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Stratified graphs for imbedding systems

Jonathan L. Gross[a,*,1], Thomas W. Tucker[b,2]

[a] Columbia University, 450, Computer Science Building, New York, NY 10027, USA
[b] Colgate University, Hamilton, NY 13346, USA

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Abstract

Two imbeddings of a graph G are considered to be adjacent if the second can be obtained from the first by moving one or both ends of a single edge within its or their respective rotations. Thus, a collection of imbeddings S of G, called a 'system', may be represented as a 'stratified graph', and denoted SG; the focus here is the case in which S is the collection of all orientable imbeddings. The induced subgraph of SG on the set of imbeddings into the surface of genus k is called the 'kth stratum', and the cardinality of that set of imbeddings is called the 'stratum size'; one may observe that the sequence of stratum sizes is precisely the genus distribution for the graph G. It is known that the genus distribution is not a complete invariant, even when the category of graphs is restricted to be simplicial and 3-connected. However, it is proved herein that the link of each point — that is, the subgraph induced by its neighbors — of SG is a complete isomorphism invariant for the category of graphs whose minimum valence is at least three. This supports the plausibility of a probabilistic approach to graph isomorphism testing by sampling higher-order imbedding distribution data. A detailed structural analysis of stratified graphs is presented.

1. Introduction

The set of imbeddings of a graph G admits a natural concept of adjacency between imbeddings. We thereby obtain a graded 'edge-colored' graph, denoted SG, that we call the 'stratified graph' for G. A few preliminaries and the formal definition of SG appear in this section, shortly below.

The stratified graph SG is very much larger than G itself. Indeed, each point of SG typically has more neighbors than G has vertices. Some of the structure of such a neighborhood is described by Cayley graphs we call 'circular arrangement graphs', which we examine in Section 2. In Section 3, we study the general structure of the

* Corresponding author.

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neighborhood of any point in SG, with particular attention paid to cliques. In Section 4, we show how to reconstruct a graph from a neighborhood of any point in the colored stratified graph, thereby establishing the colored stratified graph as a complete invariant of isomorphism type over the category of all graphs of minimum valence at least 3. The uncolored stratified graph is considered in Section 5 and related to the medial graph of an imbedding. The cubic case is analyzed completely and is shown to provide a large supply of constant link (Zykov-regular) graphs.

Beyond the inherent topological interest in the formulation of this nonsuperficial complete invariant for isomorphism type, one might well wonder about the usefulness of something so large in isomorphism testing. In Section 6 we illustrate how two graphs might be 'nearly isomorphic', yet distinguishable by accessible properties of their stratified graphs.

Throughout this paper a *graph* is 'simplicial', that is it has no multiple adjacencies or self-adjacencies. It is taken to be connected, unless one can readily infer otherwise from the immediate context.

The closed orientable surface of genus j is denoted by S_j. By an *imbedding* we mean a cellular imbedding of a (labeled) graph into a closed orientable surface. Some authors call this a 'labeled' imbedding. In general, the methods described here are readily adaptable to the non-orientable surfaces and to the collection of all closed surfaces.

In the present exposition, it is assumed that the reader is familiar with the fundamentals of topological graph theory, as described by Gross and Tucker [15], or — with minor terminological differences — by White [32].

We regard two imbeddings as *adjacent* if one can be obtained from the other either by moving an edge-end in the rotation at its vertex to somewhere else in that rotation, or by moving both ends of the same edge within their respective rotations. With this definition of adjacency, the set of all imbeddings of a graph G forms itself a graph, which we denote SG and call the *stratified graph* of G. For clarity, we refer to 'vertices' and 'edges' in G, and to 'points' and 'lines' in SG. Each point (imbedding of G) in SG is labeled by the genus of the corresponding imbedding surface. We think of the point-labels as altitudes. The two kinds of imbedding-adjacency (i.e., one edge-end or both) are called VM-lines and EM-lines, for 'vertex modification' and 'edge modification', respectively.

The induced subgraph of SG on all points labeled j (all imbeddings of G of genus j) is called the *jth stratum* of SG and is denoted S_jG. Lines of SG that lie within a single stratum of G are called *level lines*. All other lines of SG run between consecutive strata and are called *transverse lines* (or *transversals*).

The size of the jth stratum is denoted $g_j(G)$, or simply g_j, if there is only one graph whose imbeddings are under consideration. Thus, the sequence

$$g_0, g_1, g_2, \ldots$$

of stratum sizes is just the genus distribution for the graph G. Conversely, we observe that the problem of describing the structure of the stratified graph SG is precisely a refinement of the problem of calculating the genus distribution of G.

Thus, stratified graphs are a proper member of the hierarchy of graph invariants that correspond to distributional information about the entire system of cellular embeddings of a graph, described by Gross and Furst [11]. There are already several calculations of formulas for genus distribution, region-size distribution, and other invariants at the low end of that hierarchy.

The first such calculation for any infinite classes of graphs is the result of Furst et al. [10] establishing the genus distributions of closed-end ladders and of 'cobblestone paths', which they prove to be strongly unimodal. Stahl [25, 28] substantially generalizes these results to 'linear families' of graphs. Stahl [24, 26] derives asymptotic estimates of the genus distribution of bouquets and then generalizes his approach to show how certain small-diameter graphs have Stirling-like genus distributions, and are therefore strongly unimodal.

Gross et al. [14] derive the genus distributions of bouquets, by using a formula of Jackson [16] concerning representations of the symmetric group.

Rieper's [21] thesis includes a computation of the region-size distributions for bouquets and several other significant results, based on enumerative methods of Redfield [20]. Mull et al. [19] enumerate the congruence classes of imbedding distributions of wheels and of complete graphs. Kwak and Lee [17] extend and refine these results by using subgroups of automorphisms in the congruence relation.

Gross and Furst [11] also initiate the study of the average genus of an individual graph, which is continued by Gross et al. [12]. Earlier work of Stahl [23] considers the average genus of graph imbeddings taken over a class of graphs. Stahl [27] provides a general upper bound for average genus. Chen et al. [8] calculate a general lower bound for average genus. Stahl [30] calculates bounds for the case of an amalgamated graph. Stahl [29] explores the average genus of random graphs. Chen and Gross [7] obtain forbidden subgraph results for average genus.

2. On circular arrangement graphs and the VM-structure of *SG*

Two cyclic permutations of d symbols are considered to be *adjacent* if one can be transformed into the other by moving a single symbol. For instance, if we move the symbol i within the 'standard d-cycle'

$$C = (1\,2 \ldots d)$$

to a new location preceding the symbol j, then we obtain the adjacent d-cycle

$$(1 \ldots i-1\;\; i+1 \ldots j-1\;\; i\; j \ldots d) \quad \text{if } i<j$$

or

$$(1 \ldots j-1\;\; i\; j \ldots i-1\;\; i+1 \ldots d \quad \text{if } i>j.$$

Under this notion of adjacency, the collection of d-cycles form what we call the *circular arrangement graph* on d symbols and denote by CA_d.

Circular arrangement graphs are highly symmetric; they are a form of Cayley graph. Given a group A and a generating set X, we recall that the (right) Cayley graph for A and X has the elements of A as its vertex set, and that for every $a \in A$ and for every $x \in X$ there is an edge from a to ax. It follows that if some generator and its inverse are both in X, then there are two edges adjoining various pairs of vertices. Let us call the result of collapsing all such edge pairs onto single edges a *reduced Cayley graph*.

Left multiplication by A on the vertices of a reduced Cayley graph G yields a subgroup of the automorphism group of G. This subgroup acts transitively and without fixed points of the vertex set. Conversely, as Sabidussi [22] demonstrated, if a group acts transitively and without fixed vertices on graph, then that graph is a reduced Cayley graph for the group.

Theorem 2.1. *Let G be the reduced Cayley graph for the full symmetric group $\sum_{d-1}$, using as a generating set the union of the collection of all cycles of consecutive integers — that is, cycles of the form $(i\, i+1 \ldots j-1\, j)$, where $1 \leqslant i < j < d$ — and the collection of all powers of $(1\, 2 \ldots d-1)$. Then the circular arrangement graph CA_d is isomorphic to G.*

Proof. To establish a bijection β from the vertex set of CA_d to the membership set of $\sum_{d-1}$, let us first represent each d-cycle C as a row in which the symbol d appears last. For instance, the cycle $(4\,3\,5\,2\,1)$ is represented as $2\,1\,4\,3\,5$. The image $\beta(C) \in \sum_{d-1}$ is obtained by regarding the row representation of C as a rearrangement of the symbols $1, \ldots, d-1$. For instance, the row $2\,1\,4\,3\,5$ corresponds to $(1\,2)\,(3\,4)$ in $\sum_4$. Thus, $\beta(4\,3\,5\,2\,1) = (1\,2)\,(3\,4)$. It is routine permutation algebra to verify that the vertex function β is a bijection.

Now suppose that an adjacency between two d-cycles $C_1, C_2 \in CA_d$ arises from moving symbol i, where $i < d$. Then their representations as rows would be identical, if the symbol i were deleted from both. We may assume that the symbol i occurs in locations j and k in the row representations of C_1 and C_2, respectively, with $j < k$. Then $\beta(C_2) \circ (j\, j+1 \ldots k) = \beta(C_1)$, which implies that this form of adjacency in CA_d corresponds to an adjacency in G. Alternatively, if the adjacency between C_1 and C_2 arises from moving the symbol d exactly k places forward in C_1, then $\beta(C_2) \circ (1\,2 \ldots d-1)^k = \beta(C_1)$. It is clear that this correspondence of edges is also invertible. □

The spanning subgraph of SG containing only the VM-lines is called the VM-subgraph. The proof (omitted) of the following structure theorem is an exercise in definitions.

Theorem 2.2. *Let G be a graph with valence sequence $d_1, \ldots, d_n$. Then the VM-subgraph of SG is isomorphic (as a graph, neglecting altitude labels) to the cartesian product of n circular arrangement graphs on $d_1, d_2, \ldots, d_n$ symbols, respectively.*

Theorem 2.2 raises the recognition problem for stratified graphs: which labelings of cartesian products of circular arrangement graphs are realizable as VM-subgraphs of stratified graphs? Since CA_3 is just the complete graph K_2 on two vertices, the case of 3-regular graphs is of particular interest: which labelings of the n-cube Q_n are isomorphic to the VM-subgraph of the stratified graph for a 3-regular graph?

3. Links of points in the stratified graph

If v is a vertex of a graph G, then the *link* of v is the subgraph of G induced by the set of all vertices adjacent to v (this does not include v itself). Given a point p in the stratified graph SG, let $\mathrm{TL}(p)$ and $\mathrm{VL}(p)$ denote, respectively, the link of p in SG, and the link of p in the VM-subgraph of SG. Call $\mathrm{TL}(p)$ the *total link* of p and $\mathrm{VL}(p)$ the *VM-link* of p. The purpose of Sections 4 and 5 of this paper is to show how to reconstruct an underlying graph G from the total link $\mathrm{TL}(p)$ of any point in the stratified graph SG. In order to do this, we must understand the adjacency structure of $\mathrm{TL}(p)$.

If two points of $\mathrm{TL}(p)$ are obtained from p by moving one or both the ends of the same edge e, then those two points are adjacent to each other, again by moving ends of the edge e. Call such an adjacency or such a line in $\mathrm{TL}(p)$ *standard*. The structure of $\mathrm{TL}(p)$ would be reasonably easy to describe if all lines in $\mathrm{TL}(p)$ were standard: each edge e in G gives rise to a clique of points in $\mathrm{TL}(p)$ corresponding to all embeddings q which agree with p except for the placement of the end of edge e. Call such a clique an *edge-clique*. Every point in $\mathrm{TL}(p)$ is in some edge-clique.

Two edge cliques in $\mathrm{TL}(p)$ share a point q if and only if the two edges e_1 and e_2 that generate those cliques are *consecutive* at some vertex v in the imbedding p (that is, e_2 immediately follows e_1 in the rotation at vertex v or vice versa), and q is obtained from p by switching e_1 and e_2 at vertex v. Call q a *switch point*.

The existence of *extra* adjacencies is a complicating factor. For example, suppose that p, q, and r are imbeddings which agree at every vertex except vertex v, where the rotations are

(p) $v.$ $\ldots e_0 e_1 e_2 e_3 e_4 \ldots$

(q) $v.$ $\ldots e_0 e_2 e_3 e_1 e_4 \ldots$

(r) $v.$ $\ldots e_0 e_3 e_1 e_2 e_4 \ldots$.

Then q and r are both in $\mathrm{TL}(p)$, the former by moving e_1 and the latter by moving e_3. However, q and r are also adjacent to each other by moving e_2. This is not a standard adjacency in $\mathrm{TL}(p)$. Call it a *triclaw* (extra) adjacency.

Suppose instead that G has edges uv, vw and wu, and that in the embedding p the edges uv and vw are consecutive at vertex v, that vw and wu are consecutive at vertex w, and that wu and uv are consecutive at u; call such a 3-cycle in the imbedding p a *consecutive tricycle*. Let q_u be the imbedding obtained from p by switching edges uv

and vw at vertex v and by switching edges vw and wu at vertex w. Thus q_u is obtained from p by moving edge vw. Define q_v and q_w similarly. Then q_u and q_v agree at vertex w but differ at vertices u and v. Thus q_u and q_v are adjacent by moving the edge uv. This is an extra adjacency in TL(p). These are also extra adjacencies between q_v and q_w and between q_w and q_u. Call these *tricycle* (extra) adjacencies.

The following theorem shows that the two types of extra adjacencies just described are the only extra adjacencies in TL(p). To help in the analysis, let us call a point in TL(p) a *VM-point* if it VM-adjacent to p and an *EM-point* otherwise.

Theorem 3.1. *Every extra adjacency in* TL(p) *is either triclaw or tricycle.*

Proof. Let q and r be adjacent points in TL(p). Since they are in TL(p), the number of vertices at G at which either differs from p is at most two.

First, suppose that q and r are both VM-points of TL(p). If they differ from p at two different vertices, u and v respectively, then the only way they can be adjacent is if they are obtained by moving opposite ends of an edge adjoining u and v; that is, they would be standardly adjacent in TL(p). Therefore, assume that q and r differ from p at the same vertex v. Let e_1 and e_3 be the respective edges by whole respective v-endmotions in p the imbeddings q and r are obtained. It follows that q and r are adjacent by moving the v-end of some edge e_2. We assume that $e_2 \neq e_1$ or e_3 , since otherwise, q and r would be standardly adjacent.

If the edge e_2 were deleted, then the imbeddings q and r would be identical, and they would agree with p for the placement of the v-ends of the edges e_1 and e_3. It follows that the v-ends of the edges e_1 and e_3 are consecutive at vertex v in p, say in the order e_1e_3, and that they are consecutive at vertex v in q and r, but in opposite order e_3e_1. Now consider the placement of edge e_2 at vertex v in p. If the order is $e_1e_2e_3$ in p, it must be $e_2e_3e_1$ in q since only e_1 moves and e_1 goes after e_3 in q. Similarly the order must be $e_3e_1e_2$ in r, since only e_3 moves this time and again e_1 goes after e_3. Therefore if the order is $e_1e_2e_3$ in p, we have a triclaw extra adjacency. If instead the order at p is $e_2e_1e_3$, then the order in q must be $e_2e_3e_1$ since only e_1 moves and e_1 goes after e_3. But then q is adjacent to p by moving e_3. Since r is already adjacent to p by moving e_3, it follows that q and r are standardly adjacent by moving e_3. Similarly, if the order in p is $e_1e_3e_2$, then q and r are standardly adjacent by moving edge e_1.

Second, suppose that q is a VM-point of TL(p) and r is an EM-point. If q agrees with p except at the vertex u and r agrees with p except at v and w, where $v \neq u$ and $w \neq u$, then there can be no way of changing q at all three vertices u, v, and w simultaneously. Thus q and r are not adjacent. If q agrees with p except at vertex u and r agrees with p and except at u and v, then the only adjacency between q and r is a standard one obtained by moving the ends of edge uv, We conclude there are no extra adjacencies between VM- and EM-points.

Finally, suppose that q and r are both EM-points of TL(p) and are obtained from p by moving both ends of edges e_1 and e_2, respectively. Then e_1 and e_2 meet at some vertex v, or it would be impossible for q and r to be adjacent. Let u be the other end of

e_1 and w th other end of e_2. Since the graph G is simplicial, the edges e_1 and e_2 cannot share both endpoints, so $u \neq w$.

Since q and r are both EM-points of TL(p), they differ from p at u and w, respectively, from which it follows that they differ from each other at u and w. Since q and r are adjacent imbeddings, it follows that they differ from each other in at most two vertex rotations, so they must agree at v. We infer that there is an edge e_3 from u and w whose endmotions account for the extra adjacency of q and r. Since q moves edge e_1 and r moves edges e_2, in order for q and r to be the same vertex v the edges e_1 and e_2 must be consecutive in p, q and r at v. Since q agrees with p at w but is adjacent to r by moving edge e_3, it must be that imbedding r at vertex w is obtained from p not only by moving e_2, as hypothesized, but also by moving e_3. Therefore, e_2 and e_3 are consecutive a vertex w in both p and r. Similarly, e_1 and e_3 are consecutive at vertex u in both p and q. Therefore, e_1, e_2, and e_3 form a consecutive 3-cycle and the extra adjacency is tricycle. □

With Theorem 3.1 in hand, we have a complete understanding of the adjacency structure of TL(p). Our method of reconstructing the graph G from TL(p) also uses the line-color distinction between VFM-lines and EM-lines. In view of Theorem 2.2, Lemma 3.2 represents further analysis of the structure of VL(p), the VM-link of p.

Lemma 3.2. *The link of a vertex in CA_3 is a single vertex. The link of a vertex in CA_4 is a 4-cycle. The link of a vertex in CA_d, for $d > 4$ consists of d copies $H_1, H_2, \ldots, H_d$ of the complete graph K_{d-2} arranged in a circle so that H_i shares exactly one vertex with H_{i-1} and exactly one vertex with H_{i+1}, and, in addition, for each i there is an extra edge joining a vertex of H_i with a vertex of H_{i+2} (the joined vertices are not shared vertices). In particular, the link of a vertex in CA_d, for all $d > 2$, is connected and nonempty.*

Proof. Consider the general case $d > 4$ first. Since CA_d is vertex symmetric, we can just look at the link of the standard d-cycle C. There are $d-2$ different positions the symbol i can occupy in an arrangement of the symbols $1, \ldots, d-1$ other than position i itself. Thus the set of vertices in CA_d obtained from C by moving symbol i induces in the link of C a complete graph H_i. The subgraphs H_i and H_j share a vertex if and only if i and j are consecutive in cycle C, that is $j = i + 1$ or $i = j + 1$. The extra edge joining vertices in H_i and H_{i+2} is that corresponding to a triclaw extra adjacency.

For $d = 3$ clearly CA_3 is a two-vertex graph so the link of a vertex is a single vertex (technically, the description for $d > 4$ still holds since 3 copies of K_1 each sharing a vertex with the other is simply a single vertex). For $d = 4$, one might expect the link to be a 4-cycle together with both diagonals as the extra edges, but again the description requires the extra edges to be between vertices in H_i and H_{i+2} that are not shared with another H_j. When $d = 4$, each of the two vertices in H_i is shared vertex. Alternatively, one can check that the two vertices (2 3 1 4) and (3 1 2 4) joined by an extra edge, although apparently obtained by moving 1 and 3, respectively, are also obtained by moving 4 and hence are already standardly adjacent. □

The clique structure of TL(p) is complicated by extra adjacencies, but it is still possible to give a complete description. The extra triclaw lines join points in edge cliques which do not share a switch point. Hence each of these lines is a clique of size two. If t is consecutive 3-cycle in the imbedding p, then the three switch points in TL(p) corresponding to t form a second 3-cycle TL(p). Finally, a third type of 3-cycle is created in TL(p) by t among any two EM-points q and r joined by extra tricycle line together with the switch point shared by the edge cliques containing q and r. Call these three types of triangles in TL(p), respectively, the *VM 3-cycle*, and *EM 3-cycle* and the *VEM 3-cycles* (there are three of them) created by the consecutive 3-cycle t. As long as G is not K_4, it is impossible to have a configuration of four consecutive 3-cycles in the imbedding p based on four vertices in G. It follows that each of the 3-cycles created by a consecutive 3-cycle is not contained in a larger clique. Thus each of these 3-cycles is a clique itself, and every clique of size larger than 3 is an edge clique. We summarize this discussion in the following theorem.

Theorem 3.3. *Let G be a graph of minimum valence 3 and p a point in SG. Then the cliques of* TL(p), *listed by size, are as follows*:

(1) *there are no cliques of size* 1;

(2) *every clique of size* 2 *is a triclaw adjacency*;

(3) *every clique of size* 3 *is a VM, EM or VEM triangle created by a consecutive triangle in p, or the edge clique of an edge in G joining two vertices of valence* 3;

(4) *all cliques of size* 4 *or greater are edge cliques.*

4. The complete invariance of colored stratified graphs

We will show a graph G can be recovered in a canonical way from the link of any point in the stratified graph SG, if we are given the coloring of lines of SG as VM or EM. An edge uv in a graph G is *combinatorially contracted* by deleting the edge, identifying u and v, and removing any resulting multiple adjacencies.

Theorem 4.1. *Let G be any graph of minimum valence at least* 3 *and let p be any point in the stratified graph SG. Then G is isomorphic to the graph obtained from the link* TL(p) *by deleting all EM-points and then combinatorially contracting all VM-lines.*

Proof. The link of a vertex in a cartesian product is the disjoint union of the links of the coordinates of that vertex in the factors of the cartesian product. Therefore, the VM-link VL(p) consists of n disjoint graphs of the form described in Lemma 3.2, one for each of the n vertices of G. Since each of these graphs is connected, again by Lemma 3.2, each in different components of VL(p) corresponds to a vertex of G. Moreover, there is a line joining points in different components of VL(p) if and only if the corresponding vertices of G are joined by an edge (extra triclaw lines only join points in the same component of VL(p) and extra tricycle lines only join EM-points in

TL(p)). Thus if EM-points are deleted and VM-lines contracted, each component of VL(p) will contract to a single point, corresponding to a single vertex of G, and the points will be joined by lines if and only if the corresponding vertices in G are joined by edges. □

Corollary 4.2. *The VM/EM-colored stratified graph is a complete isomorphism invariant for simplicial graphs of minimum valence at least* 3.

If the graph at hand were not simplicial, one could also recover multiple adjacencies and self-adjacencies. The number of points in a component of VL(p) determines the degree of the corresponding vertex of G. The number of EM-lines joining different components of VL(p) determines the number of edges joining the corresponding vertices of G. Once the degree of each vertex and a number of multiple adjacencies have been determined, the number of self-adjacencies at each vertex is determined. The simplicial structure of G is already determined by Theorem 4.1. We therefore have the following corollary for non-simplicial graphs.

Corollary 4.3. *The VM/EM-colored stratified graph is a complete isomorphism invariant for all graphs of minimum valence at least* 3.

5. The uncolored stratified graph of a cubic graph

We would like to be able to recover G from its stratified graph SG without using the VM-coloring, but purely from the adjacency structure alone. In this section, we show how this can be done for a 3-regular simplicial graphs. We also consider the case when G has minimum valence 4.

Suppose that G is a 3-regular and simplicial and that p is a point in SG. Each component of VL(p) is a single point, corresponding to a reversal of rotation at some vertex v of G, which implies that in applying the conclusion of Theorem 4.1, no edge contractions are necessary. Since any two of these points of VL(p) are EM-adjacent if and only if their corresponding vertices are adjacent, and since no two points of VL(p) are VM-adjacent, it follows that the graph G is isomorphic to the subgraph TL(p) induced by the VM-points. The trouble is that, without seeing the VM-coloring, it is not obvious which points of TL(p) are VM-points. Nevertheless, the entire adjacency structure of TL(p) is not difficult to describe. Each EM-point in TL(p) is standardly adjacent to two VM-points: each edge-clique in TL(p) consists of a 3-cycle containing one EM-point (corresponding to moving both ends of the edge) and the two VM-points (corresponding to moving either end of the edge). There are no triclaw extra adjacencies. However, since every 3-cycle in a cubic graph is a consecutive triangle, for every 3-cycle t in G there is a 3-cycle of extra line of tricycle in TL(p) joining the three EM-points corresponding to moving both ends of each of the three edges of t. We can summarize this discussion as follows.

Theorem 5.1. *Let G be a 3-regular simplicial graph and p any point in SG. Then* TL(p) *is isomorphic to a graph obtained from G as follows: first double every edge; next, subdivide each new edge by inserting an extra vertex in its interior; then for each 3-cycle t of G, add an extra 3-cycle by mutually adjoining the three new vertices on the doubled edges. In particular, unless it is isomorphic to* K_4, *the graph G is isomorphic to the subgraph of* TL(p) *induced by the set of all 6-valent points that are themselves adjacent to at least three 6-valent points.*

Proof. The first assertion of the conclusion is simply a summary of the discussion immediately preceding the theorem. To verify the second assertion, we observe that the VM-point p_v corresponding to reversing the rotation at vertex v is EM-adjacent in TL(p) to the three VM-points corresponding to reversals of the rotations at the three respective neighbors of v and VM-adjacent in TL(p) to the three EM-points that correspond to reversing the rotations at both ends of the respective edges incident on v. Thus, not only is p_v 6-valent in TL(p), but also the three VM-points to which it is EM-adjacent are 6-valent in TL(p).

Now consider the EM-point p_{vw} of TL(p), corresponding to reversing the rotations at both ends of the edge vw of G. Its only VM-neighbors in TL(p) are p_v and p_w, both of which are 6-valent in TL(p). It has zero, two, or four EM-neighbours, depending on whether the edge vw lies on zero, one, or two 3-cycles in G. (More than 3-cycles would be impossible, since G is 3-regular.) In particular, the point p_{vw} is 6-valent if and only if the edge vw lies in two 3-cycles, say uvw and wvy. If p_{vw} has at least three 6-valent neighbors, then at least one of its EM-neighbors, say p_{vy}, is also 6-valent. It follows that in addition to the 3-cycle uvy, the edge vy lies on some other 3-cycle, say on vyz. It follows in turn that $z = u$, since otherwise the vertex v would have four distinct neighbors, viz., u, w, y and z. This leads to the conclusions that G is isomorphic to K_4. □

Corollary 5.2. *Let G be any 3-regular simplicial graph. Then every point in the stratified graph has the same link.*

Various authors (see, for example, [3]) have studied the question of which graphs can be the link in a constant link ('Zykov-regular') graph. Corollary 5.2 provides a large supply of such graphs: for any triangle-free, 3-regular simplicial graph G, double every edge of G, and insert a vertex of valence two in each added edge.

Corollary 5.3. *The uncolored, unlabeled stratified graph is a complete isomorphism invariant for 3-regular simplicial graphs.*

Proof. By Theorem 5.1, it suffices to distinguish the stratified graph K_4 from the stratified graphs for other cubic graphs. This is simply a matter of counting vertices: if G has n vertices, then SG has 2^n points. Therefore, K_4 is the only cubic graph whose stratified graph has 16 points. □

6. The general case of an uncolored stratified graph

Given any graph G, we recall that the *line graph* LG has the edges of G for its vertex set, and that two edges of G are considered to be adjacent if they share at least one endpoint. When G is simplicial, adjacent edges share exactly one endpoint.

For any imbedding p of the graph G, there is an interesting spanning subgraph of LG known as the *medial graph*, and denoted MG_p. Two edges of G are considered to be adjacent in MG_p if and only if those edges are consecutive in p at some vertex of G. Obviously, when G is 3-regular, all the medial graphs, regardless of the choice of an imbedding, are isomorphic to the line graph. In general, however MG_p depends on the imbedding p. It also depends on G, but does not determine G; for example, the medial graphs of an imbedding and its dual imbedding are isomorphic. For an interesting application of medial graphs to self-dual graphs, see [2]. Our next theorem shows how to recover the medial graph MG_p from the link of a point p in the uncolored stratified graph SG, for mst graphs G.

Theorem 6.1. *Let G be a graph of minimum valence 3 such that no 3-cycle in G contains more than 3-valent vertex. Then for any point p of SG, the medial graph MG_p is ismorphic to the subgraph of the clique graph of* $\mathrm{TL}(p)$ *induced by the subset of cliques that either include more than three points of* $\mathrm{TL}(p)$ *or include exactly three points, at least one of which is 2-valent.*

Proof. By Theorem 3.3, every clique in $\mathrm{TL}(p)$ of size greater than 3 is an edge-clique. Conversely, every edge in G incident to a vertex of valence greater than 3 gives rise to a clique in $\mathrm{TL}(p)$ of size greater than 3. By Theorem 3.3, every clique in $\mathrm{TL}(p)$ of size 3 containing a point of valence 2 is an edge clique corresponding to an edge in G between two vertices of valence 3. Since by hypothesis e does not lie on a 3-cycle in G, the point in $\mathrm{TL}(p)$ corresponding to moving both ends of edge e is not involved in any extra adjacencies and hence has valence two in $\mathrm{TL}(p)$. It follows that the edge-clique for e has size 3 and contains a point of valence 2. We conclude that the vertices in the graph constructed in the statement of this theorem correspond to the edges of G. Since the two edge-cliques in $\mathrm{TL}(p)$ share a vertex if and only if the corresponding edges are consecutive at some vertex in the imbedding p, the constructed graph is the medial graph MG_p. □

We believe that the restriction in Theorem 5.4 on triangles and vertices of valence 3 is not necessary, and that even for general G the clique structure decribed in Theorem 3.5 can be used to identify which cliques of size 3 in $\mathrm{TL}(p)$ are edge cliques. On the other hand, we do not see how to recover the original graph G, not just the medial graph, from $\mathrm{TL}(p)$ alone. It is conceivable that nonisomorphic graphs may have some isomorphic links in their uncolored stratified graphs. If that is the case, we cannot count alone on a local structure of the uncolored stratified graph SG to determine the isomorphic type of G. We nevertheless conjecture that the uncolored stratified graph is a complete isomorphism invariant.

7. Strata for two 'nearly isomorphic' graphs

To draw an entire stratified graph would be quite laborious. After all, the number of imbeddings of an n-vertex graph might be about as large as $(n!)^n$, the average VM-valence about n^3, and the average EM-valence abut n^4. Even to draw the strata tends to be a formidable task, and to compute the strata sequence of a graph is evidently more difficult than to compute the genus distribution, which is simply the sequence of strata sizes. However, if our objective is to distinguish isomorphism types, we cannot content ourselves with genus distributions.

Although Gross et al. [12] use elementary methods to construct arbitrarily many nonisomorphic 2-connected graphs with the same genus distribution, the construction of nonisomorphic 3-connected simplicial graphs with the same genus distribution was

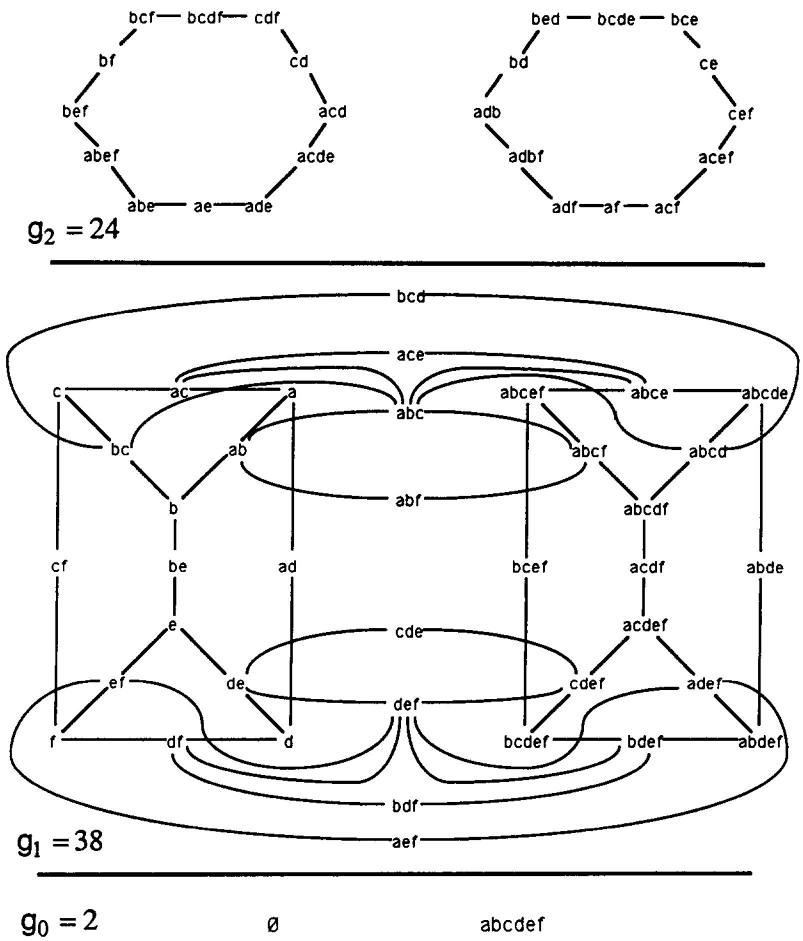

Fig. 1. The VM-strata of CL_3.

resistant until Rieper successfully used Redfield enumeration. Even if such examples were not known, the similarity in the genus distribution

2, 38, 34

of the circular ladder CL_3 with the three rungs (a.k.a. $K_2 \times K_3$, see [11]) and the genus distribution

0, 40, 24

of the Möbius ladder ML_3 on three rungs (a.k.a. $K_{3,3}$, see [11]) is disquieting, since one could not expect to distingusih the two easily with a small sample of imbeddings.

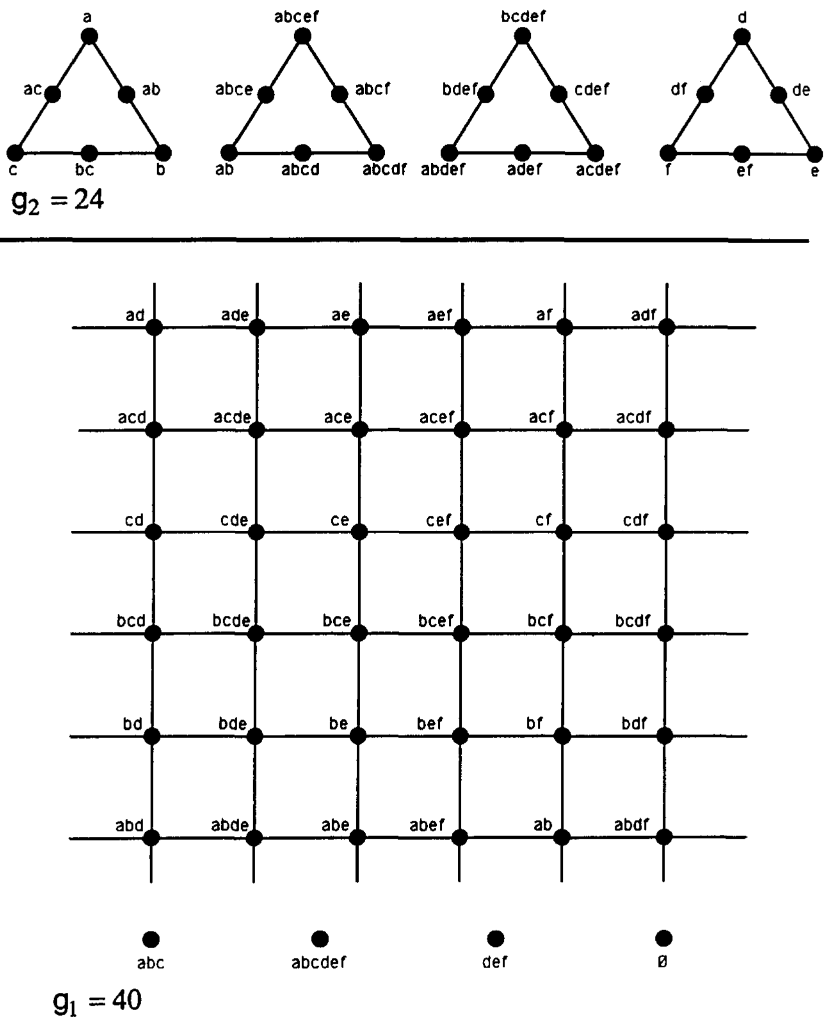

Fig. 2. The VM-Strata of ML_3.

Moreover, McGeoch [18] has proved that, in general, circular ladders and Möbius ladders with the same number of rungs have nearly identical genus distributions. In particular, they have the same number of imbeddings in all surfaces of genus two or larger, and differ elsewhere only in that the circular ladder has two sphere imbeddings and the Möbius ladder none, but two fewer toroidal imbeddings than the Möbius ladder.

Chen and Gross [4–6] investigate the occurrence of limit points in the set of values of average genus. They prove that every upper limit point represents an instance of 'ear-adding' and that there are no lower limit points. Moreover, they prove that each possible value of average genus is shared by at most finitely many cut-edges (a.k.a. bridgeless) graphs.

Having explained our motivation for examining such large objects, we now consider the VM-Strata of CL_3 and of ML_3. As illustrated by Figs. 1 and 2, the VM-strata are overtly different in various readily apparent respects. Details of the derivations of these illustrations are omitted because, although numerous, they are not difficult once the imbedding labels are explained. In Fig. 1, we imagine that the vertices on one 3-cycle of CL_3 are labeled *a*, *b*, and *c* and on the other 3-cycle *d*, *e*, and *f* so that *ad*, *be*, and *cf* are rungs. Then the imbedding label $\emptyset$ refers to a fixed imbedding of CL_3 in S_0, and each other imbedding label $v_1 \ldots v_r$ in Fig. 1 means the imbedding in which the rotations at vertices $v_1, \ldots, v_r$ have been reversed. In Fig. 2, recalling that $ML_3 \cong K_{3,3}$ we imagine that labels *a*, *b*, *c* are assigned to the three vertices in one part of the bipartition and labels *d*, *e*, *f* to the three vertices in the other part. Then imbedding label $\emptyset$ refers to a fixed 3-hexagon imbedding in S_1, and the convention for other imbedding labels is the same as for Fig. 1.

8. Algorithmic implications

We conclude this investigation by considering how stratified graphs relate to the determination of maximum and minimum genus and to the implicit program of Gross and Furst [11] for probabilistic isomorphism testing.

Gross and Rieper [13] have established that there are no false strict maxima in the uncolored stratified graph, which complements the polynomial-time maximum-genus algorithm of Furst et al. [9]. By way of contrast, they also construct arbitrarily deep local maxima, which complements the excellent result of Thomassen [31] that minimum genus is NP-complete.

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