Analytic Invariant Curves for a Planar Map

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Abstract — In this paper, we study the existence of analytic invariant curves for two-dimensional maps of the form

\[ F(x, y) = (x + y, y + G(x) + H(x + y)) \]

in complex field \( \mathbb{C} \). Observe that if \( y = f(x) \) is an invariant curve of (1), then \( f(x) \) satisfies the functional equation

\[ f(x + f(x)) = f(x) + G(x) + H(x + f(x)), \quad x \in \mathbb{C}. \]

1. INTRODUCTION

Invariant curves of the area preserving maps play an important role in the theory of periodic stability of discrete dynamical systems. The existence of analytic invariant curves for two-dimensional area preserving maps has been studied by many authors [1-6]. In particular, Diamond [3] pointed out that invariant curves are of some significance in the theory of fractional iteration, and studied the existence of analytic invariant curves for two-dimensional maps of the form

\[ T(x, y) = (x + y, y (1 + \beta x^k) + F(x, y)). \]

In this paper, we will be concerned with analytic invariants for another planar map of the form

\[ F(x, y) = (x + y, y + G(x) + H(x + y)) \] (1)

in complex field \( \mathbb{C} \). Observe that if \( y = f(x) \) is an invariant curve of (1), then \( f(x) \) satisfies the functional equation

\[ f(x + f(x)) = f(x) + G(x) + H(x + f(x)), \quad x \in \mathbb{C}. \] (2)
We will prove the existence of analytic solutions for (2) by locally reducing the equation to another functional equation without iteration of unknown function
\[ \varphi (\alpha^2 x) = 2\varphi (\alpha x) - \varphi (x) + G(\varphi (x)) + H(\varphi (\alpha x)), \quad x \in \mathbb{C}, \] called the auxiliary equation of (2), where \( \alpha \) satisfies one of the following conditions:
(H1) \( 0 < |\alpha| < 1; \)
(H2) \( |\alpha| > 1; \)
(H3) \( |\alpha| = 1, \alpha \) is not a root of unity, and
\[
\log \frac{1}{|\alpha^n - 1|} \leq K \log n, \quad n = 2, 3, \ldots
\]
for some positive constant \( K \).

We always assume that \( G(x), H(x) \) are analytic on a neighborhood of the origin, \( G(0) = 0, G'(0) = s, H(0) = 0, \) and \( H'(0) = ((\alpha - 1)^2 - s)/\alpha. \)

### 2. SOME PREPARATORY LEMMAS

In this section, we will state and prove some preparatory lemmas which will be used in the proof of our main result.

**Lemma 1.** Assume that (H1) holds and that \( s \neq 1 \), or that (H2) holds. For any \( \eta \in \mathbb{C} \), equation (3) has an analytic solution \( \varphi(x) \) in a neighborhood of the origin such that \( \varphi(0) = 0 \) and \( \varphi'(0) = \eta \).

**Proof.** Fix an \( \eta \in \mathbb{C} \). If \( \eta = 0 \), then (3) has a trivial solution \( \varphi(x) = 0 \). So assume that \( \eta \neq 0 \). Let
\[
G(x) = \sum_{n=1}^{\infty} c_n x^n, \quad H(x) = \sum_{n=1}^{\infty} d_n x^n. \tag{4}
\]
Since \( G(x) \) and \( H(x) \) are analytic on a neighborhood of the origin, there exists a positive \( \rho \) such that
\[
|c_n| \leq \rho^{n-1}, \quad |d_n| \leq \rho^{n-1}, \quad n = 2, 3, \ldots. \tag{5}
\]
Introducing new functions \( \tilde{\varphi}(x) = \rho \varphi(\rho^{-1} x), \tilde{G}(x) = \rho G(\rho^{-1} x), \) and \( \tilde{H}(x) = \rho H(\rho^{-1} x) \), we obtain from (3),
\[
\varphi (\alpha^2 x) = 2\tilde{\varphi}(\alpha x) - \tilde{\varphi}(x) + \tilde{G}(\varphi(x)) + \tilde{H}(\varphi(\alpha x)),
\]
which is an equation of form (3). From (4) and \( \tilde{G}(x) = \rho G(\rho^{-1} x), \tilde{H}(x) = \rho H(\rho^{-1} x) \), we have
\[
\tilde{G}(x) = \rho G(\rho^{-1} x) = sx + \sum_{n=2}^{\infty} c_n \rho^{1-n} x^n,
\]
\[
\tilde{H}(x) = \rho H(\rho^{-1} x) = d_1 x + \sum_{n=2}^{\infty} d_n \rho^{1-n} x^n.
\]
By (5), it follows that
\[
\left| \frac{c_n}{\rho^{n-1}} \right| \leq 1, \quad \left| \frac{d_n}{\rho^{n-1}} \right| \leq 1, \quad n = 2, 3, \ldots.
\]
Therefore, without loss of generality, we assume that
\[
|c_n| \leq 1, \quad |d_n| \leq 1, \quad n = 2, 3, \ldots. \tag{6}
\]
Let
\[ \varphi(x) = \sum_{n=1}^{\infty} b_n x^n \quad (7) \]
be the expansion of a formal solution \( \varphi(x) \) of equation (3). Inserting (4) and (7) into (3), we have
\[
\sum_{n=1}^{\infty} \alpha^{2n} b_n x^n = 2 \sum_{n=1}^{\infty} \alpha^n b_n x^n - \sum_{n=1}^{\infty} b_n x^n + \sum_{n=1}^{\infty} \left( \sum_{\substack{n_1 + \cdots + n_t = n \geq 1; \ t=1,2,3,\ldots,n}} c_t b_{n_1} b_{n_2} \ldots b_{n_t} \right) x^n \\
+ \beta \sum_{n=1}^{\infty} \alpha^n \left( \sum_{\substack{n_1 + \cdots + n_t = n; \ t=1,2,3,\ldots,n}} d_t b_{n_1} b_{n_2} \ldots b_{n_t} \right) x^n.
\]
Comparing coefficients, we obtain
\[
\alpha^{2n} b_n = (2\alpha^n - 1)b_n + \sum_{\substack{n_1 + \cdots + n_t = n; \ t=1,2,3,\ldots,n}} (c_t + \alpha^n d_t) b_{n_1} b_{n_2} \ldots b_{n_t}, \quad n = 1,2,\ldots.
\]
This implies that
\[
[(\alpha - 1)^2 - s - \alpha d_1] b_1 = 0
\]
and
\[
[(\alpha^n - 1)^2 - s - \alpha^n d_t] b_n = \sum_{\substack{n_1 + \cdots + n_t = n; \ t=2,3,\ldots,n}} (c_t + \alpha^n d_t) b_{n_1} b_{n_2} \ldots b_{n_t}, \quad n = 2,3,\ldots \quad (8)
\]
Noting that \( d_1 = ((\alpha - 1)^2 - s)/\alpha \), we see that \( (\alpha - 1)^2 - s - \alpha d_1 = 0 \). Hence, we choose \( b_1 = \eta \) and by (8), we get
\[
(\alpha^{n-1} - 1) (\alpha^{n+1} - 1 + s) b_n = \sum_{\substack{n_1 + \cdots + n_t = n; \ t=2,3,\ldots,n}} (c_t + \alpha^n d_t) b_{n_1} b_{n_2} \ldots b_{n_t}, \quad n = 2,3,\ldots \quad (9)
\]
Now, we show that the power series (7) converges in a neighborhood of the origin. First of all, since
\[
\lim_{n \to \infty} \left| \frac{1 + |\alpha|^n}{(\alpha^{n-1} - 1)(\alpha^{n+1} - 1 + s)} \right| = \begin{cases} \frac{1}{|1-s|}, & 0 < |\alpha| < 1, \\ 0, & |\alpha| > 1, \end{cases}
\]
there exists a positive number \( M \), such that for \( n \geq 2 \),
\[
\left| \frac{1 + |\alpha|^n}{(\alpha^{n-1} - 1)(\alpha^{n+1} - 1 + s)} \right| \leq M
\]
when either (H1) or (H2) is fulfilled. Thus, if we define a sequence \( \{B_n\}_{n=1}^{\infty} \) by \( B_1 = |\eta| \) and
\[
B_n = M \sum_{\substack{n_1 + \cdots + n_t = n; \ t=2,3,\ldots,n}} B_{n_1} B_{n_2} \ldots B_{n_t}, \quad n = 2,3,\ldots,
\]
then in view of (9) and inequality (6),
\[
|b_n| \leq B_n, \quad n = 1,2,\ldots
\]
Now we define

\[ R(x) = \sum_{n=1}^{\infty} B_n x^n. \]

Since \( R(0) = 0 \), there is a positive number \( \sigma_1 \) such that \( |R(x)| < 1 \) for \( |x| < \sigma_1 \). Then

\[
R(x) = \sum_{n=1}^{\infty} B_n x^n = |\eta|x + \sum_{n=2}^{\infty} B_n x^n \\
= |\eta|x + M \sum_{n=2}^{\infty} \left( \sum_{n_1 + \ldots + n_t = n; t=2,3,\ldots} B_{n_1} B_{n_2} \ldots B_{n_t} \right) x^n \\
= |\eta|x + M \sum_{n=2}^{\infty} |R(x)|^{n-2} = |\eta|x + M \frac{|R(x)|^2}{1 - R(x)}.
\]

Hence,

\[
R(x) = \frac{1}{2(1 + M)} \left\{ 1 + |\eta|x + \sqrt{1 - 2(1 + 2M)|\eta|x + |\eta|^2x^2} \right\}.
\]

But since \( R(0) = 0 \), only the negative sign of the square root is possible, so that

\[
R(x) = \frac{1}{2(1 + M)} \left\{ 1 + |\eta|x - \sqrt{1 - 2(1 + 2M)|\eta|x + |\eta|^2x^2} \right\}.
\]

It follows that the power series \( R(x) = \sum_{n=1}^{\infty} B_n x^n \) converges for \( |x| < \sigma = \min\{\sigma_1, (1/|\eta|)(1 + 2M - 2\sqrt{M + M^2})\} \), which implies that (7) is also convergent in a neighborhood of the origin. The proof is complete.

Now, we introduce the following lemma, the proof of which can be found in [7, Chapter 6] or [8, pp. 166–174].

**Lemma 2.** Assume that (H3) holds. Then there is a positive number \( \delta \) such that \( |\alpha^n - 1|^{-1} < (2n)^\delta \) for \( n = 1, 2, \ldots \). Furthermore, the sequence \( \{d_n\}_{n=1}^{\infty} \) defined by \( d_1 = 1 \) and

\[
d_n = \frac{1}{|\alpha^{n-1} - 1|} \max_{\sum_{i=1}^{n_t} n_t = n; 0 < n_1 \leq \ldots \leq n_t; t \geq 2} \{d_{n_1} \ldots d_{n_t}\}, \quad n = 2, 3, \ldots
\]

will satisfy

\[
d_n \leq (2^{5\delta+1})^{n-1} n^{-2\delta}, \quad n = 1, 2, \ldots.
\]

**Lemma 3.** Suppose (H3) holds and that \( s \neq 0, 2 \). Then when \( 0 < |\eta| \leq 1 \), equation (3) has an analytic solution \( \varphi(z) \) of form (7) in a neighborhood of the origin such that \( \varphi(0) = 0 \) and \( \varphi'(0) = \eta \).

**Proof.** As in the proof of Lemma 1, we seek a power series solution of form (7). Then defining \( b_1 = \eta \), (9) again holds so that

\[
|b_n| \leq \frac{1}{|\alpha^{n-1} - 1| |\alpha^{n+1} - 1 + s|} \sum_{\sum_{i=2,3,\ldots} n_t = n; t=2,3,\ldots} |c_t + \alpha^n d_t| |b_{n_1}| |b_{n_2}| \ldots |b_{n_t}|,
\]

for \( n = 2, 3, \ldots \).

For convenience in writing, we put

\[
N = \frac{2}{|1 - |1 - s||}.
\]
Therefore, in view of (6), the above inequality changes into

\[ |b_n| \leq \frac{N}{|\alpha^{n-1} - 1|} \sum_{|t|_1 + \ldots + |n_t| = n; \, t=2,3,\ldots,n} |b_{n_1}| |b_{n_2}| \ldots |b_{n_t}|, \quad n = 2, 3, \ldots. \]

Let us now consider the function

\[ R(x) = \frac{1}{2(1+N)} \left\{ 1 + x - \sqrt{1 - 2(1 + 2N)x + x^2} \right\}, \]

which in view of the binomial series expansion can be written in the form

\[ R(x) = x + \sum_{n=2}^{\infty} C_n x^n \]

for \(|x| < 1 + 2N - 2\sqrt{N + N^2}\). Since \(R(0) = 0\), there is a positive number \(\tau_1\) such that \(|R(x)| < 1\) for \(|x| < \tau_1\), and \(R(x)\) satisfies the equation

\[ R(x) = x + N \frac{|R(x)|^2}{1 - R(x)}. \]

By the method of undetermined coefficients, it is not difficult to see that the coefficient sequence \(\{C_n\}_{n=1}^{\infty}\) will satisfy \(C_1 = 1\) and

\[ C_n = N \sum_{|n_1| + \ldots + |n_t| = n; \, t=2,3,\ldots,n} C_{n_1} C_{n_2} \ldots C_{n_t}, \quad n = 2, 3, \ldots. \]

Hence, we easily see that

\[ |b_n| \leq C_n d_n, \quad n = 1, 2, \ldots, \]

where the sequence \(\{d_n\}_{n=1}^{\infty}\) is defined in Lemma 2. Indeed, \(|b_1| = |\eta| \leq 1 = C_1 d_1\). Assume by induction that the above inequality holds for \(n = 1, 2, \ldots, l\). Then

\[ |b_{l+1}| \leq \frac{N}{|\alpha^{l-1} - 1|} \sum_{|n_1| + \ldots + |n_{l+1}| = l+1} |b_{n_1}| |b_{n_2}| \ldots |b_{n_l}| \]

\[ \leq \frac{N}{|\alpha^{l-1} - 1|} \sum_{|n_1| + \ldots + |n_{l+1}| = l+1} C_{n_1} d_{n_1} C_{n_2} d_{n_2} \ldots C_{n_l} d_{n_l} \]

\[ \leq \frac{C_{l+1}}{|\alpha^{l-1} - 1|} \max_{0 < n_1 \leq \ldots \leq n_l, t \geq 2} \{d_{n_1} \ldots d_{n_l}\} \]

\[ = C_{l+1} d_{l+1} \]

as desired.

Since \(R(x)\) converges in the open disc \(|x| < \tau = \min \{\tau_1, 1 + 2N - 2\sqrt{N + N^2}\}\), there is a positive \(A\) such that

\[ C_n \leq A^n \]

for \(n = 1, 2, \ldots\). In view of this and Lemma 2, we finally see that

\[ |b_n| \leq A^n \left(2^{5\delta + 1}\right)^n n^{-2\delta}, \quad n = 1, 2, \ldots, \]

that is,

\[ \lim_{n \to \infty} \sup \{|b_n|\}^{1/n} \leq \lim_{n \to \infty} \sup A \left(2^{5\delta + 1}\right)^{n-1} n^{-2\delta} = A \left(2^{5\delta + 1}\right), \]

which shows that series (7) converges for \(|x| < \min \{\tau, (A2^{5\delta + 1})^{-1}\}\). The proof is complete. \(\blacksquare\)

**Remark 1.** Whether equation (3) has an analytic solution remains an open problem when \(|\alpha| = 1\) and \(|s| = 0, 2\).
3. EXISTENCE OF ANALYTIC SOLUTIONS FOR EQUATION (2)

In this section, we will state and prove our main result in this note.

**Theorem 1.** Suppose the conditions of Lemmas 1 or 3 are fulfilled. Then equation (2) has an analytic solution of the form

\[ f(x) = \varphi(\alpha \varphi^{-1}(x)) - x \]  

in a neighborhood of the origin, where \( \varphi(x) \) is an analytic solution of equation (3).

**Proof.** In view of Lemmas 1 and 3, we find a sequence \( \{b_n\}_{n=1}^\infty \) such that the function \( \varphi(x) \) of form (7) is an analytic solution of (3) in a neighborhood of the origin. Since \( \varphi'(0) = \eta \neq 0 \), the function \( \varphi^{-1}(x) \) is analytic in a neighborhood of the point \( \varphi(0) = 0 \). If we now define \( f(x) \) by means of (10), then

\[
\begin{align*}
  f(x + f(x)) &= \varphi(\alpha \varphi^{-1}(\varphi(\alpha \varphi^{-1}(x)))) - \varphi(\alpha \varphi^{-1}(x)) \\
  &= \varphi(\alpha^2 \varphi^{-1}(x)) - \varphi(\alpha \varphi^{-1}(x)) \\
  &= \varphi(\alpha \varphi^{-1}(x)) - x + G(x) + H(\varphi(\alpha \varphi^{-1}(x))) \\
  &= f(x) + G(x) + G(x + f(x))
\end{align*}
\]

as required. The proof is complete. \( \Box \)

We now show how to explicitly construct an analytic solution of an equation of form (2) by means of an example. Consider the following equation:

\[
f(x + f(x)) = f(x) + G(x) + H(x + f(x)),
\]

where \( G(x) = 2(e^x - 1) = \sum_{n=1}^\infty (2x^n/n!) \), \( H(x) = 1 - e^x = -\sum_{n=1}^\infty (x^n/n!) \). Obviously, \( G(0) = 0 \), \( H(0) = 0 \), \( s = G'(0) = 2 \), \( d_1 = H'(0) = -1 \), and the equation

\[
(\alpha - 1)^2 - s - \alpha d_1 = \alpha^2 - \alpha - 1 = 0
\]

has two roots

\[
\alpha_+ = \frac{1 + \sqrt{5}}{2}, \quad \alpha_- = \frac{1 - \sqrt{5}}{2}
\]

and \( |\alpha_+| > 1, \ 0 < |\alpha_-| < 1 \). For \( \alpha_+ \), by means of Lemma 1, the auxiliary equation

\[
\varphi(\alpha_+^2 x) = 2\varphi(\alpha_+ x) - \varphi(x) + G(\varphi(x)) + H(\varphi(\alpha_+ x))
\]

has an analytic solution \( \varphi(x) \) in a neighborhood of the origin such that \( \varphi(0) = 0 \) and \( \varphi'(0) = \eta \neq 0 \). Let

\[
\varphi(x) = \sum_{n=1}^\infty b_n x^n, \quad b_1 = \eta.
\]

Inserting this series into (12) and equating the coefficients, we get

\[
b_n = \frac{2 - \alpha_+^2}{(\alpha_+^2 - 1)(\alpha_+^n + 1)} \sum_{n_1 + \cdots + n_t = n, \ t = 2, 3, \ldots} \frac{1}{t!} b_{n_1} b_{n_2} \cdots b_{n_t}, \quad n = 2, 3, \ldots
\]

It is not difficult to calculate the coefficients \( b_n \) by means of (13), indeed the first few terms are as follows:

\[
b_2 = \frac{\varphi''(0)}{2!} = \frac{(1 - \alpha_+)\eta^2}{4\alpha_+},
\]

\[
b_3 = \frac{\varphi'''(0)}{3!} = \frac{5(1 - \alpha_+)\eta^3}{36\alpha_+^4},
\]

\[ \ldots \]
Next, recall from the proof of the above theorem that \( \varphi^{-1}(x) \) is analytic in a neighborhood of the point \( \varphi(0) = 0 \). Therefore, it can also be determined once its derivatives at \( x = 0 \) have been determined

\[
(p^{-1})'(0) = \frac{1}{\varphi'(\varphi^{-1}(0))} = \frac{1}{\varphi'(0)} = \frac{1}{\eta},
\]

\[
(p^{-1})''(0) = -\frac{\varphi''(\varphi^{-1}(0)) (p^{-1})'(0)}{(\varphi'(\varphi^{-1}(0)))^2} = -\frac{\varphi''(0) (p^{-1})'(0)}{(\varphi'(0))^2} = -\frac{1 - \alpha_+}{2\alpha_+ \eta},
\]

\[
(p^{-1})'''(0) = -\frac{[\varphi'''(\varphi^{-1}(0))] [(p^{-1})'(0)]^2 + \varphi''(\varphi^{-1}(0)) (p^{-1})''(0) [\varphi'(\varphi^{-1}(0))]^2}{[\varphi'(\varphi^{-1}(0))]^4} + \frac{\varphi''(\varphi^{-1}(0)) (p^{-1})'(0) \cdot 2 \cdot \varphi'(\varphi^{-1}(0)) \varphi''(\varphi^{-1}(0)) (p^{-1})''(0)}{[\varphi'(\varphi^{-1}(0))]^4}
\]

\[
= -\frac{10\alpha_+ - 7}{12\alpha_+^4 \eta},
\]

e tc.

Finally, we determine a solution \( f(z) \) of (11) by finding its derivatives at \( z = 0 \)

\[
f(0) = \varphi(\alpha_+ \varphi^{-1}(0)) - 0 = \varphi(\alpha_+ \cdot 0) = 0,
\]

\[
f'(0) = \varphi'(\alpha_1 \varphi^{-1}(0)) \cdot \alpha_+ (p^{-1})'(0) - 1 = \alpha_+ \varphi'(0) (p^{-1})'(0) - 1 = \alpha_+ - 1,
\]

\[
f''(0) = \alpha_+^2 \varphi''(\alpha_+ \varphi^{-1}(0)) \left[(p^{-1})'(0)\right]^2 + \alpha_+ \varphi'(\alpha_+ \varphi^{-1}(0)) (p^{-1})''(0)
\]

\[
= \alpha_+^2 \varphi''(0) \eta^{-2} + \alpha_+ \varphi'(0) (p^{-1})''(0) = \frac{\alpha_+ - 2}{2},
\]

\[
f'''(0) = \alpha_+^3 \varphi'''(\alpha_+ \varphi^{-1}(0)) \left[(p^{-1})'\right]^3 + 2\alpha_+^2 \varphi''(\alpha_+ \varphi^{-1}(0)) (p^{-1})'(0)(p^{-1})''(0)
\]

\[
+ \alpha_+^2 \varphi''(\alpha_+ \varphi^{-1}(0)) (p^{-1})'(0)(p^{-1})''(0) + \alpha_+ \varphi'(\alpha_+ \varphi^{-1}(0))(p^{-1})'''(0)
\]

\[
= \alpha_+^3 \varphi'''(0) \eta^{-3} + 3\alpha_+^2 \varphi''(0) \eta^{-1} \cdot (p^{-1})'''(0) = -\frac{9\alpha_+ + 7}{12\alpha_+^3},
\]

e tc.

Thus, the desired solution is

\[
f(z) = (\alpha_+ - 1)z + \frac{\alpha_+ - 2}{4} z^2 - \frac{9\alpha_+ + 7}{72\alpha_+^3} z^3 + \cdots.
\]

For \( \alpha_- \), by means of the same method, we can obtain

\[
f(z) = (\alpha_- - 1)z + \frac{\alpha_- - 2}{4} z^2 - \frac{9\alpha_- + 7}{72\alpha_-^3} z^3 + \cdots.
\]

It is clear from the above procedure that with the help of commercial software which is capable of doing exact differentiations, an arbitrary number of terms of the above series can be obtained.

REFERENCES