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Randomized nonuniform sampling and reconstruction in fractional Fourier domain

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A B S T R A C T

The fractional Fourier transform (FRFT) is one of the most useful tools for the nonstationary signal processing. In this paper, the randomized nonuniform sampling and approximate reconstruction of the nonstationary random signals in the fractional Fourier domain (FRFD) are developed. The nonuniform samples are treated as random perturbations from a uniform grid. The samples used for the sinc interpolation reconstruction are placed on another nonuniform grid which is not necessarily equal to the samples originally acquired. When considering the second-order random statistic characters, the nonuniform sampling is equivalent to the uniform sampling of the signal after a pre-filter in the FRFD, where the frequency response is related to the characteristic function (with its argument scaled by csc α) of the perturbations. The effectiveness of the reconstruction is analyzed and the mean square error (MSE) is computed by utilizing the equivalent filter system. Furthermore, the randomized reconstruction of the chirp period stationary random signal is proposed. At last, the minimum MSE on the special cases of the randomized sampling and reconstruction is discussed. The effectiveness of the proposed reconstruction method is verified by the simulation.

1. Introduction

As the nonstationary signals, especially for the chirp signals, have compact support in the fractional Fourier domain (FRFD), the fractional Fourier transform (FRFT) became a powerful tool in the nonstationary signal processing [1–3]. The FRFT is a generalized form of the Fourier transform (FT), and its properties have been derived in a number of research papers [4–8]. The sampling theorem of the nonstationary signals plays an important role in digital signal processing, and arouses researchful interest in the literatures. The expansions of the uniform sampling theorem about the nonstationary signals in the FRFD have been investigated in [9–13].

Shannon sampling theorem is the classical uniform sampling theorem. However, sometimes samples cannot be collected uniformly in a variety of applications, such as in the fields of synthetic aperture Radar (SAR), astronomies and geophysics. The issue of approximate reconstruction using nonuniform samples or its transformed version has attracted much attention in the signal processing community [14–21]. Yao and Thomas [14] stated that when the sampling instants t_n did not derived by more than $T/4$ from a uniform grid with spacing of T, the bandlimited signal in the Fourier domain (FD) could be completely reconstructed by the Lagrange interpolation. The generalized nonuniform sampling theorem similar to [14] was also derived in the FRFD [15]. The complexity of the

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Lagrange interpolation motivated other much simpler approximate reconstruction methods. Tao et al. [17] derived the periodic nonuniform sampling and reconstruction in the FRFD at first, and other early works on nonuniform sampling and reconstruction in the FRFD were developed in [18–21]. These sampling theorems verified that if a signal was bandlimited or compact in a FRFD, it could be sampled by using the FRFT instead of the FT with a larger sampling interval. In fact, some signals have random characters. Some related results on random variables and fractional stationarity concepts in the FRFD were derived in [22–26]. The above generalized sampling theorems are suitable for the class of deterministic signals which have compact support in the FRFD. The reference [27] derived the sampling methods for random signals with known spectral densities in the mean square sense, including uniform and nonuniform sampling. The sampling theorem in the FRFD and its relation to the von Neumann ergodic theorem were also discussed in [28].

The reconstructions of the signals from nonuniform samples are more difficult to complete than that from uniform samples, and the complexity of the implementation of the nonuniform sampling theorem makes it difficult to be used in practice [15]. Although the periodic nonuniform sampling is a quite fascinating sampling theorem, not all of the nonuniform sampling cases can be modeled by it. In this paper, we propose a randomized nonuniform sampling and approximate sinc interpolation reconstruction method for the chirp stationary signals, which are nonstationary in the usual sense or in the FD [25]. The nonuniform samples are regarded as random perturbations from a uniform sampling grid in the analysis. Therefore, the nonuniform sampling sequence has random characters. On the basis of the second-order statistic theory, the randomized nonuniform sampling is equivalent to the uniform sampling of the signal after a pre-filter. The pre-filter is a fractional multiplicative filter, whose frequency response is the characteristic function (with its argument scaled by $\csc\alpha$) of the random sampling perturbations. Thus, the randomized sampling and approximate reconstruction can be considered by an equivalent filter system. The mean square error (MSE) which represents the performance of the reconstruction is analyzed by the equivalent system. Furthermore, the case of the chirp period stationary random signal is developed. Special cases of the sinc interpolation reconstruction are discussed and simulated at last.

The paper is organized as follows: the preliminaries are reviewed in Section 2. In Section 3, for nonstationary signals, nonuniform sampling with random perturbations and approximate reconstruction are proposed. Then the error of the reconstruction is analyzed in detail. Section 4 investigates the randomized sinc interpolation reconstruction of the chirp period stationary signals, which can be implemented by a finite summation. The special cases and simulation results of the reconstruction are discussed in Section 5. Conclusions are drawn in Section 6.

2. Preliminaries

2.1. The fractional Fourier transform

The FRFT of a signal $x(t)$ with angle α is defined as [1]

$$X_\alpha(u) = F^\alpha[x(t)](u) = \int_{-\infty}^{\infty} x(t)K_\alpha(t,u)\,dt, \tag{1}$$

where

$$K_\alpha(t,u) = \begin{cases} A_\alpha e^{j(1/2)(t^2+u^2)\cot\alpha - jut\csc\alpha}, & \alpha \neq n\pi \\ \delta(t-u), & \alpha = 2n\pi \\ \delta(t+u), & \alpha = (2n\pm 1)\pi, \end{cases} \tag{2}$$

and $A_\alpha = \sqrt{(1-j\cot\alpha)/2\pi}$. Obviously, the FRFT reduces to the identity transform and the conventional FT for $\alpha = 0$ and $\alpha = \pi/2$, respectively. The relationship between the FRFT and the FT is proposed in [6] as

$$F^\alpha[x(t)](u) = A_\alpha e^{j(1/2)u^2\cot\alpha} \cdot F^{\pi/2}[e^{j(1/2)t^2\cot\alpha} \cdot x(t)](u\csc\alpha), \tag{3}$$

where $F^{\pi/2}$ denotes the FT operation.

2.2. The fractional power spectral density and fractional multiplicative filter

$E\{\cdot\}$ is used to denote the mathematical expectation. For a zero-mean random signal $x(t)$, if the autocorrelation function of the signal does not change over time, i.e. $R_{xx}(t_1,t_2) = R_{xx}(\tau)|_{\tau=t_1-t_2}$, the signal is said to be *wide sense stationary* or *stationary* for short. The fractional autocorrelation function is defined as [22]

$$R^\alpha_{xx}(t_1,t_2) = E\{x(t_1)x^*(t_2)e^{jt_2(t_1-t_2)\cot\alpha}\} = R_{xx}(t_1,t_2)e^{jt_2(t_1-t_2)\cot\alpha}, \tag{4}$$

where the superscript $*$ means the conjugate operation. It is clear that when the signal $x(t)$ is nonstationary, $R^\alpha_{xx}(t_1,t_2)$ is also related to the instant time. If the chirp modulated form $\tilde{x}(t) = x(t)e^{j(1/2)t^2\cot\alpha}$ of the signal $x(t)$ is stationary, i.e. $R_{\tilde{x}\tilde{x}}(t_1,t_2) = R_{\tilde{x}\tilde{x}}(\tau)|_{\tau=t_1-t_2}$, the signal $x(t)$ is called the *α chirp stationary signal*. The definition results from the fact that $R^\alpha_{xx}(t_1,t_2) = R_{\tilde{x}\tilde{x}}(t_1,t_2)e^{(-j(1/2))(t_1-t_2)^2\cot\alpha}$, i.e. $R^\alpha_{xx}(t_1,t_2) = R^\alpha_{xx}(\tau)|_{\tau=t_1-t_2}$, which has been discussed in the reference [28]. The fractional power spectral density $P^\alpha_{xx}(u)$ and the fractional correlation function form a FRFT pair [22], i.e.

$$P^\alpha_{xx}(u) = A_{-\alpha}F^\alpha[R^\alpha_{xx}(\tau)](u) \cdot e^{-j(1/2)u^2\cot\alpha} \tag{5}$$

and

$$R^\alpha_{xx}(\tau) = \int_{-u_r}^{u_r} P^\alpha_{xx}(u)e^{ju\tau\csc\alpha}e^{-j(1/2)\tau^2\cot\alpha}\,du. \tag{6}$$

In particular, when $\alpha = \pi/2$, (5) becomes the Wiener–Khinchine theorem and (6) is the standard autocorrelation function.

For the multiplicative filter $h_\alpha(t)$ in the FRFD, let $x(t)$ and $y(t)$ be the random input and output of the filter, $P^\alpha_{xx}(u)$ and $P^\alpha_{yy}(u)$ be the fractional power spectrums of $x(t)$ and $y(t)$, respectively. We have the following input–output relationships [22,27]:

$$F^\alpha[y(t)](u) = F^\alpha[x(t)](u) \cdot H_\alpha(u), \tag{7}$$

and

$$P_{yy}^{\alpha}(u) = |H_\alpha(u)|^2 P_{xx}^{\alpha}(u), \tag{8}$$

where $H_\alpha(u)$ is the frequency response of the fractional filter, and $h_\alpha(t) = F^{-\pi/2}[H_\alpha(u)](t \csc\alpha)$.

2.3. The bandlimited random signal

A random signal $x(t)$ is bandlimited in the αth FRFD if its fractional power spectral density satisfies [27]

$$P_{xx}^{\alpha}(u) = 0, \quad |u| > u_r, \tag{9}$$

where u_r is the bandwidth of the signal in the FRFD. When $\alpha = \pi/2$, the random signal is bandlimited in the FD.

3. The randomized nonuniform sampling and reconstruction

First, a useful conclusion associated with the fractional bandlimited random signal is shown here for convenience.

Lemma 1. *Assume a zero-mean nonstationary random signal $x(t)$ is bandlimited to u_r in the αth FRFD, if its phase modulated signal $\tilde{x}(t) = x(t)e^{j(1/2)t^2 \cot\alpha}$ is stationary, $\tilde{x}(t)$ is a signal bandlimited to $u_r \csc\alpha$ in conventional FD (or frequency domain).*

Proof. Based on the definition of the fractional autocorrelation function (4), we have that [22]

$$R_{xx}^{\alpha}(\tau) = R_{\tilde{x}\tilde{x}}(\tau)e^{-j(1/2)\tau^2 \cot\alpha}\big|_{\tau = t_1 - t_2}. \tag{10}$$

According to the relationship between the correlation function and the power spectral density $P_{\tilde{x}\tilde{x}}(u) = (1/2\pi)\int_{-\infty}^{\infty} R_{\tilde{x}\tilde{x}}(\tau)e^{ju\tau}\, d\tau$, by utilizing (5), it is easy to obtain

$$P_{xx}^{\alpha}(u) = 2\pi A_{-\alpha}A_\alpha P_{\tilde{x}\tilde{x}}(u \csc\alpha). \tag{11}$$

Because $P_{xx}^{\alpha}(u)$ is bandlimited to u_r, the power spectral density $P_{\tilde{x}\tilde{x}}(u\csc\alpha)$ of the signal $\tilde{x}(t)$ is bandlimited to $u_r \csc\alpha$.□

3.1. Nonuniform sampling with random perturbations

For the nonstationary signal $x(t)$ which is bandlimited to u_r in the αth FRFD, the general nonuniform sampling theorem shows that if the sampling instants meet the following constraint [15]:

$$|t_n - nT_N| \le d < \frac{T_N}{4}, \quad d \in \mathbb{R}, \tag{12}$$

where $T_N = \pi/(u_r \csc\alpha)$ is the Nyquist sampling interval, the original signal can be reconstructed by the Lagrange interpolation formula as

$$x(t) = e^{-j(1/2)t^2 \cot\alpha} \sum_{n=-\infty}^{\infty} x(t_n)e^{j(1/2)t_n^2 \cot\alpha} \frac{G(t)}{G'(t_n)(t-t_n)}, \tag{13}$$

where

$$G(t) = e^{at}(t - t_0)\prod_{n \ne 0}\left(\frac{1-t}{t_n}\right)e^{t/t_n}, \quad a = \sum_{n \ne 0}\frac{1}{t_n}, \tag{14}$$

and $G'(t_n)$ is the derivative of $G(t)$ evaluated at $t = t_n$.

From the above description, the reconstruction of the fractional bandlimited signal from the nonuniform samples using the Lagrange interpolation requires a lot of computations as the interpolating functions have different forms at different sampling times. Motivated by the complexity of the implementation and the strict condition of (12), a simpler approximate reconstruction method will be developed in this paper. Before analyzing the reconstruction formula, the nonuniform sampling model is established first.

The zero-mean α chirp stationary random signal $x(t)$, which is bandlimited in the FRFD, is analyzed in the following. The nonuniform sampling sequence of $x(t)$ is denoted as $x[n]$, i.e. $x[n] = x(t_n)$. The set $\{t_n\}$ represents the nonuniform sampling grid which is modeled as random perturbations from a uniform grid, i.e. $t_n = nT + \xi_n$, which is described in Fig. 1. Where T denotes the average sampling interval and will be assumed not to exceed the Nyquist sampling interval in this paper, i.e. $T \le T_N$. $\{\xi_n\}$ is an independent identically distributed (i.i.d.) sequence with zero mean in the interval $\xi_n \in (-T/2, T/2)$. The probability density function (PDF) of the random variable ξ_n is $f_\xi(\xi)$. One natural distribution of ξ_n is uniform distribution, i.e. $f_\xi(\xi) = 1/T$, $\xi \in (-T/2, T/2)$, and $\int_{-T/2}^{T/2} f_\xi(\xi')d\xi' = 1$. It is also possible to imagine a truncated Gaussian distribution or other bounded distribution. The randomized sampling can be represented in Fig. 2.

Firstly, from the perspective of the second-order statistics analysis, the following theorem can be derived.

Theorem 1. *For a zero-mean α chirp stationary random signal $x(t)$, which is bandlimited in the FRFD, the nonuniform sampling with random perturbations of the signal can be equivalently represented by the uniform sampling after a prefilter $h_\alpha(t)$ shown in Fig. 3 with respect to the second-order statistics, where the filter is $h_\alpha(t) = F^{-\pi/2}[H_\alpha(u)](t\csc\alpha)$, T is the uniform sampling interval, $t_n = nT + \xi_n$ is the sampling*

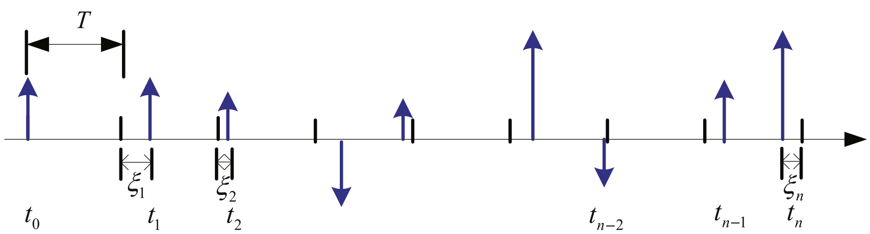

Fig. 1. The nonuniform sampling sequence.

instant time, $v(t)$ is the zero-mean additive noise and uncorrelated with $x(t)$. The fractional power spectral density of $v(t)$ is $P^{\alpha}_{vv}(u) = P^{\alpha}_{xx}(u)(1 - |H_{\alpha}(u)|^2)$, and $v[n] = v(t_n)$.

Proof. From the definition of the α chirp stationary signal, we know that the signal $\tilde{x}(t) = x(t)e^{(1/2)jt^2 \cot\alpha}$ is stationary. The randomized sampling can be represented by the form in Fig. 4. According to the fractional multiplicative filter, the input–output relationship of the fractional power spectrum in Fig. 3 is

$$P^{\alpha}_{yy}(u) = |H_{\alpha}(u)|^2 P^{\alpha}_{xx}(u). \tag{15}$$

As $v(t)$ is the zero-mean additive noise and uncorrelated with $x(t)$, the autocorrelation function of $z[n]$ is equivalent to that of $y[n]$. By utilizing the formula (6), the fractional autocorrelation function of $z[n]$ is expressed as

$$R^{\alpha}_{zz}(nT, (n-k)T) = \int_{-u_r}^{u_r} P^{\alpha}_{xx}(u)|H_{\alpha}(u)|^2 e^{jukT \csc\alpha} e^{-j(1/2)(kT)^2 \cot\alpha}\, du. \tag{16}$$

Therefore,

$$R_{\tilde{z}\tilde{z}}(nT, nT-kT) = R^{\alpha}_{zz}(nT, nT-kT)e^{j(1/2)(kT)^2 \cot\alpha} = \int_{-u_r}^{u_r} P^{\alpha}_{xx}(u)|H_{\alpha}(u)|^2 e^{jukT \csc\alpha}\, du. \tag{17}$$

On the other hand, because both of $x(t_n)$ and ξ_n are random variables, the autocorrelation function of $x[n]$ is given by

$$R^{\alpha}_{xx}(t_n, t_{n-k}) = E\{R^{\alpha}_{xx}(kT+\xi_n-\xi_{n-k})\} = \int_{-u_r}^{u_r} P^{\alpha}_{xx}(u)E\{e^{ju(kT+\xi_n-\xi_{n-k})\csc\alpha} e^{-j(1/2)(kT+\xi_n-\xi_{n-k})^2 \cot\alpha}\}\, du. \tag{18}$$

As $\tilde{x}(t) = x(t)e^{j(1/2)t^2 \cot\alpha}$ is a stationary random signal, by utilizing (10), the autocorrelation function of $\tilde{x}[n]$ is

$$E\{R_{\tilde{x}\tilde{x}}(kT+\xi_n-\xi_{n-k})\} = E\{R^{\alpha}_{xx}(kT+\xi_n-\xi_{n-k})e^{j(1/2)(kT+\xi_n-\xi_{n-k})^2 \cot\alpha}\}. \tag{19}$$

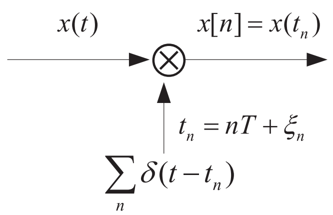

Fig. 2. The randomized sampling representation.

Using Eq. (6), the above formula can be rewritten as

$$E\{R_{\tilde{x}\tilde{x}}(kT+\xi_n-\xi_{n-k})\} = \int_{-u_r}^{u_r} P^{\alpha}_{xx}(u)e^{jukT \csc\alpha} E\{e^{ju(\xi_n-\xi_{n-k})\csc\alpha}\}\, du. \tag{20}$$

Based on the definition of the PDF, for the two i.i.d. random variables X and Y, the PDF of $Z = X + Y$ is the convolution of the PDFs of X and Y, i.e.

$$f_Z(z) = f_X \odot f_Y(z), \tag{21}$$

where $\odot$ is the classical convolution operation. As the perturbations ξ_n are i.i.d. random variables, the PDF of the term $Z = \xi_n - \xi_{n-k}$ can be obtained as

$$f_Z(\xi) = f_{\xi_n - \xi_{n-k}}(\xi) = f_{\xi}(\xi) \odot f_{\xi}(-\xi). \tag{22}$$

According to the definition of the mathematical expectation $E\{X\} = \int_{-\infty}^{\infty} x f_X(x)\, dx$, we have the result that

$$E\{e^{ju(\xi_n-\xi_{n-k})\csc\alpha}\} = \int_{-\infty}^{\infty} e^{ju\xi \csc\alpha} f_Z(\xi)\, d\xi = \int_{-\infty}^{\infty} e^{ju\xi \csc\alpha}[f_{\xi}(\xi) \odot f_{\xi}(-\xi)]\, d\xi. \tag{23}$$

The characteristic function of the random perturbations is defined as

$$\phi_{\xi}(u) = \int f_{\xi}(\xi')e^{ju\xi'}\, d\xi', \tag{24}$$

which is the FT of the PDF. By using the convolution theorem, the autocorrelation function in (20) can be deduced as

$$E\{R_{\tilde{x}\tilde{x}}(kT+\xi_n-\xi_{n-k})\} = \int_{-u_r}^{u_r} P^{\alpha}_{xx}(u)e^{jukT \csc\alpha} \int_{-\infty}^{\infty} e^{ju\xi \csc\alpha}[f_{\xi}(\xi) \odot f_{\xi}(-\xi)]d\xi\, du = \int_{-u_r}^{u_r} P^{\alpha}_{xx}(u)|\phi_{\xi}(u \csc\alpha)|^2 e^{jukT \csc\alpha}\, du. \tag{25}$$

Obviously, when the frequency response of the fractional filter equals to the characteristic function (with its argument scaled by $\csc\alpha$) of the random perturbations, i.e.

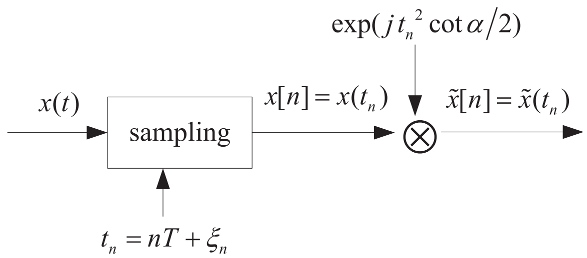

Fig. 4. Another form of the randomized sampling.

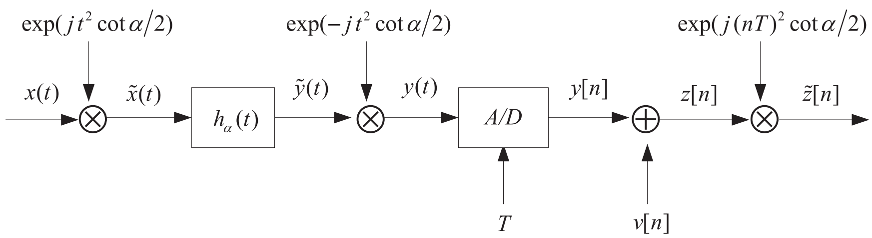

Fig. 3. The equivalent system of the randomized sampling.

$H_\alpha(u) = \phi_\xi(u \csc\alpha)$, Eq. (17) is equivalent to (25) which indicates that the autocorrelation function of the samples $\tilde{x}[n]$ in Fig. 4 is identical to that of the output in the system of Fig. 3.

Hence, in the sense of second-order statistic characters, the nonuniform sampling with random perturbations can be equivalent to the uniform sampling in the system of Fig. 3.□

According to Theorem 1, the PDF $f_\xi(\xi)$ can be designed to make $\phi_\xi(u \csc\alpha)$ act as an equivalent anti-aliasing low-pass filter in Fig. 3. Correspondingly, the random perturbations can still manifest themselves through the additive noise. Thus, the system in Fig. 3 suggests that aliasing can be traded off with uncorrelated noise by an appropriate design of the PDF of the random sampling perturbations.

3.2. Approximate reconstruction with randomized sinc interpolation

The reconstructions of the signals from nonuniform samples are more difficult to complete than that from uniform samples which have been explained before. In the case of uniform sampling, the Lagrange interpolation reduces to the sinc interpolation. Motivated by the fact that the sinc interpolation results in perfect reconstruction for uniform sampling, the approximate reconstruction of bandlimited random signal $x(t)$ (in the FD) from its nonuniform samples using randomized sinc interpolation is expressed as [29]

$$\hat{x}(t) = \frac{T}{T_N} \sum_{n=-\infty}^{\infty} x(t_n) \cdot h(t - \tilde{t}_n), \tag{26}$$

where $h(t) = \text{sinc}(t \cdot \pi/T_N)$, π/T_N is the bandwidth of the signal $x(t)$, and the samples used in the sinc interpolation are placed on the nonuniform grid $\tilde{t}_n = nT + \zeta_n$ which is not necessarily equal to the samples originally acquired. However, the theorem may lead to the wrong conclusion for the signal which is nonbandlimited in the FD, such as the chirp signals. As the nonbandlimited signals in the FD may be bandlimited in the FRFD for a certain value of angle α, the signal reconstruction by randomized sinc interpolation in the FRFD need to be proposed.

Theorem 2. *Let a zero-mean α chirp stationary random signal $x(t)$ is bandlimited to u_r in the FRFD, the approximate reconstruction of $x(t)$ from its nonuniform samples using randomized sinc interpolation is*

$$\hat{x}(t) = \frac{T}{T_N} e^{-j(1/2)t^2 \cot\alpha} \sum_{n=-\infty}^{\infty} x(t_n) e^{j(1/2)t_n^2 \cot\alpha} \cdot h(t - \tilde{t}_n), \quad n \in \mathbb{Z}, \tag{27}$$

where $h(t) = \text{sinc}(u_r t \csc\alpha)$, T denotes the nominal sampling interval, T_N is the Nyquist sampling interval, $\{t_n\}$ is the sampling instants sequence, and the samples used for interpolation are placed on another nonuniform grid $\tilde{t}_n = nT + \zeta_n$ that is not necessarily equal to the samples originally acquired, i.e. ζ_n is not necessarily same as ξ_n.

Proof. As the random signal $x(t)$ is α chirp stationary, its phase modulated signal $\tilde{x}(t) = x(t)e^{j(1/2)t^2 \cot\alpha}$ is stationary. As $x(t)$ is bandlimited to u_r, $\tilde{x}(t)$ is bandlimited to $u_r \csc\alpha$ in the FD from the Lemma 1. Thus, the signal $\tilde{x}(t)$ can be reconstructed by the randomized sinc interpolation based on Eq. (26),

$$\overline{x}(t) = \frac{T}{T_N} \sum_{n=-\infty}^{\infty} \tilde{x}(t_n) \cdot \text{sinc}\big((t - \tilde{t}_n)\pi/T_N\big), \tag{28}$$

where T denotes the nominal sampling interval, T_N is the Nyquist sampling interval, $u_r \csc\alpha = \pi/T_N$ is the bandwidth of the signal $\tilde{x}(t)$. The instant times are $t_n = nT + \xi_n$ and $\tilde{t}_n = nT + \zeta_n$, where ξ_n and ζ_n are random perturbations of sampling and reconstruction, respectively. It is worth noting that $\overline{x}(t) = \hat{x}(t)e^{j(1/2)t^2 \cot\alpha}$. And then substituting $\tilde{x}(t) = x(t)e^{j(1/2)t^2 \cot\alpha}$ and its bandwidth $u_r \csc\alpha$ into Eq. (28), (27) can be derived.

Theorem 2 shows that the reconstruction using the sinc interpolation with another randomized nonuniform grid can be represented as the system in Fig. 5. Nevertheless, the reconstruction with randomized sinc interpolation is an approximate method. How to evaluate the performance of the method will be given in the next section.

3.3. Mean square error analysis of the reconstruction

We consider the performance of the reconstruction from the perspective that the sampling and reconstruction process can be equivalent to the system whose frequency response is related to the PDF of the perturbations. Based on Theorem 1, the system in Fig. 6 is equivalent to the process which consists of nonuniform sampling discussed in Section 3.1 and the randomized sinc interpolation suggested in Fig. 5 in the sense of second-order statistics, when the average sampling rate meets or exceeds the Nyquist rate. In detail, the frequency response of the fractional filter can be obtained as $\phi_{\xi\zeta}(u \csc\alpha, -u \csc\alpha)$, which is the joint characteristic function of the random perturbations ξ_n and ζ_n. In addition, $v(t)$ is the zero-mean additive colored noise and uncorrelated with the original signal, with power spectrum:

$$P_{vv}(u) = \frac{T}{2\pi} \int_{-u_r}^{u_r} P_{xx}^{\alpha}(u_1) \left[1 - |\phi_{\xi\zeta}(u_1 \csc\alpha, -u)|^2\right] du_1,$$

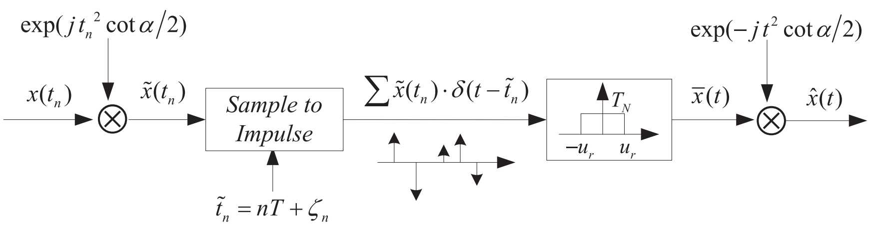

Fig. 5. The reconstruction with randomized sinc interpolation.

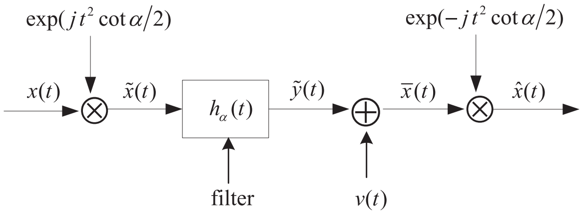

Fig. 6. The randomized sampling and reconstruction system.

$$|u| < u_r. \tag{29}$$

The equivalence will be proved afterward.

We denote $e(t) = \hat{x}(t) - x(t)$ as the error between $x(t)$ and its approximation $\hat{x}(t)$ obtained by the reconstruction with randomized sinc interpolation. The MSE of the randomized approximate reconstruction will be derived by the power spectrums of the original signal and the reconstruction according to the system in Fig. 6.

From Theorem 2, we have $\overline{x}(t) = (T/T_N) \sum_{n=-\infty}^{\infty} \tilde{x}(t_n) \cdot h(t - \tilde{t}_n)$. The autocorrelation function of $\overline{x}(t)$ is

$$\begin{aligned} R_{\overline{x}\overline{x}}(t, t-\tau) &= E\left\{ \frac{T}{T_N} \sum_{n=-\infty}^{\infty} \tilde{x}(t_n) \cdot h(t - \tilde{t}_n) \cdot \frac{T}{T_N} \sum_{k=-\infty}^{\infty} \tilde{x}^*(t_k) \cdot h^*(t-\tau-\tilde{t}_k) \right\} \\ &= \left(\frac{T}{T_N}\right)^2 E\left\{ \sum_{n=-\infty}^{\infty} \tilde{x}(nT+\xi_n) \cdot h(t-nT-\zeta_n) \cdot \sum_{k=-\infty}^{\infty} \tilde{x}^*(kT+\xi_k) \cdot h^*(t-\tau-kT-\zeta_k) \right\} \\ &= \left(\frac{T}{T_N}\right)^2 \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} E\{R_{\tilde{x}\tilde{x}}(nT-kT+\xi_n-\xi_k) \cdot h(t-nT-\zeta_n) h^*(t-\tau-kT-\zeta_k)\}. \end{aligned} \tag{30}$$

Furthermore, the above equation can be represented by the sum of the following two terms:

$$\begin{aligned} R_{\overline{x}\overline{x}}(t, t-\tau) &= \left(\frac{T}{T_N}\right)^2 R_{\tilde{x}\tilde{x}}(0) \sum_{n=-\infty}^{\infty} E\{h(t-nT-\zeta_n)h^*(t-\tau-nT-\zeta_n)\} \\ &+ \left(\frac{T}{T_N}\right)^2 \sum_{n \neq k} E\{R_{\tilde{x}\tilde{x}}(nT-kT+\xi_n-\xi_k) h(t-nT-\zeta_n) \cdot h^*(t-\tau-kT-\zeta_k)\}. \end{aligned} \tag{31}$$

Expressing $h(t)$ by $H(u)$ using their FT pair relationship

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} E\{h(t-nT-\zeta_n)h^*(t-\tau-nT-\zeta_n)\} \\ &= \left(\frac{1}{2\pi}\right)^2 \int\int H(u_1)H^*(u_2) e^{j(u_1-u_2)t} e^{ju_2\tau} \cdot \sum_n e^{j(u_2-u_1)nT} \cdot E\{e^{j(u_2-u_1)\zeta_n}\}\, du_1\, du_2, \end{aligned} \tag{32}$$

and by utilizing the following Poisson summation equivalent formula:

$$\sum_n e^{(j(u_2-u_1)nT)} = 2\pi \sum_k \delta((u_2-u_1)T - 2\pi k), \tag{33}$$

the first term of Eq. (31) is derived as

$$\begin{aligned} &\left(\frac{T}{T_N}\right)^2 R_{\tilde{x}\tilde{x}}(0) \sum_{n=-\infty}^{\infty} E\{h(t-nT-\zeta_n)h^*(t-\tau-nT-\zeta_n)\} \\ &= \frac{1}{2\pi}\left(\frac{T}{T_N}\right)^2 R_{\tilde{x}\tilde{x}}(0) \int_{-u_r \csc\alpha}^{u_r \csc\alpha} \frac{1}{T} |H(u)|^2 e^{ju\tau}\, du. \end{aligned} \tag{34}$$

The second term can be simplified as (the detailed calculating process is given in Appendix A)

$$\begin{aligned} &\left(\frac{T}{T_N}\right)^2 \sum_{n \neq k} E\{R_{\tilde{x}\tilde{x}}(nT-kT+\xi_n-\xi_k) h(t-nT-\zeta_n) h^*(t-\tau-kT-\zeta_k)\} = \int_{-u_r \csc\alpha}^{u_r \csc\alpha} e^{ju\tau} \cdot \Big[P_{\tilde{x}\tilde{x}}(u) |\phi_{\xi\zeta}(u,-u)|^2 \\ &- \frac{T}{2\pi} \int_{-u_r \csc\alpha}^{u_r \csc\alpha} P_{\tilde{x}\tilde{x}}(u_1) |\phi_{\xi\zeta}(u_1,-u)|^2 du_1 \Big]\, du. \end{aligned} \tag{35}$$

Substituting (34) and (35) into (31) results in

$$\begin{aligned} R_{\overline{x}\overline{x}}(t,t-\tau) &= \int_{-u_r \csc\alpha}^{u_r \csc\alpha} P_{\tilde{x}\tilde{x}}(u) |\phi_{\xi\zeta}(u,-u)|^2 e^{ju\tau}\, du \\ &+ \frac{T}{2\pi} \int_{-u_r \csc\alpha}^{u_r \csc\alpha} \left(\int_{-u_r \csc\alpha}^{u_r \csc\alpha} P_{\tilde{x}\tilde{x}}(u_1)(1 - |\phi_{\xi\zeta}(u_1,-u)|^2) du_1 \right) \cdot e^{ju\tau}\, du. \end{aligned} \tag{36}$$

Similar to the calculation in Appendix A, the cross-correlation function of $\overline{x}(t)$ and $\tilde{x}(t)$ can be expressed as

$$R_{\overline{x}\tilde{x}}(t,t-\tau) = \int_{-u_r \csc\alpha}^{u_r \csc\alpha} P_{\tilde{x}\tilde{x}}(u) \cdot \phi_{\xi\zeta}(u,-u) \cdot e^{ju\tau}\, du. \tag{37}$$

Hence, the corresponding power spectrums of the autocorrelation function and cross-correlation function can be obtained as

$$\begin{aligned} P_{\overline{x}\overline{x}}(u) &= P_{\tilde{x}\tilde{x}}(u) |\phi_{\xi\zeta}(u,-u)|^2 \\ &+ \frac{T}{2\pi} \int_{-u_r \csc\alpha}^{u_r \csc\alpha} P_{\tilde{x}\tilde{x}}(u_1) \left(1 - |\phi_{\xi\zeta}(u_1,-u)|^2\right) du_1, \end{aligned} \tag{38}$$

and

$$P_{\overline{x}\tilde{x}}(u) = P_{\tilde{x}\tilde{x}}(u) \cdot \phi_{\xi\zeta}(u,-u). \tag{39}$$

From the above theoretical results, it is clear that the first term in (38) is the power spectrum of $\tilde{y}(t)$ in the system of Fig. 6, when the frequency response of the filter is $\phi_{\xi\zeta}(u,-u)$. The power spectrum of $v(t)$ given in (29) is the same as the second term of (38). Therefore, the randomized sampling and approximate reconstruction process of the original signal can be equivalent to the system in Fig. 6 with respect to the second-order statistic character of the power spectrum.

Based on Eq. (11), the fractional auto-power spectrum can be obtained:

$$\begin{aligned} P^{\alpha}_{\hat{x}\hat{x}}(u) &= 2\pi A_\alpha A_{-\alpha} P_{\overline{x}\overline{x}}(u \csc\alpha) \\ &= 2\pi A_\alpha A_{-\alpha} P_{\tilde{x}\tilde{x}}(u \csc\alpha) |\phi_{\xi\zeta}(u\csc\alpha, -u\csc\alpha)|^2 \\ &+ T \cdot A_\alpha A_{-\alpha} \cdot \int_{-u_r \csc\alpha}^{u_r \csc\alpha} P_{\tilde{x}\tilde{x}}(u_1)(1 - |\phi_{\xi\zeta}(u_1, -u\csc\alpha)|^2)\, du_1 \end{aligned}$$

$$= P_{xx}^{\alpha}(u)\left|\phi_{\xi\zeta}(u \csc \alpha, -u \csc \alpha)\right|^2 + \frac{T \csc \alpha}{2\pi} \int_{-u_r}^{u_r} P_{xx}^{\alpha}(u_1)\left(1 - \left|\phi_{\xi\zeta}(u_1 \csc \alpha, -u \csc \alpha)\right|^2\right) du_1. \tag{40}$$

As the fractional cross-correlation function has the similar definition and properties with (11) of the fractional autocorrelation function [22], the cross-power spectrum can be expressed as

$$P_{\hat{x}x}^{\alpha}(u) = 2\pi A_\alpha A_{-\alpha} P_{\overline{x}\tilde{x}}(u \csc \alpha) = 2\pi A_\alpha A_{-\alpha} P_{\tilde{x}\tilde{x}}(u \csc \alpha) \cdot \phi_{\xi\zeta}(u \csc \alpha, -u \csc \alpha) = P_{xx}^{\alpha}(u) \cdot \phi_{\xi\zeta}(u \csc \alpha, -u \csc \alpha). \tag{41}$$

Consequently, the power spectrum of the reconstruction error in Fig. 6 is

$$P_{ee}^{\alpha}(u) = P_{\hat{x}\hat{x}}^{\alpha}(u) - P_{\hat{x}x}^{\alpha}(u) - P_{x\hat{x}}^{\alpha}(u) + P_{xx}^{\alpha}(u) = P_{xx}^{\alpha}(u)\left|\phi_{\xi\zeta}(u \csc \alpha, -u \csc \alpha)\right|^2 + \frac{T \csc \alpha}{2\pi} \int_{-u_r}^{u_r} P_{xx}^{\alpha}(u_1) \left(1 - \left|\phi_{\xi\zeta}(u_1 \csc \alpha, -u \csc \alpha)\right|^2\right) du_1 - P_{xx}^{\alpha}(u) \cdot \phi_{\xi\zeta}(u \csc \alpha, -u \csc \alpha) - [P_{xx}^{\alpha}(u) \cdot \phi_{\xi\zeta}(u \csc \alpha, -u \csc \alpha)]^* + P_{xx}^{\alpha}(u) = P_{xx}^{\alpha}(u)\left|1 - \phi_{\xi\zeta}(u \csc \alpha, -u \csc \alpha)\right|^2 + \frac{T \csc \alpha}{2\pi} \int_{-u_r}^{u_r} P_{xx}^{\alpha}(u_1)\left(1 - \left|\phi_{\xi\zeta}(u_1 \csc \alpha, -u \csc \alpha)\right|^2\right) du_1. \tag{42}$$

The MSE is given by integrating the power spectrum of the error over frequency

$$E\{e^2(t)\} = \int_{-u_r}^{u_r} P_{ee}^{\alpha}(u)\, du = \int_{-u_r}^{u_r} P_{xx}^{\alpha}(u)\left|1 - \phi_{\xi\zeta}(u \csc \alpha, -u \csc \alpha)\right|^2 du + \frac{T \csc \alpha}{2\pi} \int_{-u_r}^{u_r} P_{xx}^{\alpha}(u) \int_{-u_r}^{u_r} (1 - |\phi_{\xi\zeta}(u \csc \alpha, -u_1 \csc \alpha)|^2)\, du_1\, du. \tag{43}$$

From the result of (43), the MSE depends on the fractional power spectrum of the original signal and the joint characteristic function of the perturbations. Designing the optimal joint characteristic function can reduce the MSE.

4. The randomized reconstruction of the chirp period stationary random signal

The randomized nonuniform sampling and reconstruction for the chirp stationary signal in the FRFD has been obtained in the above section. From (27), the reconstruction can be regarded as the weighted summation of an infinite number of shifted sinc functions. The numerical implementation of this method on a computer is not possible because an infinite number of samples are needed. Moreover, only a finite number of samples can be acquired in the practical applications, which will not make the reconstruction as perfect as possible [30–32]. Masry [32] has analyzed the truncation error of the sampling expansion for the stationary bandlimited processes in detail. Nevertheless, for a special class of signals, i.e. the stationary random signals with periodicity, there is a similar but interesting reconstruction method. In this section, we will consider the randomized reconstruction of the chirp stationary random signal with periodicity, which is widely used in some important applications, especially in analyzing and processing the information got from the period time-varying character. Furthermore, we will investigate that this kind of reconstruction can be rewritten as a finite summation, which will avoid the error caused by the truncation.

For the more detailed analysis, we firstly develop the definition of the chirp period stationary random signal, which is similar to the definition of the chirp period deterministic signal in the FRFD [12], i.e. a random signal $x(t)$ is said to be α *chirp period stationary* with period T_p if its phase modulated signal $\tilde{x}(t) = x(t)e^{j(1/2)t^2 \cot \alpha}$ is stationary and

$$x(t)e^{j(1/2)t^2 \cot \alpha} = x(t+T_p)e^{j(1/2)(t+T_p)^2 \cot \alpha}. \tag{44}$$

Apparently the fractional autocorrelation function of the random signal also has the chirp period property with period T_p, i.e.

$$R_{xx}^{\alpha}(\tau)e^{j(1/2)\tau^2 \cot \alpha} = R_{xx}^{\alpha}(\tau+T_p)e^{j(1/2)(\tau+T_p)^2 \cot \alpha}. \tag{45}$$

We consider the period random sampling sequence $x(t_n) = x(nT+\xi_n)$, which satisfies the following property:

$$x(nT+\xi_n)e^{j(1/2)(nT+\xi_n)^2 \cot \alpha} = x(nT+\xi_n+T_p)e^{j(1/2)(nT+\xi_n+T_p)^2 \cot \alpha}. \tag{46}$$

For simplicity, the sampling interval $T = T_N = 1$ will be used and the reconstruction formula (27) is obtained as

$$\hat{x}(t) = e^{-j(1/2)t^2 \cot \alpha} \sum_{n=-\infty}^{\infty} x(n+\xi_n)e^{j(1/2)(n+\xi_n)^2 \cot \alpha} \cdot \frac{\sin(\pi t - n\pi - \zeta_n \pi)}{\pi t - n\pi - \zeta_n \pi}. \tag{47}$$

Further simplification is possible if the following equation is used:

$$\sin(\pi t - n\pi - \zeta_n \pi) = (-1)^n \sin(\pi t - \zeta_n \pi). \tag{48}$$

As a result, (47) is rewritten as

$$\hat{x}(t) = e^{-j(1/2)t^2 \cot \alpha} \sum_{n=-\infty}^{\infty} x(n+\xi_n)e^{j(1/2)(n+\xi_n)^2 \cot \alpha} \cdot \frac{(-1)^n}{\pi} \frac{\sin(\pi t - \zeta_n \pi)}{t - n - \zeta_n}. \tag{49}$$

For the α chirp stationary random signal $x(t)$ with finite samples N_0 in one period, from [15], rearranging the summation in (49), we obtain

$$\hat{x}(t) = e^{-j(1/2)t^2 \cot \alpha} \sum_{k=-\infty}^{\infty} \sum_{n=-L+kN_0}^{M-1+kN_0} x(n+\xi_n)e^{j(1/2)(n+\xi_n)^2 \cot \alpha} \cdot \frac{(-1)^n}{\pi} \frac{\sin(\pi t - \zeta_n \pi)}{t - n - \zeta_n} = e^{-j(1/2)t^2 \cot \alpha} \sum_{k=-\infty}^{\infty} \sum_{n=-L}^{M-1} x(n+kN_0+\xi_{n+kN_0})e^{j(1/2)(n+kN_0+\xi_{n+kN_0})^2 \cot \alpha} \cdot \frac{(-1)^{n+kN_0}}{\pi} \frac{\sin(\pi t - \zeta_{n+kN_0}\pi)}{t - n - kN_0 - \zeta_{n+kN_0}}, \tag{50}$$

where L and M are arbitrary integers that obey the relation

$L+M=N_0$. If the random perturbations meet the conditions $\xi_{n+kN_0}=\xi_n$ and $\zeta_{n+kN_0}=\zeta_n$, and the sampling sequence is periodic with period N_0, the reconstruction formula can be simplified as

$$\hat{x}(t)=e^{-j(1/2)t^2\cot\alpha}\sum_{k=-\infty}^{\infty}\sum_{n=-L}^{M-1}x(n+\xi_n)e^{j(1/2)(n+\xi_n)^2\cot\alpha}\cdot\frac{(-1)^{n+kN_0}}{\pi}\frac{\sin(\pi t-\zeta_n\pi)}{t-n-kN_0-\zeta_n}. \tag{51}$$

Exchanging the summation, formula (51) can be rewritten as

$$\hat{x}(t)=e^{-j(1/2)t^2\cot\alpha}\sum_{n=-L}^{M-1}x(n+\xi_n)e^{j(1/2)(n+\xi_n)^2\cot\alpha}(-1)^n\frac{\sin(\pi t-\zeta_n\pi)}{\pi}\cdot\sum_{k=-\infty}^{\infty}\frac{1}{t-n-kN_0-\zeta_n},\quad N_0\text{ even}, \tag{52}$$

$$\hat{x}(t)=e^{-j(1/2)t^2\cot\alpha}\sum_{n=-L}^{M-1}x(n+\xi_n)e^{j(1/2)(n+\xi_n)^2\cot\alpha}(-1)^n\frac{\sin(\pi t-\zeta_n\pi)}{\pi}\cdot\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{t-n-kN_0-\zeta_n},\quad N_0\text{ odd}. \tag{53}$$

As the inner summations are the decompositions into partial fraction of cotangent and cosecant, respectively, (52) and (53) can be rewritten as

$$\hat{x}(t)=e^{-j(1/2)t^2\cot\alpha}\sum_{n=-L}^{M-1}x(n+\xi_n)e^{j(1/2)(n+\xi_n)^2\cot\alpha}(-1)^n\frac{\sin(\pi t-\zeta_n\pi)}{N_0}\cdot\cot\left(\pi\frac{t-n-\zeta_n}{N_0}\right),\quad N_0\text{ even}, \tag{54}$$

$$\hat{x}(t)=e^{-j(1/2)t^2\cot\alpha}\sum_{n=-L}^{M-1}x(n+\xi_n)e^{j(1/2)(n+\xi_n)^2\cot\alpha}(-1)^n\frac{\sin(\pi t-\zeta_n\pi)}{N_0}\cdot\csc\left(\pi\frac{t-n-\zeta_n}{N_0}\right),\quad N_0\text{ odd}. \tag{55}$$

By utilizing the trigonometric relations, the above two formulas can be joined into one equation as

$$\hat{x}(t)=e^{-j(1/2)t^2\cot\alpha}\sum_{n=-L}^{M-1}x(n+\xi_n)e^{j(1/2)(n+\xi_n)^2\cot\alpha}(-1)^n\frac{\sin(\pi t-\zeta_n\pi)}{2N_0}\cdot\left[(-1)^{N_0+1}\tan\left(\pi\frac{t-n-\zeta_n}{2N_0}\right)+\cot\left(\pi\frac{t-n-\zeta_n}{2N_0}\right)\right]. \tag{56}$$

The analysis in this section can be concluded as Theorem 3.

Theorem 3. *For the zero-mean chirp period stationary random signal with period $T_p=N_0T$, which is bandlimited in the FRFD, the original signal can be approximately reconstructed by the sinc interpolation placed on $\tilde{t}_n=nT+\zeta_n$ from the finite random nonuniform samples at $t_n=nT+\xi_n$, $n=1,2,\dots,N_0$. The reconstruction formula is presented as* (56), *which is a finite summation that can be implemented by the computer.*

5. Discussion

5.1. Special cases of the randomized approximate reconstruction

In Section 3.3, we have analyzed the MSE of the randomized reconstruction and obtained the conclusion that the performance of the result depended on the joint characteristic function of the random perturbations. When the random perturbations of the sampling and reconstruction meet particular conditions, special forms of the reconstruction occur:

(1) *Chirp Modulated Uniform Sinc Interpolation* (*CMUSI*): when the perturbations in the interpolation are zero, i.e. $\zeta_n=0$, the interpolation is applied to the samples placed on the uniform grid which is described in Fig. 7, where the solid and dotted lines are the random sampling and reconstruction instants, respectively. So the reconstruction formula can be rewritten as

$$\hat{x}(t)=\frac{T}{T_N}e^{-j(1/2)t^2\cot\alpha}\sum_{n=-\infty}^{\infty}x(t_n)\cdot e^{j(1/2)(t_n)^2\cot\alpha}\mathrm{sinc}(u_r(t-nT)\csc\alpha), \tag{57}$$

where T is the nominal sampling interval which does not exceed the Nyquist sampling interval T_N. The reconstruction can be treated as the weighted sum of the samples modulated by the functions $e^{j(1/2)t_n^2\cot\alpha}\,\mathrm{sinc}(u_r(t-nT)\csc\alpha)$, thus it can be called the chirp modulated uniform sinc interpolation.

(2) *Chirp Modulated Independent Sinc Interpolation* (*CMISI*): when the perturbations ζ_n are independent of the sampling perturbations ξ_n, the reconstruction formula is the same as (27). The random sampling and

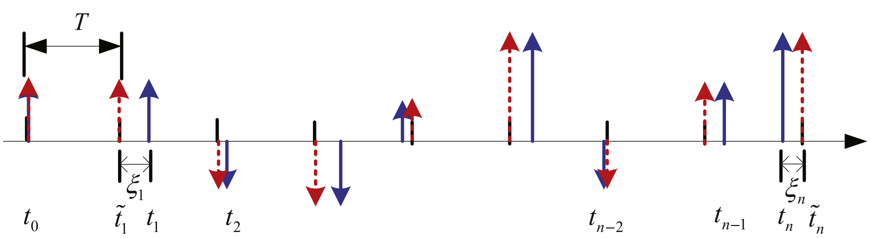

Fig. 7. The random sampling and reconstruction instants in CMUSI.

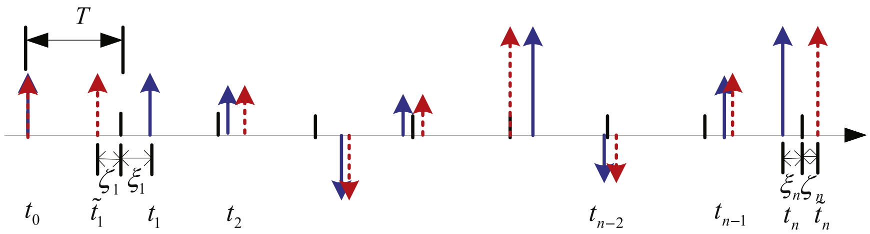

Fig. 8. The random sampling and reconstruction instants in CMISI.

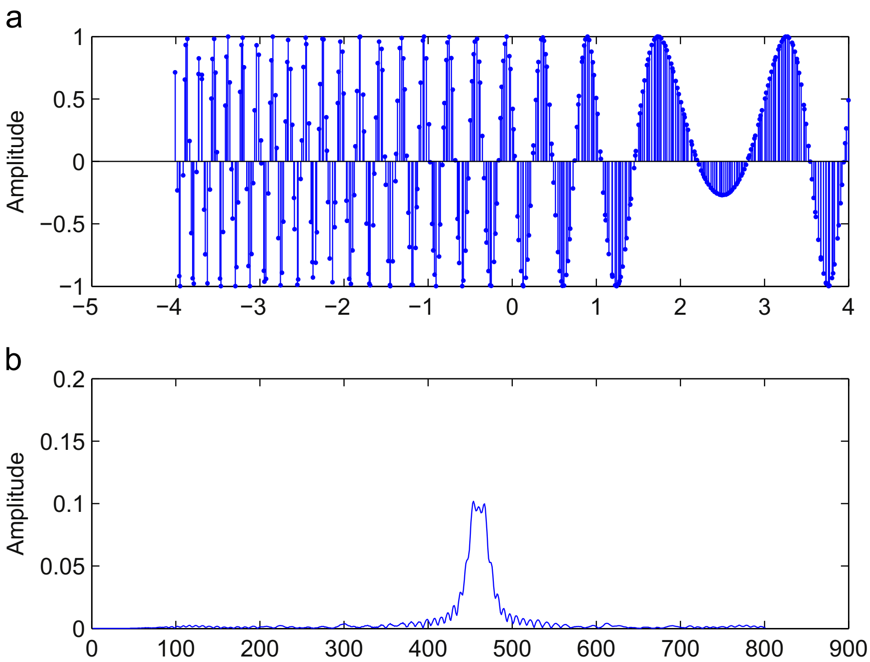

Fig. 9. (a) The samples. (b) The original fractional power spectrum.

reconstruction instants are presented in Fig. 8, where the solid and dotted lines are the random sampling and reconstruction instants, respectively.

(3) *Chirp Modulated Nonuniform Sinc Interpolation* (*CMNSI*): when the interpolation perturbations are just equivalent to the sampling perturbations, i.e. the adjacent solid and dotted lines in Fig. 8 are overlapping, by utilizing the sampling instants, the reconstruction is

$$\hat{x}(t)=\frac{T}{T_N}e^{-j(1/2)t^2\cot\alpha}\sum_{n=-\infty}^{\infty}x(t_n)e^{j(1/2)t_n^2\cot\alpha}\cdot\mathrm{sinc}(u_r(t-t_n)\csc\alpha). \tag{58}$$

For the chirp period stationary random signal, the reconstruction formula (56) also have the similar three special cases. Since the above forms have different kinds of the joint characteristic functions of the perturbations, the corresponding MSE will be different. Next, we will consider the case of small, zero-mean perturbations from a uniform grid and discuss the approximated characteristic function and the MSE of the reconstruction.

According to the relationship of the characteristic function and the statistical moment function

$$\frac{d^n\phi_\xi(u)}{du^n}\Big|_{u=0}=\int_{-\infty}^{\infty}f_\xi(\xi')(j\xi')^n d\xi'=j^nE\{\xi^n\}, \tag{59}$$

the joint characteristic function can be approximated by the second-order Taylor expansion as follows:

$$\phi_{\xi\zeta}(u_1,u_2)\approx 1-\sigma_{\xi\zeta}u_1u_2-\frac{1}{2}\sigma_\xi^2u_1^2-\frac{1}{2}\sigma_\zeta^2u_2^2, \tag{60}$$

where σ_ξ and σ_ζ are the corresponding standard deviations of the random variables ξ_n and ζ_n, respectively, and $\sigma_{\xi\zeta}$ is the joint standard deviation. Substituting (60) into (43) yields

$$E\{e^2(t)\}\approx\left(\sigma_\xi^2\cdot B_{xx}+\frac{1}{3}u_r^2\sigma_\zeta^2\right)(\csc\alpha)^3R_{xx}^\alpha(0), \tag{61}$$

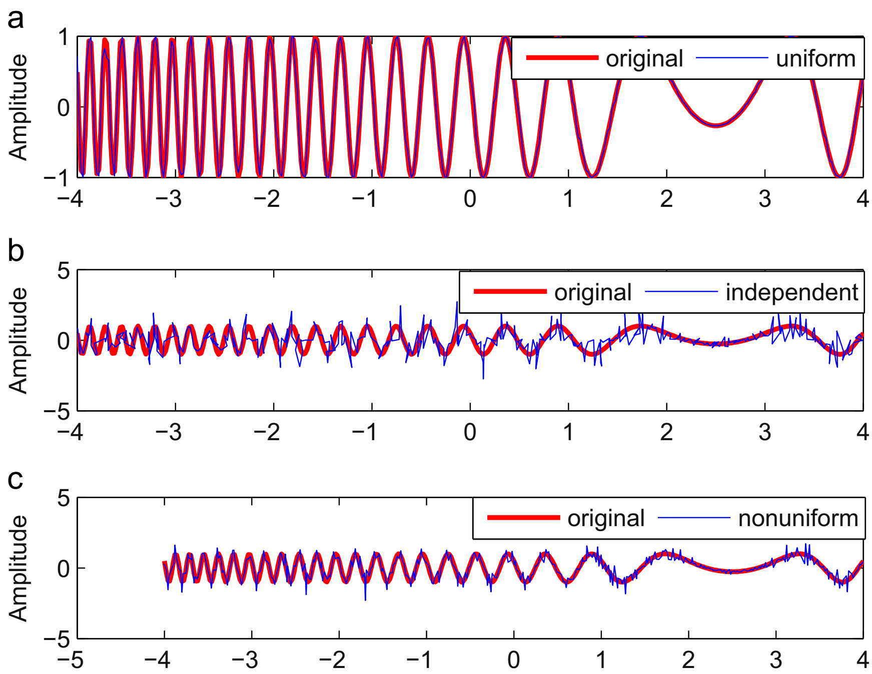

Fig. 10. The signals reconstructed by different methods. (a) CMUSI. (b) CMISI. (c) CMNSI.

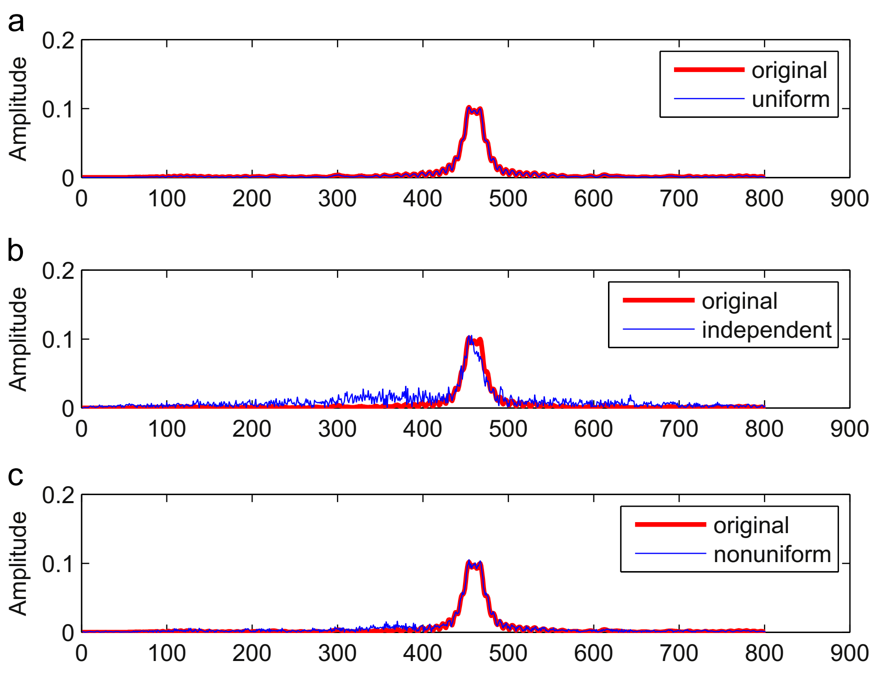

Fig. 11. The reconstructed fractional power spectrums compared with the original.

where B_{xx} is a measure of the bandwidth of the signal and defined as

$$B_{xx} = \int_{-u_r}^{u_r} u^2 \left(\frac{P_{xx}^{\alpha}(u)}{\int_{-u_r}^{u_r} P_{xx}^{\alpha}(u')\, du'} \right) du. \tag{62}$$

Formula (61) manifests that the MSE is independent of the detailed distribution characteristics of the perturbations and the spectrum of the signal. It is preferable to reconstruct the signal using the CMUSI, i.e. corresponding to $\zeta_n = 0$, as long as the perturbations around the uniform grid are small enough so that (60) holds.

Actually, the above is still an approximate solution, and it is not possible to claim which method is perfect in general. In fact, the randomized sinc interpolation in the FRFD derived in this paper is related to the exact sampling instants and the PDF of the perturbations. As the above three methods are all special cases of the randomized sinc interpolation, a lower MSE may be obtained when an appropriate joint PDF $f_{\xi\zeta}(\xi,\zeta)$ is chosen.

5.2. Simulation

In this section, a chirp stationary random signal $x(t)=\exp(jk_0\pi t+j\varphi-j\pi t^2)$ with the initial value $k_0=5$ and the initial uniformly distributed random phase φ is considered. Assume that the perturbations of the sampling all follow the uniform distribution in the interval $[-0.01, 0.01]$. The randomized samples of the original signal and the fractional power spectrum in the $\alpha=0.1578\pi/2$ FRFD are plotted in Fig. 9. According to the discussions in Section 5.1, Fig. 10 corresponds to the reconstruction results of the three different sinc interpolation methods compared with the original signal. In detail, the sampling rate in the CMUSI exceeds the Nyquist sampling rate. The perturbations which are used in the CMISI follow the normal distribution, i.e. $\zeta_n\sim N(0,1)$. The fractional power spectrums of the reconstructions are compared with that of the original signal in Fig. 11. From the results in Figs. 10 and 11, the reconstruction from the CMUSI is preferable than the other two methods. The simulation results are consistent to the analysis in Section 5.1.

6. Conclusion

For nonstationary signals, which are chirp stationary in the FRFD, a randomized nonuniform sampling and sinc interpolation reconstruction method is proposed in this paper. Where the nonuniform samples are regarded as the random perturbations of the uniform samples. The reconstruction based on the chirp modulated sinc interpolation uses the sampling instants from another random perturbations in average sampling period. The randomized sampling and reconstruction process have been proven to be equivalent to a fractional filter system in the sense of random statistic character. The frequency response of the filter is the characteristic function (with scaled argument) of the perturbations. The error of the approximate reconstruction can be controlled by the random perturbations at some level. The reconstruction of the chirp period stationary random signal is also discussed. Besides, our results can be extended to multichannel sampling theorem for multicomponent chirp stationary signals or to other transform domains.

Acknowledgment

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Appendix A. Proof of Eq. (35)

$$\left(\frac{T}{T_N}\right)^2\sum_{n\neq k}E\{R_{\tilde{x}\tilde{x}}(nT-kT+\xi_n-\xi_k)h(t-nT-\zeta_n)h^*(t-\tau-kT-\zeta_k)\}$$
$$=\left(\frac{1}{2\pi}\right)^2\left(\frac{T}{T_N}\right)^2\sum_{n\neq k}E\left\{\int_{-u_r\csc\alpha}^{u_r\csc\alpha}P_{\tilde{x}\tilde{x}}(u)e^{ju(nT-kT+\xi_n-\xi_k)}\,du\right.$$
$$\left.\cdot\int H(u_1)e^{ju_1(t-nT-\zeta_n)}du_1\int H^*(u_2)e^{-ju_2(t-\tau-kT-\zeta_k)}\,du_2\right\}$$
$$=\left(\frac{1}{2\pi}\right)^2\left(\frac{T}{T_N}\right)^2\sum_{n\neq k}\int_{-u_r\csc\alpha}^{u_r\csc\alpha}\int\int P_{\tilde{x}\tilde{x}}(u)H(u_1)H^*(u_2)e^{ju_2\tau}$$
$$\cdot e^{j(u_1-u_2)t}e^{j(u-u_1)nT}e^{-j(u-u_2)kT}$$
$$E\{e^{ju\xi_n}e^{-ju\xi_k}e^{-ju_1\zeta_n}e^{ju_2\zeta_k}\}\,du\,du_1\,du_2$$
$$=\left(\frac{1}{2\pi}\right)^2\left(\frac{T}{T_N}\right)^2\int_{-u_r\csc\alpha}^{u_r\csc\alpha}\int\int P_{\tilde{x}\tilde{x}}(u)H(u_1)H^*(u_2)\phi_{\xi\zeta}$$
$$(u,-u_1)\phi^*_{\xi\zeta}(u,-u_2)\cdot e^{ju_2\tau}e^{j(u_1-u_2)t}\sum_n e^{j(u-u_1)nT}$$
$$\sum_k e^{-j(u-u_2)kT}\,du\,du_1\,du_2-\left(\frac{1}{2\pi}\right)^2\left(\frac{T}{T_N}\right)^2$$
$$\int_{-u_r\csc\alpha}^{u_r\csc\alpha}\int\int P_{\tilde{x}\tilde{x}}(u)H(u_1)H^*(u_2)\phi_{\xi\zeta}(u,-u_1)\phi^*_{\xi\zeta}(u,-u_2)$$
$$\cdot e^{ju_2\tau}e^{j(u_1-u_2)t}\sum_n e^{j(u_2-u_1)nT}\,du\,du_1\,du_2 \quad (A.1)$$

$$\left(\frac{T}{T_N}\right)^2\sum_{n\neq k}E\{R_{\tilde{x}\tilde{x}}(nT-kT+\xi_n-\xi_k)h(t-nT-\zeta_n)h^*$$
$$(t-\tau-kT-\zeta_k)\}=\left(\frac{1}{T_N}\right)^2$$
$$\int_{-u_r\csc\alpha}^{u_r\csc\alpha}P_{\tilde{x}\tilde{x}}(u)|\phi_{\xi\zeta}(u,-u)|^2|H(u)|^2e^{ju\tau}\,du$$
$$-\frac{T}{2\pi}\left(\frac{1}{T_N}\right)^2\int_{-u_r\csc\alpha}^{u_r\csc\alpha}\int_{-u_r\csc\alpha}^{u_r\csc\alpha}P_{\tilde{x}\tilde{x}}(u_1)|\phi_{\xi\zeta}(u_1,-u)|^2$$
$$|H_\alpha(u)|^2e^{ju\tau}\,du_1\,du \quad (A.2)$$

$$\left(\frac{T}{T_N}\right)^2\sum_{n\neq k}E\{R_{\tilde{x}\tilde{x}}(nT-kT+\xi_n-\xi_k)h(t-nT-\zeta_n)h^*$$
$$(t-\tau-kT-\zeta_k)\}=\left(\frac{1}{T_N}\right)^2\int_{-u_r\csc\alpha}^{u_r\csc\alpha}|H(u)|^2e^{ju\tau}$$
$$\cdot\left[P_{\tilde{x}\tilde{x}}(u)|\phi_{\xi\zeta}(u,-u)|^2\right.$$
$$\left.-\frac{T}{2\pi}\int_{-u_r\csc\alpha}^{u_r\csc\alpha}P_{\tilde{x}\tilde{x}}(u_1)|\phi_{\xi\zeta}(u_1,-u)|^2\,du_1\right]du \quad (A.3)$$

$$\left(\frac{T}{T_N}\right)^2\sum_{n\neq k}E\{R_{\tilde{x}\tilde{x}}(nT-kT+\xi_n-\xi_k)h(t-nT-\zeta_n)h^*(t-\tau-kT-\zeta_k)\}$$
$$=\int_{-u_r\csc\alpha}^{u_r\csc\alpha}e^{ju\tau}\cdot\left[P_{\tilde{x}\tilde{x}}(u)|\phi_{\xi\zeta}(u,-u)|^2\right.$$
$$\left.-\frac{T}{2\pi}\int_{-u_r\csc\alpha}^{u_r\csc\alpha}P_{\tilde{x}\tilde{x}}(u_1)|\phi_{\xi\zeta}(u_1,-u)|^2\,du_1\right]du. \quad (A.4)$$

From (A.1) to (A.2), the Poisson summation formula (33) is utilized. Because $|H(u)|^2=(T_N)^2$, (A.3) is reduced to (A.4).

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