JOURNAL OF ALGEBRA 22, 241–253 (1972)

Modules with Decompositions That Complement Direct Summands

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Communicated by A. W. Goldie

Received January 31, 1971 (revised April 22, 1971)

In this paper we study a generalization of a fundamental property of semisimple modules. A decomposition

$$M = \bigoplus_A M_\alpha$$

of a module M as a direct sum of nonzero submodules $(M_\alpha)_{\alpha \in A}$ is said to *complement direct summands* in case for each direct summand K of M there is a subset $B \subseteq A$ with

$$M = K \oplus \left(\bigoplus_B M_\beta\right).$$

It is easy to see that such a decomposition is automatically an *indecomposable decomposition*; that is, one in which each term M_α is indecomposable.

Every semisimple module admits a decomposition (with simple terms) that complements every submodule. Indeed this property characterizes such modules. Here we shall be concerned with modules that admit a decomposition that complements direct summands. A ring is (artinian) semisimple if and only if each of its left modules is semisimple. More generally it would be of interest to characterize those rings over which all (left) modules have decompositions that complement direct summands. We do not attempt to obtain such a characterization here; rather (recalling that over semisimple rings all modules are projective and injective), we characterize those rings over which every projective left module has such a decomposition and those over which

* Research partially supported by National Science Foundation grant GP-18828.

every injective left module has such a decomposition. These are the left perfect rings and the left noetherian rings, respectively. This provides at least a partial answer to the general problem: if every module over a ring has a decomposition that complements direct summands, then the ring is artinian.

Perhaps the most important theorem on (the large) direct decompositions of modules is Azumaya's extension of the Krull–Schmidt Theorem [1]. Before stating it we recall that two direct decompositions of a module

$$M = \bigoplus_A M_\alpha = \bigoplus_B N_\beta$$

are equivalent in case there is a bijection $\sigma : A \to B$ with

$$M_\alpha \cong N_{\sigma(\alpha)} \qquad (\alpha \in A).$$

THEOREM [Azumaya]. *If a module M has a direct decomposition $M = \bigoplus_A M_\alpha$ such that for each $\alpha \in A$ the endomorphism ring* end(M_α) *is a local ring, then*

(1) *every nonzero direct summand of M has an indecomposable direct summand*;

(2) *if $M = K \oplus N$ with N indecomposable, then there is a $\beta \in A$ such that $M = K \oplus M_\beta$*;

(3) *every indecomposable decomposition of M is equivalent to the given decomposition $M = \bigoplus_A M_\alpha$.*

We say that a direct decomposition $M = \bigoplus_A M_\alpha$ of a module that satisfies condition (2) of Azumaya's Theorem *complements maximal direct summands.* Of course any decomposition of a module that complements direct summands must also complement maximal direct summands. As we shall see the converse fails.

Suppose that a module M has a direct decomposition $M = \bigoplus_A M_\alpha$ that complements maximal direct summands. Then a routine inductive argument shows that for every decomposition

$$M = K \oplus (N_1 \oplus \cdots \oplus N_n)$$

with each $N_1 ,..., N_n$ indecomposable, there exist $\alpha_1 ,..., \alpha_n \in A$ such that

$$M_{\alpha_i} \cong N_i \qquad (i = 1,..., n)$$

and

$$M = K \oplus (M_{\alpha_1} \oplus \cdots \oplus M_{\alpha_n}).$$

Of course a module need not have any indecomposable direct summands (e.g., the regular representation of the ring of continuous two-valued functions on the rationals) and so for such a module every decomposition complements maximal direct summands.

It is easy to see that a direct sum of homologically independent modules $(H_\gamma)_{\gamma \in C}$ (i.e., $\hom(H_\gamma, H_\delta) = 0$ whenever $\gamma \neq \delta$) has a decomposition that complements (maximal) direct summands if each H_γ has such a decomposition. And clearly any indecomposable module has a decomposition complementing direct summands. Thus in particular, for a decomposition $M = \bigoplus_A M_\alpha$ to complement direct summands it is not necessary that the endomorphism rings $\operatorname{end}(M_\alpha)$ be local. Nevertheless, as we shall show, for an indecomposable decomposition, condition (2) of Azumaya's Theorem does imply condition (3). First, however, we have

1. Lemma. *Suppose M has a decomposition $M = \bigoplus_A M_\alpha$ that complements (maximal) direct summands. If $A' \subseteq A$, then the decomposition*

$$M' = \bigoplus_{A'} M_{\alpha'}$$

of M' complements (maximal) direct summands. Moreover, if M has a decomposition that complements direct summands, then so does every direct summand of M.

Proof. Suppose that $A' \subseteq A$ and that $M' = \bigoplus_{A'} M_{\alpha'}$. Suppose also that K is a (maximal) direct summand of M'. Then

$$K \oplus \left(\bigoplus_{A \setminus A'} M_\alpha \right)$$

is a (maximal) direct summand of $M = \bigoplus_A M_\alpha$; so by hypothesis there is a subset $B' \subseteq A$ such that

$$M = K \oplus \left(\bigoplus_{A \setminus A'} M_\alpha \right) \oplus \left(\bigoplus_{B'} M_{\beta'} \right).$$

But then clearly we must have $B' \subseteq A'$ and

$$M' = K \oplus \left(\bigoplus_{B'} M_{\beta'} \right).$$

This proves the first statement. The last statement now follows from the first in view of the fact that if

$$M = \bigoplus_A M_\alpha ,$$

and if $B \subseteq A$ with

$$M = N \oplus \left(\bigoplus_{B} M_\beta \right),$$

then

$$N \cong \bigoplus_{A \backslash B} M_\alpha .$$

We do not know whether the last assertion of this lemma holds for decompositions complementing maximal direct summands. The best we can say is that if

$$M = \bigoplus_{A} M_\alpha$$

with each $\mathrm{end}(M_\alpha)$ a local ring, and if either A is finite [1] or each M_α is countably generated [12], then the direct summands of M all have decompositions that complement maximal direct summands.

2. THEOREM. *If a module M has an indecomposable decomposition that complements maximal direct summands, then all indecomposable decompositions of M are equivalent.*

Proof. Suppose that $M = \oplus_A M_\alpha$ and $M = \oplus_C N_\gamma$ are indecomposable decompositions and that the first complements maximal direct summands. For each indecomposable module L set

$$A(L) = \{\alpha \in A \mid M_\alpha \cong L\} \qquad \text{and} \qquad C(L) = \{\gamma \in C \mid N_\gamma \cong L\}.$$

Then virtually copying from the proof of Azumaya's Theorem [1] we see that

$$\mathrm{card}\ A(L) \geqslant \mathrm{card}\ C(L).$$

In particular, there is an injection $\sigma : C \to A$ such that

$$N_\gamma \cong M_{\sigma(\gamma)} \qquad (\gamma \in C).$$

Therefore there is an isomorphism

$$f : M = \bigoplus_{C} N_\gamma \to \bigoplus_{C} M_{\sigma(\gamma)}$$

such that $f(N_\gamma) = M_{\sigma(\gamma)}$ $(\gamma \in C)$. Thus, by Lemma 1, the decomposition $M = \oplus_C N_\gamma$ also complements maximal direct summands. But then reversing the roles of A and C we have that for each indecomposable module L,

$$\mathrm{card}\ A(L) \leqslant \mathrm{card}\ C(L).$$

Thus for each L we have equality, and the proof is complete.

We note that this is very nearly identical with Azumaya's proof [1] of condition (3) of his theorem. The difference is that Azumaya obtained the second inequality, card $A(L) \leqslant$ card $C(L)$, by observing that condition (2) and the hypothesis of local endomorphism rings for the M_α force local endomorphism rings for the N_γ.

It is clear that if two decompositions of a module are equivalent and if one of them complements (maximal) direct summands, then so does the other. Therefore we have at once the following corollary.

3. COROLLARY. *If a module M has an indecomposable decomposition $M = \bigoplus_A M_\alpha$ that complements (maximal) direct summands, then every indecomposable decomposition of M complements (maximal) direct summands.*

As we noted earlier the hypothesis of local endomorphism rings is not necessary for an indecomposable decompositions to complement maximal direct summands. However, using a clever argument of Warfield [12] we see that it is almost necessary; for a consequence of the following result is that if both M and $M^{(2)} = M \times M$ have indecomposable direct decompositions that complement maximal summands, then the endomorphism rings of the terms in these decompositions must be local.

4. PROPOSITION. *Let $M = \bigoplus_A M_\alpha$ be an indecomposable decomposition that complements maximal direct summands. If M_α appears at least twice in this decomposition* (i.e., *if there is a $\beta \neq \alpha$ in A such that $M_\beta \cong M_\alpha$*), *then* end(M_α) *is a local ring.*

Proof. According to Lemma 1 it will suffice to show that if M is indecomposable and the decomposition

$$M^{(2)} = M_1 \oplus M_2,$$

where

$$M_1 = \{(m, 0) \mid m \in M\} \qquad \text{and} \qquad M_2 = \{(0, m) \mid m \in M\},$$

complements maximal direct summands, then end(M) is local. We shall do this by using Warfields argument in the proof of [12, Proposition 1]. So suppose that the above decomposition complements maximal direct summands and let

$$\pi_i : M^{(2)} \to M \quad (i = 1, 2)$$

be the natural projections with $\ker \pi_i = M_j$, $i \neq j$. Now let $f, g \in$ end(M) with

$$f - g = 1_M.$$

Set

$$M' = \{(f(m), g(m)) \mid m \in M\} \qquad \text{and} \qquad M_d = \{(m, m) \mid m \in M\}.$$

Then, as Warfield observed

$$M^{(2)} = M' \oplus M_d ,$$

whence M' is a maximal direct summand of M^2. Thus, either

$$M^{(2)} = M' \oplus M_1 \qquad \text{or} \qquad M^{(2)} = M' \oplus M_2 ,$$

and therefore either

$$\pi_1 |_{M'} : M' \to M \qquad \text{or} \qquad \pi_2 |_{M'} : M' \to M$$

is an isomorphism. That is, either f or g is an automorphism; so end(M) is a local ring.

A ring R is semiperfect (see [2]) if and only if the identity of R is a sum of pairwise orthogonal idempotents

$$1 = e_1 + \cdots + e_n$$

such that e_iRe_i is a local ring for each $i = 1,..., n$ (see [9], or the proof of [10, Lemma 3] and [11]). Also Müller [9], Klatt [7], and Warfield [12], generalizing a theorem of Kaplansky [6], have each proved that if R is semiperfect then every projective left R-module P has a decomposition

$$P = \bigoplus_A M_\alpha ,$$

where each of the summands M_α is isomorphic to one of the primitive left ideals Re_i. Thus, applying Azumaya's Theorem, we see that every projective left module over a semiperfect ring has an indecomposable decomposition that complements maximal direct summands. As we shall see, unless the ring is left perfect, such an indecomposable decomposition need not complement all direct summands. However, for finitely generated modules we do have the following result.

5. THEOREM. *For a ring R the following statements are equivalent:*

(a) *R is semiperfect.*

(b) *Every projective (left) R-module has an indecomposable decomposition that complements maximal direct summands.*

(c) *Every finitely generated projective (left) R-module has a decomposition that complements direct summands.*

(d) *The free (left) R-module $R^{(2)}$ has a decomposition that complements direct summands.*

Proof. (a) ⇒ (b). This follows from the preceding remarks.

(b) ⇒ (c). Assume (b). Then if P is a finitely generated projective R-module, every direct summand of P is a finite direct sum of indecomposable modules. Thus, a decomposition of P that complements maximal direct summands must complement all direct summands.

(c) ⇒ (d). This is clear.

(d) ⇒ (a). Assume (d). Then by Lemma 1 the direct summand R of $R^{(2)}$ also has a decomposition that complements direct summands. Say

$$R = Re_1 \oplus \cdots \oplus Re_n$$

is such a decomposition. Then the decomposition

$$R^{(2)} = Re_1 \oplus Re_1 \oplus \cdots \oplus Re_n \oplus Re_n$$

complements direct summands by Corollary 3. Since every term in this latter decomposition appears at least twice, we have by Proposition 4 that the endomorphism rings e_iRe_i of the Re_i must be local. Thus, R is semiperfect.

For arbitrary projective modules we have the following result. In its proof we freely use Bass's well-known results on perfect rings in [2].

6. THEOREM. *For a ring R the following statements are equivalent*:

(a) *R is left perfect.*

(b) *Every projective left R-module has a decomposition that complements direct summands.*

(c) *The free left R-module on countably many free generators has a decomposition that complements direct summands.*

Proof. (a) ⇒ (b). Let R be left perfect with (Jacobson) radical J, and let P be a projective left R-module. Then there exist primitive submodules M_α $(\alpha \in A)$ of P such that

$$P = \bigoplus_A M_\alpha .$$

We claim that this decomposition complements direct summands. Indeed, let

$$P = P' \oplus P''.$$

Since P/JP is semisimple,

$$P/JP = \bigoplus (M_\alpha + JP)/JP$$

complements direct summands. Thus, since $(P' + JP)/JP$ is a direct summand of P/JP, there is a subset $B \subseteq A$ such that

$$P/JP = ((P' + JP)/JP) \oplus \left(\bigoplus_B (M_\beta + JP)/JP\right).$$

We claim that $P = P' \oplus (\oplus_B M_\beta)$. Clearly,

$$P = P' + \left(\sum_B M_\beta\right) + JP.$$

So since JP is superfluous in P, we have $P = P' + \sum_B M_\beta$. Set

$$L = \sum_B M_\beta .$$

Then to establish our claim it will suffice to show that $P' \cap L = 0$. Since P' and L are direct summands of P, we have that

$$JP' = P' \cap JP \qquad \text{and} \qquad JL = L \cap JP;$$

thus, it is clear that

$$P' \cap L \subseteq JP' \cap JL.$$

Now

$$P' \times L \xrightarrow{f} P = P' + L$$

via

$$f : (p', l) \to p' + l$$

is a split epimorphism whose kernel is

$$K = \{(p', -p') \mid p' \in P' \cap L\}.$$

Since $J(P' \times L) = JP' \times JL$, we have that $K \leqslant J(P' \times L)$, whence K is superfluous in $P' \times L$. Thus, $K = 0$. Therefore, $P' \cap L = 0$, and our claim is established.

(b) $\Rightarrow$ (c). This is clear.

(c) $\Rightarrow$ (a). Now suppose that the free module on countably many generators has a decomposition that complements direct summands. Then by Theorem 5 and Lemma 1 we have that R is semiperfect. So it will suffice to prove that the radical J of R is left T-nilpotent. Suppose J is not left T-nilpotent. Then there is a sequence $b_1, b_2, \ldots$ in J such that

$$b_1 b_2 \cdots b_n \neq 0 \qquad (n = 1, 2, \ldots).$$

Since R is semiperfect, it has an orthogonal set $e_1, e_2, \ldots, e_m$ of primitive idempotents with

$$1 = e_1 + e_2 + \cdots + e_m .$$

Thus, as Müller has shown [9], since

$$1b_1 1b_2 1 \cdots 1b_n 1 \neq 0 \qquad (n = 1, 2, \ldots),$$

it follows from the König Graph Theorem that there is at least one of the idempotents $e_1, e_2, \ldots, e_m$, say e, and a sequence $a_1, a_2, \ldots$ in eJe with

$$a_1 a_2 \cdots a_n \neq 0 \qquad (n = 1, 2, \ldots).$$

(Note that here each a_i is of the form

$$eb_{i_1} e_{j_1} b_{i_1+1} e_{j_2} \cdots b_{i_1+k} e.)$$

We shall obtain a contradiction by using a modification of an argument given by Bass [2]. Let F be the free left R-module with free basis $z_1, z_2, \ldots$. For each n set

$$x_n = ez_n .$$

Then we have a direct summand P of F with a decomposition

$$P = \bigoplus_{n=1}^{\infty} Rx_n$$

that, by Lemma 1 and Corollary 3, complements direct summands. For each n let

$$\pi_n : P \to Rx_n$$

be the projection on Rx_n along $\sum_{m \neq n} Rx_m$. Next, for each n set

$$y_n = x_n - a_n x_{n+1} .$$

Then clearly,

$$Ry_n \cong Rx_n \cong Re;$$

so each Ry_n and each Rx_n is indecomposable. It is easy to check that

$$\begin{aligned} P &= \bigoplus_{n=1}^{\infty} Rx_n \\ &= \left(\bigoplus_{n=1}^{\infty} Rx_{2n}\right) \oplus \left(\bigoplus_{n=1}^{\infty} Ry_{2n-1}\right) \\ &= \left(\bigoplus_{n=1}^{\infty} Rx_{2n-1}\right) \oplus \left(\bigoplus_{n=1}^{\infty} Ry_{2n}\right) \end{aligned}$$

are indecomposable decompositions of P. Therefore, by Corollary 3, they all complement direct summands. In particular, there must be subsets H and K of the natural numbers such that

$$P = \left(\bigoplus_{n=1}^{\infty} Ry_{2n}\right) \oplus \left(\bigoplus_{h\in H} Ry_{2h-1}\right) \oplus \left(\bigoplus_{k\in K} Rx_{2k}\right).$$

If there is a natural number $m \notin H$, then

$$\pi_{2m-1}(P) = Ra_{2m-2}x_{2m-1}$$

is in the radical of Rx_{2m-1} which is clearly impossible. Therefore, H must contain all natural numbers and

$$P = \left(\bigoplus_{n=1}^{\infty} Ry_{n}\right) \oplus \left(\bigoplus_{k\in K} Rx_{2k}\right).$$

If K is finite, then there is a natural number m such that

$$P = \left(\bigoplus_{n=1}^{\infty} Ry_{n}\right) + Rx_1 + \cdots + Rx_{m-1}\,.$$

Thus, there exist elements $r_1 ,..., r_n , s_1 ,..., s_{m-1} \in eRe$ with

$$x_m = r_1 y_1 + \cdots + r_n y_n + s_1x_1 + \cdots + s_{m-1}x_{m-1}\,.$$

Applying π_m we deduce that

$$e = -r_{m-1}a_{m-1} + r_m$$

whence r_m is invertible in eRe. Applying $\pi_{m+1} ,..., \pi_n$, we have

$$-r_ma_m + r_{m+1} = \cdots = -r_{n-1}a_{n-1} + r_n = -r_na_n = 0,$$

so that $r_ma_m \cdots a_n = 0$ and $a_m \cdots a_n = 0$. Therefore, if the sequence is not left T-nilpotent, the set K is infinite. But if k, $m \in K$ with $k > m$, then it is easy to check that

$$Rx_m \leqslant Rx_k + \sum_{n=1}^{\infty} Ry_n$$

contrary to our requirement of independence.

Now we turn to the study of decompositions of injective modules.

7. PROPOSITION. *If an injective module E has an indecomposable decomposition, then that decomposition complements direct summands.*

Proof. Recall [8, p. 103] that indecomposable injective modules have local endomorphism rings. Suppose then that E is injective and that

$$E = \bigoplus_{A} E_\alpha$$

is an indecomposable direct decomposition of E. In particular then each E_α has a local endomorphism ring. Let K be a direct summand of E. Then we can choose a subset $B \subseteq A$ maximal with respect to

$$K \cap \left(\bigoplus_{B} E_\beta\right) = 0.$$

Then the submodule

$$K \oplus \left(\bigoplus_{B} E_\beta\right) = K \oplus \left(\bigoplus_{B} E_\beta\right)$$

of E is injective whence for some $E' \leqslant E$,

$$E = K \oplus \left(\bigoplus_{B} E_\beta\right) \oplus E'.$$

We claim that $E' = 0$. For if $E' \neq 0$, then by (1) and (2) of Azumaya's Theorem there is a $\gamma \in A$ and $E'' < E'$ such that

$$E = K \oplus \left(\bigoplus_{B} E_\beta\right) \oplus E_\gamma \oplus E''$$

contrary to the maximality of B. Thus, $E' = 0$, and the given decomposition complements K.

By results of Matlis and Papp (see [5]) it is known that a ring R is left noetherian if and only if every injective left R-module has an indecomposable decomposition. Thus applying Proposition 7 we have

8. THEOREM. *A ring is left noetherian if and only if each of its injective left modules has a decomposition that complements direct summands.*

It is well-known that left perfect left noetherian rings are left artinian. (By Levitzki's Theorem [8, p. 70] the radical of such a ring is nilpotent). Thus, we infer

9. Corollary. *A ring is left artinian if and only if each of its left injective modules and each of its left projective modules has a decomposition that complements direct summands.*

Remark. Let M be a module and let K be a direct summand of M. We say that K has the *exchange property in M* in case for each direct decomposition

$$M = \bigoplus_C L_\gamma$$

of M there are submodules

$$L_\gamma' \leqslant L_\gamma \qquad (\gamma \in C)$$

such that

$$M = K \oplus \left(\bigoplus_C L_\gamma'\right).$$

Crawley and Jønsson [4] to whom the basic idea is due, say that a module K has *the exchange property* in case in our terminology it has the exchange property in every module in which it embeds as a direct summand. Suppose that M has an indecomposable decomposition. Then using Corollary 3 it is not difficult to show that M has a decomposition that complements direct summands if and only if every direct summand of M has the exchange property in M.

Acknowledgment

The authors wish to thank the School of Mathematics, Trinity College, Dublin, and the Mathematics Institute, Aarhus University, for their generous hospitality during the preparation of this paper.

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Added in proof. (1) We have recently learned that our Proposition 7 was obtained earlier in [Warfield, *Pacific J. Math.* **31** (1969), 263–276].

(2) Crawley and Jønsson [4] showed that, over any ring, the class of modules possessing the exchange property is closed under finite, but not necessarily infinite, direct products. In view of our concluding remark and Theorems 5 and 6, any non-finitely generated free module over a ring that is semiperfect but not perfect provides an example which shows that this class need not be closed under infinite direct sums either.