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Approximating minimum-power edge-covers and 2, 3-connectivity ☆

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ABSTRACT

Given a graph with edge costs, the power of a node is the maximum cost of an edge leaving it, and the power of a graph is the sum of the powers of its nodes. Motivated by applications in wireless networks, we consider several fundamental undirected network design problems under the power minimization criteria. The Minimum-Power Edge-Cover (MPEC) problem is: given a graph $G = (V, E)$ with edge costs $\{c(e) : e \in E\}$ and a subset $S \subseteq V$ of nodes, find a minimum-power subgraph H of G containing an edge incident to every node in S. We give a 3/2-approximation algorithm for MPEC, improving over the 2-approximation by [M.T. Hajiaghayi, G. Kortsarz, V.S. Mirrokni, Z. Nutov, Power optimization for connectivity problems, Mathematical Programming 110 (1) (2007) 195–208]. For the Min-Power k-Connected Subgraph (MPkCS) problem we obtain the following results. For $k = 2$ and $k = 3$, we improve the previously best known ratios of 4 [G. Calinescu, P.J. Wan, Range assignment for biconnectivity and k-edge connectivity in wireless ad hoc networks, Mobile Networks and Applications 11 (2) (2006) 121–128] and 7 [M.T. Hajiaghayi, G. Kortsarz, V.S. Mirrokni, Z. Nutov, Power optimization for connectivity problems, Mathematical Programming 110 (1) (2007) 195–208] to $3\frac{2}{3}$ and $5\frac{2}{3}$, respectively. Finally, we give a $4r_{max}$-approximation algorithm for the Minimum-Power Steiner Network (MPSN) problem: find a minimum-power subgraph that contains $r(u, v)$ pairwise edge-disjoint paths for every pair u, v of nodes.

1. Introduction

1.1. Motivation and problems considered

Wireless networks are studied extensively due to their wide applications. The power consumption of a station determines its transmission range, and thus also the stations it can send messages to; the power typically increases at least quadratically in the transmission range. Assigning power levels to the stations (nodes) determines the resulting communication network. Conversely, given a communication network, the cost required at v only depends on the furthest node that is reached directly by v. This is in contrast with wired networks, in which every pair of stations that need to communicate directly incurs a cost. We study the design of symmetric wireless networks that meet some prescribed degree or connectivity properties, and such that the total power is minimized. An important network property is fault-tolerance, which is often measured by node-connectivity of the network. Node-connectivity is much more central here than edge-connectivity, as it models stations failures. Such power minimization problems were vastly studied. See [1,3,6,13,17,21,22,24] for a small sample of papers in this area. The first problem we consider is finding a low power network that "covers" a specified set S of

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nodes. This is the power variant of the fundamental Edge-Cover problem, c.f., [7]. The second problem is the Min-Power k-Connected Subgraph problem which is the power variant of the classic Min-Cost k-Connected Subgraph problem. We give approximation algorithms for these problems that significantly improve the previously best known ones.

Definition 1.1. Let $H = (V, I)$ be a graph with edge-costs $\{c(e) : e \in I\}$. For $v \in V$, the *power* $p(v) = p_H(v)$ *of* v in H (w.r.t. c) is the maximum cost of an edge in I leaving v (or zero, if no such edge exists), i.e., $p(v) = p_I(v) = \max_{vu \in I} c(vu)$. The power of the graph is the sum of the powers of its nodes. (Note that in directed graphs the edges entering v do not affect its power.)

Note that $p(H)$ differs from the ordinary cost $c(H) = \sum_{e \in I} c(e)$ of H even for unit costs; for unit costs, if H is undirected, then $c(H) = |I|$ and (if H has no isolated nodes) $p(H) = |V|$. For example, if I is a perfect matching on V then $p(H) = 2c(H)$. If H is a clique then $p(H)$ is roughly $c(H)/\sqrt{|I|/2}$. For directed graphs, the ratio of the cost over the power can be equal to the maximum outdegree, e.g., for stars with unit costs. The following statement, parts of which appeared in [4,13], shows that these are the extremal cases for general edge costs.

Proposition 1.1 (*[4,13]*). *$c(H)/\sqrt{|I|/2} \leq p(H) \leq 2c(H)$ for any undirected graph $H = (V, I)$, and if H is a forest then $c(H) \leq p(H) \leq 2c(H)$. For any directed graph H holds: $c(H)/\Delta(H) \leq p(H) \leq c(H)$, where $\Delta(H)$ is the maximum outdegree of a node in H.*

Minimum-power problems are usually harder than their minimum-cost versions. The Minimum-Power Spanning Tree problem is APX-hard [6]. The problem of finding minimum-cost k pairwise edge-disjoint paths is in P (this is the Minimum-Cost k-Flow problem, c.f., [7]) while both directed and undirected minimum-power variants are unlikely to have even a polylogarithmic approximation [13,21]. Another example is finding an arborescence rooted at s, that is, a subgraph that contains an sv-path for every node v. The minimum-cost case is in P (c.f., [7]), while the minimum-power variant is at least as hard as the Set-Cover problem.

Unless stated otherwise, graphs are assumed to be undirected and simple. Let $H = (V, I)$ be a graph. For $X \subseteq V$, $\Gamma_I(X) = \Gamma_H(X) = \{u \in V - X : \exists v \in X, vu \in I\}$ is the set of neighbors of X, $\delta_I(X) = \delta_H(X)$ is the set of edges leaving X, and $d_I(X) = |\delta_H(X)|$ is the degree of X in H. Given a graph $G = (V, E)$ with edge-costs, we seek to find a low power *communication network*, that is, a low power subgraph of G (that is, an edge subset of E) that satisfies some prescribed property. Two such fundamental properties are: edge-cover and fault-tolerance/connectivity.

Definition 1.2. Given a subset $S \subseteq V$ of nodes, we say that an edge set I on V is an S*-cover* if for every $v \in S$ there is an edge in I incident to v.

Finding a minimum-cost S-cover is a fundamental problem in Combinatorial Optimization, as this is essentially the Edge-Cover problem, c.f., [7]. The following problem is the power variant.

Minimum-Power Edge-Cover (MPEC):

Instance: A graph $G = (V, E)$ with edge costs $\{c(e) : e \in E\}$, and a subset $S \subseteq V$ of nodes.

Objective: Find a min-power S-cover $I \subseteq E$.

We now define our connectivity problems. A graph is k-connected if it contains k internally-disjoint uv-paths for all $u, v \in V$. We consider the min-power variant of the extensively studied Min-Cost k-Connected Subgraph (MCkCS) problem.

Minimum-Power k-Connected Subgraph (MPkCS):

Instance: A graph $G = (V, E)$ with edge costs $\{c(e) : e \in E)$, and an integer k.

Objective: Find a minimum-power k-connected spanning subgraph H of G.

We also consider min-power variant of the min-cost Steiner Network problem.

Minimum-Power Steiner Network (MPSN):

Instance: A graph $G = (V, E)$ with edge costs $\{c(e) : e \in E)$, and requirements $\{r(u, v) : u, v \in V\}$.

Objective: Find a minimum-power subgraph H of G so that H contains $r(u, v)$ pairwise edge-disjoint uv-paths for every $u, v \in V$.

1.2. Previous and related work

Results on edge-cover problems: The following generalization of MPEC was considered in [13,17]. Given a degree requirement function r on V, an edge set I on V is an r*-cover* if $d_I(v) \geq r(v)$ for every $v \in V$. The Minimum-Power Edge-Multi-Cover problem seeks to find an r-cover of minimum power; MPEC is a particular case when r is a 0, 1-valued function, namely, when $r(v) = 1$ if $v \in S$ and $r(v) = 0$ otherwise. In [13] the approximation ratio $\min\{r_{\max} + 1, O(\log^4 n)\}$ was derived, where $r_{\max} = \max_{v \in V} r(v)$; this is improved to $O(\log n)$ in [17]. However, for the fundamental case MPEC, the best ratio was 2.

Results on connectivity problems: The simplest connectivity problem is when we require the network to be connected. In this case, the minimum-cost variant is just the Minimum-Cost Spanning Tree problem, while the minimum-power variant is APX-hard. A 5/3-approximation algorithm for the Minimum-Power Spanning Tree problem is given by Althaus et al. [1]. Minimum-cost connectivity problems were extensively studied, see surveys in [16,18]. The best known approximation ratio for the Minimum-Cost k-Connected Subgraph (MCkCS) problem is $O\left(\log k \cdot \log \frac{n}{n-k}\right)$ [25] for both directed and undirected graphs, see also [2,5,8,9,19,20] for various algorithms for the problem. It turns out that (for undirected graphs) approximating MPkCS is closely related to approximating MCkCS and the Min-Power k-Cover problem – a particular case of the Min-Power Edge-Multi-Cover problem when $r(v) = k$ for all $v \in V$. The following statement was observed independently in [13,15].

Theorem 1.2 (*[13,15]*).

(i) *An α-approximation for* MCkCS *and a β-approximation for Min-Power $(k-1)$-Cover implies a $(2\alpha + \beta)$-approximation for* MPkCS.
(ii) *A ρ-approximation for* MPkCS *implies a $(2\rho + 1)$-approximation for* MCkCS.

One can combine various values of α, β with Theorem 1.2 to get approximation algorithms for MPkCS. In [13,17] the bound $\beta = \min\{k+1, O(\log n)\}$ was derived. The best known values for α are: $\alpha = \lceil (k+1)/2 \rceil$ for $2 \le k \le 7$ (see [2] for $k = 2, 3$, [8] for $k = 4, 5$, and [19] for $k = 6, 7$); $\alpha = k$ for other small values of k [19], and $\alpha = O\left(\log k \cdot \log \frac{n}{n-k}\right)$ for large values of k [25]; note that the latter ratio is $O(\log k)$ for all k but $k = n - o(n)$, and is $O(\log^2 k) = O(\log^2 n)$ for $k = n - o(n)$. Thus for undirected MPkCS the following ratios follow: $3k$ for any k, $k + 2\lceil (k+1)/2 \rceil$ for $2 \le k \le 7$, and $O(\log n)$ unless $k = n - o(n)$. Improvements over the above bounds are known only for $k \le 2$. As was mentioned, the Minimum-Power Spanning Tree problem (this is the case $k = 1$ of MPkCS) admits a 5/3-approximation algorithm [1]. Calinescu and Wan [3] gave a 4-approximation algorithm for the case $k = 2$ of MPkCS.

For further results on other minimum-power connectivity problems, among them results for problems on directed graphs, see [13,21,24].

1.3. Our results

The previous best approximation ratio for MPEC was 2 [13]. We prove:

Theorem 1.3. *MPEC admits a 3/2-approximation algorithm.*

For MPkCS we improve the best known ratios for $k = 2, 3$:

Theorem 1.4. *Undirected* MPkCS *with $k \in \{2, 3\}$ admit a $(2k - 1/3)$-approximation algorithm.*

For $k = 2$, Theorem 1.4 improves the previously best known ratio of 4 [3] to $3\frac{2}{3}$. For $k = 3$ the improvement is from 7 to $5\frac{2}{3}$.

We note that recently in [26] the ratios were improved to 3 for $k = 2$ and to 4 for $k = 3$. However, the algorithms presented in this paper still have the best ratio of 2 for the problem of adding a minimum-power edge set to increase the connectivity from $k - 1$ to k, for $k = 2, 3$.

We also consider the MPSN problem. Williamson et al. [28] gave a $2r_{\max}$-approximation algorithm for the min-cost case, and then this was improved to $2H(r_{\max})$ in [12]. The currently best known ratio for the min-cost case is 2 [14]. We show that the algorithm of [28] for the min-cost case, has approximation ratio $4r_{\max}$ for the minimum-power variant MPSN.

Theorem 1.5. *Undirected MPSN admits a $4r_{\max}$-approximation algorithm.*

Theorems 1.3–1.5 are proved in Sections 2–4, respectively.

1.4. Techniques

Our approach for MPEC is inspired by the decomposition method used by Zelikovsky [29] and Prömel and Steger [27] for the Minimum-Cost Steiner Tree problem: decomposing solutions into small parts, and then reducing the problem to the Min-Cost Spanning Tree problem in 3-uniform hypergraphs, with loss of 5/3 in the approximation ratio. A similar method was used in [1] for the Minimum-Power Spanning Tree problem. In our case, to prove Theorem 1.3, we reduce MPEC to the Min-Cost Edge-Cover problem in *graphs*, and the loss in the approximation ratio is 3/2.

For MPkCS with $k = 2, 3$ we show a 2-approximation algorithm for the "augmentation problem" of increasing the connectivity by 1. Combining with the 5/3-approximation algorithm of [1] for the Minimum-Power Spanning Tree gives the ratio in Theorem 1.4. We note that the augmentation version admits an easy 4-approximation by combining three facts:

(i) Any minimal solution to the augmentation problems is a forest [23].

(ii) The min-cost augmentation problem admits a 2-approximation [2].

(iii) $c(F) \le p(F) \le 2c(F)$ if F is a forest, see Proposition 1.1.

These facts are also valid for an appropriate augmentation version of MPSN, c.f., [12,28]; this is how we obtain a $4r_{\max}$-approximation for MPSN in Theorem 1.5. However, getting a ratio of 2 for the augmentation version of MPkCS with $k = 2, 3$ is done by using a different approach. Specifically, we consider *directed* solutions to a related "k-inconnectivity problem", and show that their underlying graphs have low power.

2. A 3/2-approximation for MPEC (Proof of Theorem 1.3)

We reduce Minimum-Power S-cover to Minimum-Cost S-Cover; the latter is solvable in polynomial time, c.f., [7]. However, the reduction is not approximation ratio preserving, but incurs a loss of 3/2 in the approximation ratio. That is, given an instance (G, c, S) of Minimum-Power S-Cover, we construct in polynomial time an instance (G', c', S) of Minimum-Cost S-Cover such that:

(i) for any S-cover I' in G' corresponds an S-cover I in G with $p(I) \le c'(I')$.

(ii) $\text{opt}' \le 3\text{opt}/2$, where opt' is the minimum cost of an S-cover in G', c';

Hence if I' is an optimal (min-cost) solution to (G', c', S), then $p(I) \le c'(I') = \text{opt}' \le 3\text{opt}/2$.

Clearly, any minimal S-cover has no paths/cycles of length 3, and thus is a union of pairwise node disjoint stars, with at most one node (the center) not in S. We now define a certain decomposition of stars, which is similar to the decompositions of trees in [1,27,29].

Definition 2.1. Let I be (an edge set of) a star. A collection $\mathcal{I} = \{I_1, \ldots, I_\ell\}$ of sub-stars of I is a *t-decomposition* of I if the following holds: the stars in $\mathcal{I}$ cover all the leaves of I, every star has at least one and most t edges, and there is at least one star in $\mathcal{I}$ with at most $t-1$ edges. The power $p(\mathcal{I}) = \sum_{I_j \in \mathcal{I}} p(I_j)$ of $\mathcal{I}$ is the sum of the powers of its parts.

Intuitively, every part in $\mathcal{I}$ is "in charge" to cover its leaves; a part with $t-1$ edges can also cover the center v_0 of I (we need this if $v_0 \in S$). In this way, every part is in charge to cover at most t nodes, and every node of I (including the center) is covered by some part of $\mathcal{I}$ in this way.

For the purpose of proving Theorem 1.3, we use only 2-decomposition, which is just a collection of single edges and pairs of edges of the star, with at least one single edge, that collectively cover all the nodes of the star. We however will consider the case of t arbitrary, as such decompositions may have other applications.

We will be interested in establishing that for any star I there exists a t-decomposition $\mathcal{I}$ so that the ratio $p(\mathcal{I})/p(I)$ is small. Namely, we would like to bound the ratio $\max_I \min_{\mathcal{I}} p(\mathcal{I})/p(I)$, where the maximum is taken over all stars I with edge costs, and the minimum is taken over all t-decompositions $\mathcal{I}$ of I. Even for stars with unit costs, this ratio cannot be asymptotically better than $1 + 1/t$. Indeed, let I be a star with $\ell t + (t-1)$ leaves and with unit edge costs. Then $p(I) = (\ell+1)t$, while $p(\mathcal{I}) \ge (\ell+1)t + \ell$ for any t-decomposition $\mathcal{I}$ of I. Thus $p(\mathcal{I})/p(I) \ge (1+1/t) - 1/(t(\ell+1))$, and this ratio can be arbitrary close to $1 + 1/t$. The following statement shows that this bound is achievable for any edge costs.

Lemma 2.1. *Any star I admits a t-decomposition $\mathcal{I}$ with $p(\mathcal{I}) \le (1 + 1/t)p(I)$.*

Proof. Let I be a star with center v_0. Let $e_1, \ldots, e_d$ be the edges of I sorted by non-decreasing costs, so $c_1 \ge c_2 \ge \cdots \ge c_d \ge 0$, where $c_j = c(e_j)$ for $j = 1, \ldots, d$. Let $\ell = \lceil (d+1)/t \rceil$.

We will define a specific t-decomposition $\mathcal{I} = \{I_1, \ldots, I_\ell\}$ of I. If $\ell = 1$, then $\mathcal{I} = \{I\}$ and we are done. If $\ell \ge 2$, then let I_1 consist of the t most expensive edges, I_2 of the next t most expensive edges, and so on. In this way we obtain stars $I_1, \ldots, I_{\ell-1}$ with exactly t edges each. If t does not divide d, then the last star I_ℓ will consist of the remaining at most $t-1$ edges. If t divides d, $I_1, \ldots, I_{\ell-1}$ already contain all the leaves of I; in this case we set $I_\ell = \{e_d\}$ to consist of the cheapest edge. We will show that $p(\mathcal{I}) \le (1 + 1/t)p(I)$.

Let $p_j(v_0) = \max_{e \in I_j} c(e)$ be the power of v_0 in I_j. Note that $p(I_j) = c(I_j) + p_j(v_0)$. The key point is that

$$p_j(v_0) \le c(I_{j-1})/t \quad j = 2, \ldots, \ell.$$

This is since every edge in I_{j-1} has cost at least $p_j(v_0)$, and since there are t edges in I_{j-1}, as $j - 1 \ne \ell$. Therefore,

$$\begin{aligned} p(\mathcal{I}) &= \sum_{j=1}^{\ell} (c(I_j) + p_j(v_0)) = c(I) + c_1 + \sum_{j=2}^{\ell} p_j(v_0) \\ &\le c(I) + c_1 + \sum_{j=2}^{\ell} c(I_{j-1})/t \le c(I) + c_1 + c(I)/t \\ &\le (1 + 1/t)p(I). \quad \square \end{aligned}$$

Given an instance $(G = (V, E), c, S)$ of MPEC, the algorithm is:

1. Construct an instance $(G' = (S, E'), c')$ of Min-Cost Edge-Cover as follows.
   For $u, v \in S$ (possibly $u = v$), among all $\{u, v\}$-covers in G that consists of one edge or of two adjacent edges, let I_{uv} be one of minimum power. The graph G' is a complete graph on S with all loops, and the edge costs are $c'(uv) = p(I_{uv})$ for every $u, v \in S$.
2. Find a minimum-*cost* edge-cover I' in G', c'.
3. Return $I = \cup \{I_{uv} : uv \in F'\}$

Clearly, all the steps in the algorithm can be implemented in polynomial time. The following statement is used to prove that the approximation ratio of the algorithm is 3/2.

Lemma 2.2. (i) *If I' is an edge-cover in G' then $I = \cup\{I_{uv} : uv \in I'\}$ is an S-cover in G and $p(I) \le c'(I')$.*
(ii) $\mathsf{opt}' \le 3\mathsf{opt}/2$, *where* opt' *is the minimum cost of an S-cover in G', c'.*

Proof. I is an S-cover since I' is an S-cover, and since I_{uv} covers $\{u, v\}$ for every $uv \in I'$; note that the latter is true also if $u = v$. Also, $p(I) \le c'(I')$ since

$$p(I) = p\left(\bigcup_{uv\in I'} I_{uv}\right) \le \sum_{uv\in I'} p(I_{uv}) = \sum_{uv\in I'} c'(uv) = c(I').$$

We now prove that $\mathsf{opt}' \le 3\mathsf{opt}/2$. Let I be an optimal solution to MPEC in (G, c, S), so $p(I) = \mathsf{opt}$. Applying Lemma 2.1 with $t = 2$ implies that there exists a 2-decomposition $\mathcal{I} = \{I_1, \ldots, I_{\ell+1}\}$ of I with $p(\mathcal{I}) \le 3p(I)/2 = 3\mathsf{opt}/2$. To every $I_j \in \mathcal{I}$ corresponds an edge e'_j in G' (e'_j is a loop if I_j consists of a single edge with only one endnode in S) and $c'(e'_j) \le p(I_j)$. Let $I' = \{e'_1, \ldots, e'_{\ell+1}\}$. Then I' is an edge-cover in G', since e'_j and I_j cover the same set of nodes in S for every j, and since I covers S. Hence $\mathsf{opt}' \le c'(I')$. Thus:

$$\mathsf{opt}' \le c'(I') = \sum_{j=1}^{\ell+1} c'(e'_j) \le \sum_{j=1}^{\ell+1} p(I_j) = p(\mathcal{I}) \le 3p(I)/2 = 3\mathsf{opt}/2. \quad \square$$

Theorem 1.3 now easily follows from Lemma 2.2. Let I, I' be as in the algorithm. Then, by Lemma 2.2, we have $p(I) \le c'(I') = \mathsf{opt}' \le 3\mathsf{opt}/2$.

The proof of Theorem 1.3 is complete.

3. Approximating 2, 3-connectivity (Proof of Theorem 1.4)

3.1. Reduction to k-inconnectivity

A (possibly directed) graph is *k-inconnected to s* if it contains k internally-disjoint vs-paths for every $v \in V$. We need to consider the following problem:

Minimum-Power k-Inconnected Subgraph (MPkIS):

Instance: A graph $G = (V, E)$ with costs $\{c(e) : e \in E\}$, an integer k, and $s \in V$.

Objective: Find a min-power k-inconnected to s subgraph $H = (V, I)$ of G.

In the next section we will prove:

Theorem 3.1. *Undirected* MPkIS *admits a* $(2k - 1/3)$*-approximation algorithm.*

Clearly, a k-connected graph is k-inconnected to every node. However, a k-inconnected to s graph H may not be even 2-connected (e.g., if H consists from two $(k + 1)$-cliques with a common node s). The following statement shows that for $k \in \{2, 3\}$, "k-connectivity" and "k-inconnectivity to s" are equivalent concepts if we restrict ourself to graphs in which the degree of s is exactly k.

Lemma 3.2 ([2]). *For $k \in \{2, 3\}$, if s is a node of degree k in an undirected graph H, then H is k-inconnected to s if, and only if, H is k-connected.*

This motivates the following auxiliary problem, which min-cost variant is the basis for the algorithms in [2,8,19].

Restricted MPkIS

Instance: A graph $G = (V, E)$, edge costs $\{c(e) : e \in E\}$, $s \in V$, an integer k.

Objective: Find a min-power k-inconnected to s spanning subgraph H of G with $d_H(s) = k$.

Lemma 3.3. *If* MPkIS *admits a ρ-approximation algorithm then* Restricted MPkIS *admits a ρ-approximation algorithm for any constant k. In particular,* Restricted MPkIS *admits a* $(2k - 1/3)$*-approximation algorithm for* $k \in \{2, 3\}$.

Proof. The algorithm for Restricted MPkIS is derived from the algorithm for MPkIS by "guessing" the k edges incident to s in some optimal solution for Restricted MPkIS. For any subset $K \subseteq E$ of k edges incident to s, remove the other edges incident to s, and compute a ρ-approximate solution H_K to MPkIS (or declare that no such H_K exists). Then, among the subgraphs H_K computed, output one H of the minimum power. The running time is $O(n^k)$ times the running time of the ρ-approximation algorithm for MPkIS, hence polynomial for any constant k. The second statement of the lemma follows from Theorem 3.1. $\square$

The following statement shows that any inclusion minimal optimal solution H^* to MPkCS has a node s (in fact, at least $|V|/3$ such nodes) so that $d_{H^*}(s) = k$.

Theorem 3.4 ([23]). *A minimally k-connected graph contains at least $\frac{(k-1)n+2}{2k-1}$ nodes of degree k.*

Summarizing, we have that for $k \in \{2, 3\}$, undirected MPkCS is equivalent (via an approximation ratio preserving reduction) to Restricted MPkIS for some $s \in V$. As s is not known to us, we simply try all possible choices. Namely, for every node $s \in V$, we compute a $(2k - 1/3)$-approximate solution H_s for Restricted MPkIS using the algorithm from Lemma 3.3, and return the cheapest graph H among the subgraphs H_s computed. Then H is k-connected by Lemma 3.2, and $p(H) \le (2k - 1/3)\text{opt}$ by Theorems 3.1 and 3.4. This gives a $(2k - 1/3)$-approximation algorithm for MPkCS with $k \in \{2, 3\}$, provided we prove Theorem 3.1.

3.2. Proof of Theorem 3.1

To prove Theorem 3.1, we need to consider the "augmentation" version of MPkIS:

Minimum-Power k-Inconnectivity Augmentation (MPkIA):

Instance: An integer k, a $(k - 1)$-inconnected to s graph $H_0 = (V, I_0)$, and an edge set E on V with costs $\{c(e) : e \in E\}$.

Objective: Find a min-power edge set $I \subseteq E$ so that $H_0 + I$ is k-inconnected to s.

Let us say that an edge e of a (possibly directed) k-inconnected to s graph H is *critical* if $H - e$ is not k-inconnected to s. In [21] it is proved:

Theorem 3.5 ([21]). *Let uv', uv'' be two distinct critical edges of a k-inconnected to s directed graph H. Then $d_H(u) = k$.*

Lemma 3.6. *If I is an inclusion minimal solution to directed* MPkIA *then $d_I(u) \le 1$ for every $u \in V$, and thus the power of I equals its cost. Consequently, directed* MPkIA *is solvable in polynomial time.*

Proof. Suppose to the contrary that $d_I(u) \ge 2$, so there are distinct edges $uv', uv'' \in I$. Since I is an inclusion minimal augmenting edge set, uv', uv'' are critical edge in $H_0 + I$, hence $d_{H_0+I}(u) = k$, by Theorem 3.5. This is a contradiction, since $d_{H_0}(u) \ge k - 1$, $d_I(u) \ge 2$, thus $d_{H_0+I}(u) \ge k + 1$.

Thus $p(I) = c(I)$, by Proposition 1.1. Consequently, directed MPkIA is equivalent to its min-cost version; the latter is solvable in polynomial time [10,11]. □

Lemma 3.7. *Undirected* MPkIA *admits a 2-approximation algorithm.*

Proof. A *bi-direction* of an undirected network H is a directed network $D(H)$ obtained by replacing every edge $e = uv$ of H by two opposite directed edges uv, vu each having the same cost as e. Note that $p(H) = p(D(H))$. The 2-approximation algorithm for MPkIA is as follows:

1. Let $D(H_0)$ and $D(E)$ be the bi-directions of H_0 and E, respectively.
2. Compute a min-cost edge set $I_D \subseteq D(E)$ so that $D(H_0) + I_D$ is k-inconnected to s.
3. Output the underlying edge set I of I_D.

Step 2 can be implemented in polynomial time [10,11]. We show that the algorithm has approximation ratio 2. Let I^* be an optimal solution to an (undirected) MPkIA instance (so $p(I^*) = \text{opt}$) and let $D(I^*)$ be the bi-direction of I^*. W.l.o.g. assume that I_D is inclusion minimal w.r.t. the property "$D(H_0) + I_D$ is k-inconnected to s". Thus $d_{I_D}(u) \le 1$ for every $u \in V$, by Lemma 3.6. This implies $p(I) \le 2p(I_D)$, since the contribution of an edge e to $p(I_D)$ is exactly $c(e)$, while the contribution of e to $p(I)$ is at most $2c(e)$. Also note that $p(I_D) \le p(D(I^*))$, since I_D is an optimal k-inconnected to s subgraph. Consequently, $p(I) \le 2p(I_D) \le 2p(D(I^*)) = 2p(I^*) = 2\text{opt}$. □

Theorem 3.1 follows by combining Lemma 3.7 with the 5/3-approximation algorithm of [1] for the Minimum-Power Spanning Tree problem. Indeed, we can apply the algorithm as in Lemma 3.7 sequentially to produce edge sets $I_1, \ldots, I_k$ so that $H_\ell = I_1 + \cdots + I_\ell$ is ℓ-inconnected to s, and $p(I_1) \le 5\text{opt}/3$ (I_1 is a spanning tree computed by the 5/3-approximation algorithm of [1]) and $p(I_\ell) \le 2\text{opt}$ for $\ell = 2, \ldots, k$. Consequently, if $I = I_1 + \cdots + I_k$ then $H = (V, I)$ is k-inconnected to s, and

$$p(I) \le p(I_1) + \sum_{\ell=2}^{k} p(I_\ell) \le \frac{5}{3}\text{opt} + \sum_{\ell=2}^{k} 2\text{opt} = (2k - 1/3)\text{opt}.$$

The proof of Theorem 3.1, and thus also of Theorem 1.4 is complete.

Remark. Calinescu and Wan [3] also gave a $2k$-approximation algorithm for the undirected Min-Power k-Edge-Connected Subgraph problem for arbitrary k. This ratio can be improved to $(2k-1/3)$, by a proof similar (in fact simpler) to the proof of Theorem 1.4. We provide a sketch of such a proof. Lemma 3.6 is valid for the edge-connectivity version of MPkIA, as is well known and was implicitly proved in [10] (the proof is different from the one of Lemma 3.6). Lemma 3.7 is also true for edge-connectivity, by the same proof (which uses only Lemma 3.6). Thus Theorem 3.1 extends to the edge-connectivity version of MPkIS, namely, the edge-connectivity version of undirected MPkIS admits a $(2k-1/3)$-approximation algorithm. It is well known that an undirected graph is k-edge-connected if, and only if, it is k-edge-inconnected to some node. Consequently, the edge-connectivity version of MPkIS, is equivalent (for undirected graphs) to the Min-Power k-Edge-Connected Subgraph problem, and both problems admit a $(2k-1/3)$-approximation algorithm.

4. Algorithm for MPSN (Proof of Theorem 1.5)

We need some definitions and a description of certain results from [12,28]. The minimum-cost/power Steiner Network problem can be formulated as a *set-function edge-cover problem*. Let $h : 2^V \to Z_+$ be a *set function* defined on a groundset V. An edge set I on V is an *h-cover*, if $d_I(X) \geq h(X)$ for every $X \subseteq V$. For Steiner Network problems, an appropriate choice of h is as follows. By Menger's Theorem, I is a feasible solution to minimum-cost/power Steiner network problem if, and only if, $d_I(X) \geq R(X)$ for all $\emptyset \subset X \subset V$, where $R(X) = \max\{r(u,v) : u \in X, v \in V - X\}$ (and $R(\emptyset) = R(V) = 0$). That is

$$d_I(X) \geq h(X) \equiv \max\{0, R(X)\} \quad \forall \emptyset \subseteq X \subseteq V. \tag{1}$$

The function h defined above is *weakly-supermodular*, that is $h(\emptyset) = 0$ and for every $X, Y \subseteq V$ with $h(X) > 0$, $h(Y) > 0$ at least one of the following holds:

$$h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) \tag{2}$$

$$h(X) + h(Y) \leq h(X - Y) + h(Y - X) \tag{3}$$

Note that h is also *symmetric*, that is, $h(X) = h(V - X)$ for all $X \subseteq V$.

Several connectivity problems can be formulated as (minimum-cost/power) edge-cover problems of a weakly-supermodular function, see [18]. A seminal paper of Jain [14] gives a 2-approximation algorithm for finding a minimum-cost edge-cover of an arbitrary weakly-supermodular set function h, provided certain queries related to h can be answered in polynomial time (note that h is usually not given explicitly). For h defined in (1) these queries can be realized in polynomial time via max-flows [14], which implies a 2-approximation algorithm for the Minimum-Cost Steiner Network problem. Earlier, Williamson et al. [28] gave an algorithm with approximation ratio $2h_{\max}$, which was improved later to $2H(h_{\max})$ by Goemans et al. [12].

Let h be a set function on V. For an edge set I, let $h_I(X) = \max\{h(X) - d_I(X), 0\}$. It is well known that if h is weakly supermodular, so is h_I for any edge set I, see [14]. Let $\hat{h}(X) = 1$ if $h(X) = h_{\max}$ and $\hat{h}(X) = 0$ otherwise, where $h_{\max} = \max_{X \subseteq V} h(X)$. It is easy to see that any inclusion minimal edge-cover of a $\{0,1\}$-valued set function, and thus also of $\hat{h}$, is a forest. Consider the following algorithm that applies on an arbitrary set function h, and begins with $I = \emptyset$.

While there is $X \subseteq V$ with $h_I(X) > 0$ do:
1. Find an $\hat{h}_I$-cover $F \subseteq E - I$;
2. $I \leftarrow I + F$.

EndWhile

The approximation ratio of the algorithm depends on step 1. A set function is called *uncrossable* if it is $\{0,1\}$-valued weakly-supermodular. It is easy to see that if h is weakly-supermodular, so is $\hat{h}$, that is, $\hat{h}$ is uncrossable. Williamson et al. [28] gave an algorithm that finds an edge cover of an arbitrary uncrossable function q of cost at most twice the optimum of the following LP-relaxation:

$$\min\left\{\sum_{e\in E} c(e)x_e : \sum_{e\in\delta_E(X)} x_e \geq q(X)\ \forall X \subseteq V, x_e \geq 0\right\}. \tag{4}$$

Williamson et al. [28] proved:

Theorem 4.1 (*[28]*). *For h defined by (1) the above algorithm can be implemented in polynomial time, so that at any iteration for $q = \hat{h}_I$ the forest F found has cost at most twice the optimal value of (4).*

Note that the number of iterations of the algorithm is at most $h_{\max}$. Thus Theorem 4.1 implies that for the Minimum-Cost Steiner Network problem the algorithm has approximation ratio $2h_{\max} \leq 2r_{\max}$.

We can show that for the minimum-power variant, the algorithm of [28] has approximation ratio $4r_{\max}$. This follows from Theorem 4.1 and the second part of Proposition 1.1. Indeed, the algorithm of [28] constructs the solution from at most $r_{\max}$ forests, where each forest has cost at most 2opt_c, where opt_c is the optimal solution value to the minimum-cost variant. By Proposition 1.1, each forest has power at most $2 \cdot 2\text{opt}_p = 4\text{opt}_p$, where opt_p is the optimal solution value to the minimum-power variant. This completes the proof of Theorem 1.5.

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References

[1] E. Althaus, G. Calinescu, I. Mandoiu, S. Prasad, N. Tchervenski, A. Zelikovsky, Power efficient range assignment for symmetric connectivity in static ad-hoc wireless networks, Wireless Networks 12 (3) (2006) 287–299.
[2] V. Auletta, Y. Dinitz, Z. Nutov, D. Parente, A 2-approximation algorithm for finding an optimum 3-vertex-connected spanning subgraph, Journal of Algorithms 32 (1) (1999) 21–30.
[3] G. Calinescu, P.J. Wan, Range assignment for biconnectivity and k-edge connectivity in wireless ad hoc networks, Mobile Networks and Applications 11 (2) (2006) 121–128.
[4] W.T. Chen, N.F. Huang, The strongly connecting problem on multihop packet radio networks, IEEE Transactions on Communications (1989).
[5] J. Cheriyan, S. Vempala, A. Vetta, An approximation algorithm for the minimum-cost k-vertex connected subgraph, SIAM Journal on Computing 32 (4) (2003) 1050–1055.
[6] A.E.F. Clementi, P. Penna, R. Silvestri, Hardness results for the power range assignment problem in packet radio networks, in: APPROX, 1999, pp. 197–208.
[7] W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, A. Scrijver, Combinatorial Optimization, Wiley, 1998.
[8] Y. Dinitz, Z. Nutov, A 3-approximation algorithm for finding optimum 4, 5-vertex-connected spanning subgraphs, Journal of Algorithms 32 (1) (1999) 31–40.
[9] J. Fakcharoenphol, B. Laekhanukit, An $O(\log^2 k)$-approximation algorithm for the k-vertex connected spanning subgraph problem, in: STOC, 2008, pp. 153–158.
[10] A. Frank, Increasing the rooted-connectivity of a digraph by one, Mathematical Programming 84 (3) (1999) 565–576.
[11] A. Frank, E. Tardos, An application of submodular flows, Linear Algebra and its Applications 114/115 (1989) 329–348.
[12] M.X. Goemans, A.V. Goldberg, S. Plotkin, D.B. Shmoys, E. Tardos, D.P. Williamson, Improved approximation algorithms for network design problems, in: SODA, 1994, pp. 223–232.
[13] M.T. Hajiaghayi, G. Kortsarz, V.S. Mirrokni, Z. Nutov, Power optimization for connectivity problems, Mathematical Programming 110 (1) (2007) 195–208.
[14] K. Jain, A factor 2 approximation algorithm for the generalized Steiner network problem, Combinatorica 21 (1) (2001) 39–60.
[15] X. Jia, D. Kim, S. Makki, P.-J. Wan, C.-W. Yi, Power assignment for k-connectivity in wireless ad hoc networks, Journal of Combinatorial Optimization 9 (2) (2005) 213–222.
[16] S. Khuller, Approximation algorithms for finding highly connected subgraphs, in: D.S. Hochbaum (Ed.), Ch. 6 in Approximation Algorithms for NP-Hard Problems, PWS, 1995, pp. 236–265.
[17] G. Kortsarz, V.S. Mirrokni, Z. Nutov, E. Tsanko, Approximating minimum-power degree and connectivity problems, in: LATIN 2008, pp. 423–435, Algorithmica (in press).
[18] G. Kortsarz, Z. Nutov, Approximating minimum-cost connectivity problems, in: T.F. Gonzalez (Ed.), Ch. 58 in Approximation Algorithms and Metaheuristics, Chapman & Hall/CRC, 2007.
[19] G. Kortsarz, Z. Nutov, Approximating node-connectivity problems via set covers, Algorithmica 37 (2003) 75–92.
[20] G. Kortsarz, Z. Nutov, Approximating k-node connected subgraphs via critical graphs, SIAM Journal on Computing 35 (1) (2005) 247–257.
[21] Y. Lando, Z. Nutov, On minimum power connectivity problems. Manuscript, Preliminary version in ESA 2007, pp. 87–98 (submitted for journal publication).
[22] E.L. Lloyd, R. Liu, M.V. Marathe, R. Ramanathan, S.S. Ravi, Algorithmic aspects of topology control problems for ad hoc networks, MONET 10 (1–2) (2005) 19–34.
[23] W. Mader, Ecken vom grad n in minimalen n-fach zusammenhängenden graphen, Archive der Mathematik 23 (1972) 219–224.
[24] Z. Nutov, Approximating minimum power covers of intersecting families and directed connectivity problems, in: Proc. APPROX 2006, pp. 236–247.
[25] Z. Nutov, An almost $O(\log k)$-approximation for k-connected subgraphs, in: SODA, 2009 (in press).
[26] Z. Nutov, Approximating minimum power k-connectivity, in: AD-HOC NOW, 2008, pp. 86–93.
[27] H.J. Prömel, A. Steger, A new approximation algorithm for the Steiner tree problem with performance ratio 5/3, Journal of Algorithms 36 (1) (2000) 89–101.
[28] D.P. Williamson, M.X. Goemans, M. Mihail, V.V. Vazirani, A primal-dual approximation algorithm for generalized steiner network problems, Combinatorica 15 (1995) 435–454.
[29] A. Zelikovsky, An 11/6-approximation algorithm for the network Steiner problem, Algorithmica 9 (5) (1993) 463–470.