrm{co}}, T_{\mathrm{ho}})\right] \end{pmatrix}. \tag{11}$$

Let us note that by using (10), the right-hand-side expressions in (8) are continuously differentiable on the system state-space domain

$$\mathbb{D} = \begin{cases} \{T_{\mathrm{o}} \in \mathbb{R}^2 \mid T_{\mathrm{ci}} < T_{\mathrm{o}j} < T_{\mathrm{hi}}, \quad j = 1, 2\} & \text{if } \alpha = 1, \\ \{T_{\mathrm{o}} \in \mathbb{R}^2 \mid T_{\mathrm{ci}} < T_{\mathrm{o}1} < T_{\mathrm{o}2} < T_{\mathrm{hi}}\} & \text{if } \alpha = -1. \end{cases}$$

A reasoning underlying such a definition of $\mathbb{D}$ is furnished in [49, Section 3.2]. Its sense will appear clear from the analysis developed in the following subsection.

4.1. Existence and uniqueness of solutions

We now prove that for every $T_{\mathrm{o}}(0) = T_{\mathrm{o}}^0 \in \mathbb{D}$, system (8) has a unique solution $T_{\mathrm{o}}(t; T_{\mathrm{o}}^0) \in \mathbb{D}$, $\forall t \geqslant 0$. Since the right-hand-side expressions of (8) are continuously differentiable on the system state-space domain, and $\mathbb{D}$ is a bounded set (implying compactness of its closure) contained in $\mathbb{R}_+^2$, a sufficient condition to guarantee global existence and uniqueness of solutions on $\mathbb{D}$ is that $\mathbb{D}$ be positively invariant with respect to (8) (see for instance [50, Theorem 2.4]). To prove that such is the case, let us define $L_1 \triangleq \{T_{\mathrm{o}} \in \mathbb{R}^2 | T_{\mathrm{co}} = T_{\mathrm{hi}} \geqslant T_{\mathrm{ho}} \geqslant T_{\mathrm{ci}}\}$, $L_2 \triangleq \{T_{\mathrm{o}} \in \mathbb{R}^2 \mid T_{\mathrm{ho}} = T_{\mathrm{ci}} \leqslant T_{\mathrm{co}} \leqslant T_{\mathrm{hi}}\}$, $L_3 \triangleq \{T_{\mathrm{o}} \in \mathbb{R}^2 \mid T_{\mathrm{co}} = T_{\mathrm{ci}} < T_{\mathrm{ho}} \leqslant T_{\mathrm{hi}}\}$, $L_4 \triangleq \{T_{\mathrm{o}} \in \mathbb{R}^2 \mid T_{\mathrm{ho}} = T_{\mathrm{hi}} > T_{\mathrm{co}} \geqslant T_{\mathrm{ci}}\}$, and $L_5 \triangleq \{T_{\mathrm{o}} \in \mathbb{R}^2 \mid T_{\mathrm{ci}} \leqslant T_{\mathrm{co}} = T_{\mathrm{ho}} \leqslant T_{\mathrm{hi}}\}$. Notice that $\partial\mathbb{D} = \bigcup_{j=2-\alpha}^{\frac{9-\alpha}{2}} L_j$. Furthermore, considering the analytical properties of $\Delta T = \Delta T_L$ stated in Appendix B (see specifically Lemma 2 and Remark 3), let us note that $f_1(T_{\mathrm{ci}}, T_{\mathrm{ho}}) = \frac{aUA}{M_{\mathrm{c}}C_{pc}}\Delta T(T_{\mathrm{ci}}, T_{\mathrm{ho}}) > 0$, $\forall T_{\mathrm{o}} \in L_3$, and $f_2(T_{\mathrm{co}}, T_{\mathrm{hi}}) = -\frac{aUA}{M_{\mathrm{h}}C_{ph}}\Delta T(T_{\mathrm{co}}, T_{\mathrm{hi}}) < 0$, $\forall T_{\mathrm{o}} \in L_4$, for both configuration cases; $f_1(T_{\mathrm{hi}}, T_{\mathrm{ho}}) = \frac{aF_{\mathrm{c}}}{M_{\mathrm{c}}}(T_{\mathrm{ci}} - T_{\mathrm{hi}}) < 0$, $\forall T_{\mathrm{o}} \in L_1$, and $f_2(T_{\mathrm{co}}, T_{\mathrm{ci}}) = \frac{aF_{\mathrm{h}}}{M_{\mathrm{h}}}(T_{\mathrm{hi}} - T_{\mathrm{ci}}) > 0$, $\forall T_{\mathrm{o}} \in L_2$, for the counterflow case; and $f_1(T_{\mathrm{ho}}, T_{\mathrm{ho}}) = -\frac{aF_{\mathrm{c}}}{M_{\mathrm{c}}}(T_{\mathrm{ci}} - T_{\mathrm{ho}}) < 0$ and $f_2(T_{\mathrm{ho}}, T_{\mathrm{ho}}) = \frac{aF_{\mathrm{h}}}{M_{\mathrm{h}}}(T_{\mathrm{hi}} - T_{\mathrm{ho}}) > 0$, $\forall T_{\mathrm{o}} \in L_5$, for the parallel flow case. This shows that at any $T_{\mathrm{o}} \in \partial\mathbb{D}$, the vector field $f(T_{\mathrm{o}})$ points inwards $\mathbb{D}$; see Figs. 3 and 4. Consequently, $\partial\mathbb{D}$ is unreachable by any orbit of (8) with initial conditions in $\mathbb{D}$, concluding that $T_{\mathrm{o}}^0 \in D \Rightarrow T_{\mathrm{o}}(t; T_{\mathrm{o}}^0) \in \mathbb{D}$, $\forall t \geqslant 0$.

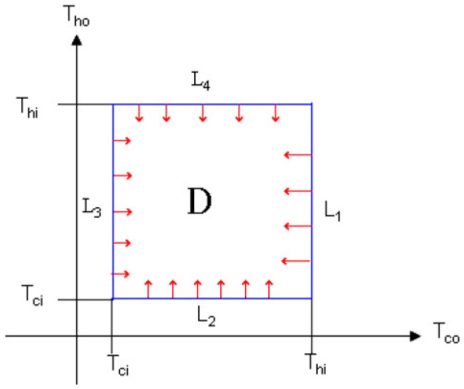

Fig. 3. Direction of f on $\partial\mathbb{D}$: $\alpha = 1$.

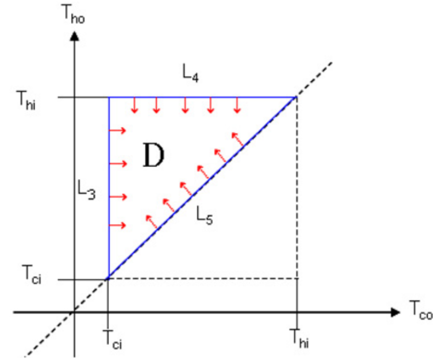

Fig. 4. Direction of f on $\partial\mathbb{D}$: $\alpha = -1$.

4.2. Equilibrium solutions

Let Q represent the set of equilibrium points of (8), *i.e.*, $Q \triangleq \{T_\mathrm{o} \in \mathbb{D} \mid f_1(T_\mathrm{o}) = f_2(T_\mathrm{o}) = 0\}$. Notice that $\frac{M_\mathrm{c}C_{pc}}{a} f_1(T_\mathrm{o}) + \frac{M_\mathrm{h}C_{ph}}{a} f_2(T_\mathrm{o}) = F_\mathrm{c}C_{pc}(T_\mathrm{ci} - T_\mathrm{co}) + F_\mathrm{h}C_{ph}(T_\mathrm{hi} - T_\mathrm{ho})$ (see (11)). Therefore, $Q \subset \{(T_\mathrm{co}, T_\mathrm{ho}) \in \mathbb{D} \mid F_\mathrm{c}C_{pc}(T_\mathrm{ci} - T_\mathrm{co}) + F_\mathrm{h}C_{ph}(T_\mathrm{hi} - T_\mathrm{ho}) = 0\}$, that is, every equilibrium point of (8), T^*_o, satisfies

$$(T^*_\mathrm{ho} - T_\mathrm{hi}) = -R(T^*_\mathrm{co} - T_\mathrm{ci}), \tag{12}$$

with $R = \frac{F_\mathrm{c}C_{pc}}{F_\mathrm{h}C_{ph}}$ (as defined in Appendix A). Then, $\forall T^*_\mathrm{o} \in Q$

$$\frac{M_\mathrm{c}C_{pc}}{a} f_1(T^*_\mathrm{o}) = \left[F_\mathrm{c}C_{pc} - UA\frac{\alpha - R}{\ln\frac{\Delta T_2}{\Delta T_1}}\right](T_\mathrm{ci} - T^*_\mathrm{co}) = 0, \tag{13}$$

(where (9) has been used) if $\alpha - R \neq 0$ (*i.e.*, for parallel flow with any $R > 0$ or counterflow with $R \neq 1$), and

$$\frac{M_\mathrm{c}C_{pc}}{a} f_1(T^*_\mathrm{o}) = F_\mathrm{c}C_{pc}(T_\mathrm{ci} - T^*_\mathrm{co}) + UA(T_\mathrm{hi} - T^*_\mathrm{co}) = 0, \tag{14}$$

if $\alpha = R = 1$ (*i.e.*, for counterflow with $R = 1$; notice that in this case, from (12), we have $T^*_\mathrm{ho} - T_\mathrm{ci} = T_\mathrm{hi} - T^*_\mathrm{co}$, *i.e.*, $\Delta T^*_2 = \Delta T^*_1$, implying, according to (10), that $\Delta T(T^*_\mathrm{co}, T^*_\mathrm{ho}) = T_\mathrm{hi} - T^*_\mathrm{co} = T^*_\mathrm{ho} - T_\mathrm{ci}$). Eq. (13) is satisfied on $\mathbb{D}$ if and only if the term within the brackets is zero. Consequently, for $\alpha - R \neq 0$, we have

$$F_\mathrm{c}C_{pc} - UA\frac{\alpha - R}{\ln\dfrac{\Delta T^*_2}{\Delta T^*_1}} = 0 \quad \Longleftrightarrow \quad \ln\frac{\Delta T^*_2}{\Delta T^*_1} = \alpha\frac{UA}{F_\mathrm{c}C_{pc}} - \frac{UA}{F_\mathrm{h}C_{ph}},$$

where from we get

$$\Delta T_2^* = e^{UA\left(\frac{\alpha}{F_c C_{pc}} - \frac{1}{F_h C_{ph}}\right)} \Delta T_1^*. \tag{15}$$

The unique solution of the system of Eqs. (12), (14), and (15) is given by (A.2) (see Appendix A). Following a similar procedure, the consideration of $\frac{M_h C_{ph}}{a} f_2(T_o^*) = 0$ leads to the same result. Consequently, $Q = \{T_o^*\}$, with $T_o^* = (T_{co}^*, T_{ho}^*)^T$ as in (A.2). Finally, it is not hard to verify that whatever (positive) value R takes, P and RP in (A.2) satisfy $0 < P < 1$ and $0 < RP < 1$, $\forall \alpha \in \{-1,1\}$, and (additionally) $P + RP < 1$ if $\alpha = -1$. Therefore, $T_{ci} < T_{co}^* < T_{pi}$ and $T_{ci} < T_{ho}^* < T_{pi}$ (see (A.2)) for both flow configurations, and (additionally) $T_{co}^* < T_{co}^* + (1 - P - RP)(T_{hi} - T_{ci}) = T_{ho}^*$ in the parallel flow case, showing that $T_o^* \in \mathbb{D}$.

Remark 1. From the equilibrium profile Eq. (A.1), one sees that $T_h(l) = T_{ho}^*$ and $T_c(\frac{1-\alpha}{2} l) = T_{co}^*$. Therefore, system (8) with ΔT as in (10) gives output temperature equilibrium values that accurately correspond to those gotten through the distributed-parameter model in steady state. This is essential since other simple models—like those using the AMTD (see equation (B.2) in Appendix B) as driving force, or the output temperature difference, *i.e.*, $\Delta T = T_{ho} - T_{co}$, for instance—may have analytical/dynamical properties that are similar, analog, or equivalent to those of the infinite-dimensional model, (3), but could hardly be able to accurately reproduce the static solutions of the outlet temperatures as (8)–(10) does.

4.3. Stability

We now prove that the unique equilibrium point of system (8), T_o^* in (A.2), is asymptotically stable and globally attractive on $\mathbb{D}$. To prove its asymptotic stability, let us consider the Jacobian matrix of f, *i.e.*,

$$\frac{\partial f}{\partial T_o} = \begin{pmatrix} \frac{\partial f_1}{\partial T_{co}} & \frac{\partial f_1}{\partial T_{ho}} \\ \frac{\partial f_2}{\partial T_{co}} & \frac{\partial f_2}{\partial T_{ho}} \end{pmatrix},$$

with

$$\frac{\partial f_1}{\partial T_{co}} = -\frac{aF_c}{M_c} + \frac{aUA}{M_c C_{pc}} \frac{\partial \Delta T}{\partial T_{co}},$$

$$\frac{\partial f_1}{\partial T_{ho}} = \frac{aUA}{M_c C_{pc}} \frac{\partial \Delta T}{\partial T_{ho}},$$

$$\frac{\partial f_2}{\partial T_{co}} = -\frac{aUA}{M_h C_{ph}} \frac{\partial \Delta T}{\partial T_{co}},$$

and

$$\frac{\partial f_2}{\partial T_{ho}} = -\frac{aF_h}{M_h} - \frac{aUA}{M_h C_{ph}} \frac{\partial \Delta T}{\partial T_{ho}}.$$

Its characteristic polynomial is given by $P(\lambda) = \lambda^2 + b(T_o)\lambda + c(T_o)$, where

$$b(T_o) = -\frac{\partial f_1}{\partial T_{co}} - \frac{\partial f_2}{\partial T_{ho}} = \frac{aF_c}{M_c} - \frac{aUA}{M_c C_{pc}} \frac{\partial \Delta T}{\partial T_{co}} + \frac{aF_h}{M_h} + \frac{aUA}{M_h C_{ph}} \frac{\partial \Delta T}{\partial T_{ho}}, \tag{16}$$

and

$$c(T_o) = \frac{\partial f_1}{\partial T_{co}} \frac{\partial f_2}{\partial T_{ho}} - \frac{\partial f_1}{\partial T_{ho}} \frac{\partial f_2}{\partial T_{co}} = \frac{4F_c F_h}{M_c M_h} - \frac{4F_h UA}{M_c M_h C_{pc}} \frac{\partial \Delta T}{\partial T_{co}} + \frac{4F_c UA}{M_c M_h C_{ph}} \frac{\partial \Delta T}{\partial T_{ho}}. \tag{17}$$

Notice that from the analytical properties of ΔT in (10) (see specifically Lemma 3 in Appendix B), we have that $\frac{\partial \Delta T}{\partial T_{ho}} > 0$ and $\frac{\partial \Delta T}{\partial T_{co}} < 0$, $\forall (T_{co}, T_{ho}) \in \mathbb{D}$. Consequently, $b(T_o) > 0$ and $c(T_o) > 0$, $\forall T_o \in \mathbb{D}$ (see (16) and (17)). Therefore, at any point on $\mathbb{D}$, both roots of $P(\lambda)$ have negative real part implying that $\frac{\partial f}{\partial T_o}$ is Hurwitz at all $T_o \in \mathbb{D}$. This proves that the unique equilibrium point, T_o^*, is asymptotically stable. To prove its global attractivity on $\mathbb{D}$, let us first note that since $\frac{\partial f_1}{\partial T_{\rm co}} + \frac{\partial f_2}{\partial T_{\rm ho}} = -b(T_{\rm o}) < 0, \forall (T_{\rm co}, T_{\rm ho}) \in \mathbb{D}$, limit cycles are excluded from $\mathbb{D}$ (according to Bendixson's criterion; see for instance [50, Theorem 7.2]). Furthermore, from the asymptotical stability of $T_{\rm o}^*$, *homoclinic orbits* are excluded from $\mathbb{D}$ too (see for instance [51, Section 1.8]). Now, from boundedness of $\mathbb{D}$ and its positive invariance with respect to the system dynamics (according to the analysis developed in Subsection 4.1 above), every solution of (8) has a nonempty, compact, and invariant *positive limit set*, Γ^+, and $T_{\rm o}(t; T_{\rm o}^0) \to \Gamma^+$ as $t \to \infty$, $\forall T_{\rm o}^0 \in \mathbb{D}$ (see for instance [50, Lemma 3.1]). Then, since limit cycles and homoclinic orbits are excluded in $\mathbb{D}$, we conclude that the unique equilibrium point, $T_{\rm o}^*$, is the only point contained in Γ^+ for every solution of (8). Consequently, $\lim_{t\to\infty} T_{\rm o}(t; T_{\rm o}^0) = T_{\rm o}^*$, $\forall T_{\rm o}^0 \in \mathbb{D}$, showing the global attractivity of $T_{\rm o}^*$ on $\mathbb{D}$.

Remark 2. From (B.4) and (B.3) in Lemma 1 (see Appendix B), it is not hard to see that ΔT in (10) is infinitely differentiable on $\mathbb{D}$. Then, $\frac{\partial f}{\partial T_{\rm o}}$ is bounded and Lipschitz on the closure of $B_r = \{T_{\rm o} \in \mathbb{R}^2 \mid \|T_{\rm o} - T_{\rm o}^*\|_2 < r\}$, and hence on B_r, for any $r > 0$ such that $B_r \subset \mathbb{D}$. Then, the asymptotic stability of $T_{\rm o}$ is actually exponential (see for instance [50, Theorem 3.13]), which agrees with the stability results exposed in Section 3 for the equilibrium profile of the distributed-parameter model.

5. Conclusions

In this work, the analytical properties of a heat exchanger 2nd-order lumped-parameter dynamic model using the LMTD as driving force, were derived and shown to coincide with those of the distributed-parameter model it is approached from. Three essential aspects were focused. First, existence and uniqueness of solutions were shown to be a common characteristic of both dynamic models. Then, equilibrium solutions of the finite-dimensional dynamics were obtained and shown to accurately agree with the outlet values of the temperature equilibrium profiles of the distributed model; this actually proves to be an essential aspect since other simple models may have analytical/dynamical properties that are similar, analog, or equivalent to those of the infinite-dimensional model, but could hardly be able to accurately reproduce the static solutions of the outlet temperatures. Furthermore, such equilibrium states were concluded to be exponentially stable and globally attractive on the system state-space domain in both dynamic representations.

The results developed in the present work permit one to conclude that, under the considered assumptions (commonly taken for the derivation of the heat exchanger infinite-dimensional dynamic equations), 2nd-order lumped-parameter models with LMTD-driving-force are reliable dynamic representations for heat exchangers. This is specially important in cases where it is not the quantitative solutions but the qualitative behavior that is important, like modelling and simulation of heat exchanger networks and complex industrial processes where heat exchangers are involved.

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Appendix A. Equilibrium profiles

The solution of (7) is given by

$$T^*(x) = \begin{pmatrix} T_{\rm c}^*(x) \\ T_{\rm h}^*(x) \end{pmatrix} = \begin{pmatrix} T_{\rm co}^* \\ T_{\rm ho}^* \end{pmatrix} + (T_{\rm hi} - T_{\rm ci}) \begin{pmatrix} g_{\rm c}(x) \\ g_{\rm h}(x) \end{pmatrix}, \tag{A.1}$$

where

$$g_{\rm c}(x) = \begin{cases} \dfrac{S - S^{x/l}}{1 + R} & \text{if } \alpha = -1, \\ \dfrac{S - S^{(1-x/l)}}{1 - RS} & \text{if } \alpha = 1, R \neq 1, \\ -\dfrac{Px}{l} & \text{if } \alpha = 1, R = 1, \end{cases}$$

$$g_{\mathrm{h}}(x) = \begin{cases} -\dfrac{R(S - S^{x/l})}{1+R} & \text{if } \alpha = -1, \\ \dfrac{R(1 - S^{(1-x/l)})}{1-RS} & \text{if } \alpha = 1, R \neq 1, \\ P\left(1 - \frac{x}{l}\right) & \text{if } \alpha = 1, R = 1, \end{cases}$$

and

$$\begin{pmatrix} T^*_{\mathrm{co}} \\ T^*_{\mathrm{ho}} \end{pmatrix} = \begin{pmatrix} 1-P & P \\ RP & 1-RP \end{pmatrix} \begin{pmatrix} T_{\mathrm{ci}} \\ T_{\mathrm{hi}} \end{pmatrix}, \tag{A.2}$$

with $R = \frac{F_{\mathrm{c}} C_{pc}}{F_{\mathrm{h}} C_{ph}}$,

$$P = \begin{cases} \dfrac{1-S}{1+R} & \text{if } \alpha = -1, \\ \dfrac{1-S}{1-RS} & \text{if } \alpha = 1, R \neq 1, \\ \dfrac{UA}{UA + F_{\mathrm{c}} C_{pc}} & \text{if } \alpha = 1, R = 1, \end{cases}$$

and $S = \mathrm{e}^{UA\left(\frac{\alpha}{F_{\mathrm{h}} C_{ph}} - \frac{1}{F_{\mathrm{c}} C_{pc}}\right)}$. In the counterflow case, algebraic manipulations were done to express the solution $T^*(x; T_{\mathrm{co}}, T_{\mathrm{hi}})$ as $T^*(x; T_{\mathrm{ci}}, T_{\mathrm{hi}})$.

Appendix B. Analytical properties of the LMTD

Throughout the following developments, the mean temperature difference is considered a bivariable function, whether the logarithmic model in (10),

$$\Delta T_L(\Delta T_1, \Delta T_2) = \begin{cases} \frac{\Delta T_2 - \Delta T_1}{\ln \frac{\Delta T_2}{\Delta T_1}} & \text{if } \Delta T_2 \neq \Delta T_1, \\ \Delta T_0 & \text{if } \Delta T_2 = \Delta T_1 = \Delta T_0, \end{cases} \tag{B.1}$$

[48], or the arithmetic one, (see for instance [44])

$$\Delta T_a(\Delta T_1, \Delta T_2) = \frac{\Delta T_1 + \Delta T_2}{2}, \tag{B.2}$$

is referred. In order for this work to be self-contained, some useful analytical properties of ΔT_L in (B.1), that were derived in [48], are proved in this Appendix. We begin by stating a useful equivalent expression.

Lemma 1 [48, Lemma 1]. *Let*

$$L(\Delta T_1, \Delta T_2) \triangleq 1 + \sum_{i=1}^{\infty} \frac{1}{2i+1} \left(\frac{\Delta T_2 - \Delta T_1}{\Delta T_2 + \Delta T_1} \right)^{2i}, \tag{B.3}$$

for all $(\Delta T_1, \Delta T_2)$ *such that* $\Delta T_1 + \Delta T_2 \neq 0$. *Then*

$$\Delta T_L(\Delta T_1, \Delta T_2) \equiv \frac{\Delta T_a(\Delta T_1, \Delta T_2)}{L(\Delta T_1, \Delta T_2)}, \tag{B.4}$$

$\forall (\Delta T_1, \Delta T_2) \in \mathbb{R}^2_+$.

Proof. We divide the proof in two parts:

(1) $\Delta T_1 \neq \Delta T_2$. From Formula 4.1.27 in[2] [52], we have

$$\ln\frac{\Delta T_2}{\Delta T_1} = 2\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{\Delta T_2-\Delta T_1}{\Delta T_2+\Delta T_1}\right)^{2i+1},$$

$\forall(\Delta T_1,\Delta T_2)\in\mathbb{R}^2_+$. Then, for all $(\Delta T_1,\Delta T_2)\in\mathbb{R}^2_+$ such that $\Delta T_1\neq\Delta T_2$, we get

$$\frac{\Delta T_2-\Delta T_1}{\ln\frac{\Delta T_2}{\Delta T_1}} = \frac{\Delta T_2-\Delta T_1}{2\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{\Delta T_2-\Delta T_1}{\Delta T_2+\Delta T_1}\right)^{2i+1}} = \frac{\Delta T_2-\Delta T_1}{2\left(\frac{\Delta T_2-\Delta T_1}{\Delta T_2+\Delta T_1}\right)\left[1+\sum_{i=1}^{\infty}\frac{1}{2i+1}\left(\frac{\Delta T_2-\Delta T_1}{\Delta T_2+\Delta T_1}\right)^{2i}\right]} = \frac{\frac{\Delta T_1+\Delta T_2}{2}}{1+\sum_{i=1}^{\infty}\frac{1}{2i+1}\left(\frac{\Delta T_2-\Delta T_1}{\Delta T_2+\Delta T_1}\right)^{2i}}$$

$$= \frac{\Delta T_a(\Delta T_1,\Delta T_2)}{L(\Delta T_1,\Delta T_2)} \quad \text{(from (B.2) and (B.3))},$$

(2) $\Delta T_1 = \Delta T_2$. Notice from (B.3) and (B.2) that $L(\Delta T_0,\Delta T_0)=1$ and $\Delta T_a(\Delta T_0,\Delta T_0)=\Delta T_0$, $\forall\Delta T_0\neq 0$. Then, for $\Delta T_1=\Delta T_2=\Delta T_0>0$, we have $\frac{\Delta T_a(\Delta T_0,\Delta T_0)}{L(\Delta T_0,\Delta T_0)}=\Delta T_0=\Delta T_L(\Delta T_0,\Delta T_0)$. □

Remark 3. Let us note that the quotient function $\frac{\Delta T_a}{L}(\Delta T_1,\Delta T_2)=\frac{\Delta T_a(\Delta T_1,\Delta T_2)}{L(\Delta T_1,\Delta T_2)}$ is defined on a subset wider than the domain of ΔT_L. Actually, $\frac{\Delta T_a}{L}:\{(\Delta T_1,\Delta T_2)\in\mathbb{R}^2 \mid \Delta T_1+\Delta T_2\neq 0\}\to\mathbb{R}$. Then $\frac{\Delta T_a}{L}$ is an extension of ΔT_L (actually Lemma 1 can be synthesized as: $\Delta T_L\equiv\frac{\Delta T_a}{L}|_{\mathbb{R}^2_+}$). Therefore, $\frac{\Delta T_a}{L}$ may be used to extrapolate ΔT_L to points on $\mathbb{R}^2$ where the latter is not defined. This is helpful for analysis purposes. For example, one sees from (B.3) that $L|_{\partial\mathbb{R}^2_+}=\sum_{i=0}^{\infty}\frac{1}{2i+1}$, and since $\frac{1}{2i+1}>\frac{1}{2(i+1)}$, $\forall i\geqslant 0$, then $\sum_{i=0}^{\infty}\frac{1}{2i+1}$ is divergent according to Theorems 3.28 and 3.25 in[3] [53]. Therefore $\frac{\Delta T_a(\Delta T_1,\Delta T_2)}{L(\Delta T_1,\Delta T_2)}\to 0$ as $(\Delta T_1,\Delta T_2)\to\partial\mathbb{R}^2_+$ which, from Lemma 1, implies that $\Delta T_L(\Delta T_1,\Delta T_2)\to 0$ as $(\Delta T_1,\Delta T_2)$ approaches $\partial\mathbb{R}^2_+$ (from the interior of $\mathbb{R}^2_+$). Then, zero can be considered the value that the LMTD (as a bivariable function) takes at any point on $\partial\mathbb{R}^2_+$. This is useful in the analysis developed in Section 4.1 of the present work.

Lemma 2 [48, Lemma 2]. *The LMTD model in (B.1) is continuously differentiable and positive on* $\mathbb{R}^2_+$.

Proof. Since $L(\Delta T_1,\Delta T_2)\geqslant 1,\forall(\Delta T_1,\Delta T_2)\in\mathbb{R}^2_+$ (see (B.3)), one sees from (B.4) that ΔT_L exists and is continuous on $\mathbb{R}^2_+$. Moreover, from (B.4), we have

$$\frac{\partial\Delta T_L}{\partial\Delta T_i} = \frac{\frac{L}{2}-\Delta T_a\frac{\partial L}{\partial\Delta T_i}}{L^2}, \tag{B.5}$$

$i=1,2$, where the arguments have been dropped for the sake of simplicity, and from (B.3), one sees that $\frac{\partial L}{\partial\Delta T_i}=\sum_{i=1}^{\infty}\frac{2i}{2i+1}S^{2i-1}\frac{\partial S}{\partial\Delta T_i}$, with $S=\frac{\Delta T_2-\Delta T_1}{\Delta T_2+\Delta T_1}$ and $\frac{\partial S}{\partial\Delta T_i}=\frac{(-1)^i 2\Delta T_{3-i}}{(\Delta T_1+\Delta T_2)^2}$. From these expressions, one observes that $\frac{\partial\Delta T_L}{\partial\Delta T_i}$, $i=1,2$, exist and are continuous on $\mathbb{R}^2_+$, proving continuous differentiability. On the other hand, notice (from (B.2)) that $\Delta T_a(\Delta T_1,\Delta T_2)>0,\forall(\Delta T_1,\Delta T_2)\in\mathbb{R}^2_+$ (the average of two positive numbers is positive). From this and the fact that $L(\Delta T_1,\Delta T_2)\geqslant 1$ on $\mathbb{R}^2_+$, we have $0<\frac{\Delta T_a(\Delta T_1,\Delta T_2)}{L(\Delta T_1,\Delta T_2)}\leqslant\Delta T_a(\Delta T_1,\Delta T_2),\forall(\Delta T_1,\Delta T_2)\in\mathbb{R}^2_+$. Then, from Lemma 1, positivity of ΔT_L follows too. □

Lemma 3 [48, Lemma 3]. *The LMTD model in (B.1) is strictly increasing in its arguments, i.e.,* $\frac{\partial\Delta T_L}{\partial\Delta T_i}>0$, $i=1,2$, $\forall(\Delta T_1,\Delta T_2)\in\mathbb{R}^2_+$.

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[2] Formula 4.1.27 in [52] states the following well-known (infinite) series expansion of the logarithmic function: $\ln z = 2\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{z-1}{z+1}\right)^{2i+1}$, $\forall z:\Re(z)\geqslant 0,\ z\neq 0$.

[3] In [53], Theorem 3.28 states that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p>1$ and diverges if $p\leqslant 1$, while point (b) of Theorem 3.25 states that if $a_n\geqslant d_n\geqslant 0$ for $n\geqslant N_0$ (for some N_0), and if $\sum_{n=1}^{\infty}d_n$ diverges, then $\sum_{n=1}^{\infty}a_n$ diverges.

Proof. From (B.1) (for $\Delta T_2 \neq \Delta T_1$) and (B.5) (for $\Delta T_2 = \Delta T_1$), we have

$$\frac{\partial \Delta T_L}{\partial \Delta T_\mathrm{i}} = \begin{cases} \dfrac{(-1)^i \left[\ln \dfrac{\Delta T_2}{\Delta T_1} - \dfrac{\Delta T_2 - \Delta T_1}{\Delta T_\mathrm{i}}\right]}{\left[\ln \dfrac{\Delta T_2}{\Delta T_1}\right]^2} & \text{if } \Delta T_2 \neq \Delta T_1, \\ \dfrac{1}{2} & \text{if } \Delta T_2 = \Delta T_1, \end{cases} \tag{B.6}$$

$i = 1, 2$, existing and being continuous on $\mathbb{R}^2_+$ according to Lemma 2. Notice from (B.6) that the proof of the lemma amounts to demonstrate positivity of $(-1)^i[\ln \frac{\Delta T_2}{\Delta T_1} - \frac{\Delta T_2 - \Delta T_1}{\Delta T_\mathrm{i}}]$, $i = 1, 2$, for all $(\Delta T_1, \Delta T_2) \in \mathbb{R}^2_+$ such that $\Delta T_1 \neq \Delta T_2$. Then, from Formula 4.1.33 in[4] [52], we have, for all such $(\Delta T_1, \Delta T_2)$: $\frac{\Delta T_2 - \Delta T_1}{\Delta T_2} < \ln \frac{\Delta T_2}{\Delta T_1} < \frac{\Delta T_2 - \Delta T_1}{\Delta T_1} \iff (-1)^i \left[\ln \frac{\Delta T_2}{\Delta T_1} - \frac{\Delta T_2 - \Delta T_1}{\Delta T_\mathrm{i}}\right] > 0$, $i = 1, 2$, proving the lemma. □

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