Criteria for Permutability to Be Transitive in Finite Groups

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A group $G$ is a $PT$-group if, for subgroups $H$ and $K$ with $H$ permutable in $K$ and $K$ permutable in $G$, it is always the case that $H$ is permutable in $G$. It is shown that a finite group is a soluble $PT$-group if and only if each subgroup of a Sylow subgroup is permutable in the Sylow normalizer. © 1999 Academic Press

1. INTRODUCTION

A subgroup $H$ of a group $G$ is said to be permutable (or quasinormal) if $HK = KH$ for all subgroups $K$ of $G$. Thus permutability is a weak form of normality. Now permutability, like normality, is not a transitive relation. Our interest here lies in groups in which permutability is transitive, that is,
groups $G$ such that $H$ permutable in $K$ and $K$ permutable in $G$ imply that $H$ is permutable in $G$. Such groups are called $PT$-groups. It is understood that all groups in this article are finite.

According to a well-known theorem of Ore [5] a permutable subgroup is subnormal. Therefore, $PT$-groups are precisely the groups in which each subnormal subgroup is permutable. Consequently $PT$-groups include all groups in which normality is transitive, the so-called $T$-groups, which have been widely studied (see for example, [1, 7, 8]).

The structure of soluble $PT$-groups was determined by Zacher [10] in 1964. He showed that these are exactly the groups with an abelian normal Hall subgroup $L$ of odd order such that $G/L$ is a nilpotent modular group and the elements of $G$ induce power automorphisms in $L$. The corresponding theorem for soluble $T$-groups is due to Gaschütz [1]; here one has to replace “nilpotent modular” in Zacher’s theorem by “Dedekind.” These results provide evidence that $PT$-groups are quite close to $T$-groups, although they are much harder to work with. It is an easy consequence of these theorems that the classes of soluble $PT$-groups and soluble $T$-groups are subgroup closed.

Our main object here is to provide necessary and sufficient conditions on the Sylow structure for a group to be a soluble $PT$-group. For this purpose we introduce the condition

$$X_p.$$  

Here a group $G$ satisfies $X_p$ if and only if each subgroup of a Sylow $p$-subgroup $P$ of $G$ is permutable in the normalizer $N_G(P)$. Since subgroups of soluble $PT$-groups are $PT$-groups, it is clear that a soluble $PT$-group must satisfy $X_p$ for all primes $p$. The interesting question is whether the converse is valid. Our main result confirms that this is true.

**Theorem A.** A group $G$ is a soluble $PT$-group if and only if it satisfies $X_p$ for all primes $p$.

To prove this result we need to study intensively the property $X_p$ and its consequences for the group structure. Some of these are quite surprising and indicate that $X_p$ “nearly” implies $p$-nilpotence.

**Theorem B.** A group $G$ satisfies $X_p$ if and only if either $G$ is $p$-nilpotent or a Sylow $p$-subgroup $P$ of $G$ is abelian and every subgroup of $P$ is normal in $N_G(P)$.

For the smallest prime divisor of the group order one has a stronger result.

**Theorem C.** Let $p$ be the smallest prime divisor of the order of $G$. Then $G$ has $X_p$ if and only if $G$ is $p$-nilpotent and Sylow $p$-subgroups of $G$ are modular.
The above theorems are to be compared with several results about soluble $T$-groups obtained by the third author [7] some 30 years ago. In these results the role of $X_p$ is played by a stronger property $C_p$: every subgroup of a Sylow $p$-subgroup is normal in the Sylow normalizer. For example, it is shown in [7] that a group $G$ is a soluble $T$-group if and only if it satisfies $C_p$ for all primes $p$.

Finally, a connection between $X_p$ and pronormality is established. Recall that a subgroup $H$ is pronormal in a group $G$ if, for any element $g$ in $G$, the subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$.

Peng [6] and the third author [7] proved that a group $G$ is a soluble $T$-group if and only if each subgroup of prime power order is pronormal in $G$. Thus one might ask whether the condition $X_p$ is related to the pronormality of certain $p$-subgroups of $G$. This question is answered in Theorem D.

**Theorem D.** A group $G$ satisfies $X_p$ if and only if a Sylow $p$-subgroup $P$ of $G$ is modular and every normal subgroup of $P$ is pronormal in $G$.

This result has the virtue of showing that $X_p$ is a subgroup closed property, a fact that is not evident from the definition.

All four theorems depend on the classification of modular $p$-groups given by Iwasawa [3] (see also 2.3.1 of [9]). These are either Dedekind groups or else $p$-groups of the form

$$G = \langle x, A \rangle$$

where $A$ is an abelian group and $a^r = a^{1 + r}$ with $r \geq 1$ (and $r \geq 2$ if $p = 2$) for all $a \in A$.

The organization of the paper is as follows. In Section 2 we give a succinct proof of Zacher's theorem with some consequences. Section 3 contains results indicating how close the classes of soluble $PT$-groups and soluble $T$-groups are. The proofs of the main theorems appear in Sections 4 and 5.

### 2. A PROOF OF ZACHER'S THEOREM

**Theorem 1 (Zacher [10]).** A soluble group $G$ is a $PT$-group if and only if it has an abelian normal Hall subgroup $L$ of odd order such that $G/L$ is a nilpotent modular group and elements of $G$ induce power automorphisms in $L$.

**Proof.** Let $G$ be a $PT$-group.

(1) If $N$ is a normal $p$-subgroup of $G$ and $p$ is a prime, then the $p'$-elements of $G$ induce power automorphisms in $N$.

For let $a \in N$ and let $x$ be a $p'$-element of $G$. Then $a^{(x)} = a^{(x)} \cap \langle a \rangle \langle x \rangle = \langle a \rangle$ since $a^{(x)} \cap \langle x \rangle \subseteq N \cap \langle x \rangle = 1$. 
(2) \( G \) is supersoluble.

Let \( A \) be a minimal normal subgroup of \( G \), with \( A \) an elementary abelian \( p \)-group, say. Put \( \overline{G} = G/C_{G}(A) \). By (1) we have that \( \overline{D} = O_{p}(\overline{G}) \) is abelian and \( \overline{G}/\overline{D} \) is a \( p \)-group. Thus \( \overline{G} = \overline{D}\overline{P} \) where \( \overline{P} \) is a Sylow \( p \)-subgroup of \( \overline{G} \). Then \( C_{A}(\overline{P}) \neq 1 \) and \( C_{A}(\overline{P}) \) is \( D \)-invariant since the \( p' \)-elements of \( G \) induce power automorphisms in \( A \). Hence \( C_{A}(\overline{P}) \) is normal in \( G \) and so \( C_{A}(\overline{P}) = A \) and \( \overline{P} = 1 \). Therefore \( |A| = p \) by (1) and so \( G \) is supersoluble by induction on \( |G| \).

(3) The hypercommutator \( L = \gamma_{s}(G) \) (i.e., the limit of the lower central series) is an abelian Hall subgroup of \( G \).

The proof is by induction on \( |G| \). Let \( p \) be the largest prime divisor of \( |G| \) and let \( P \) be a Sylow \( p \)-subgroup of \( G \). By (2) \( P \) is normal in \( G \) and by induction \( LP/P \) is an abelian Hall subgroup of \( G/P \). Notice that either \( G = PC_{G}(P) \) or else \( G/C_{G}(P) \) contains non-trivial \( p' \)-elements.

Assume first that \( G = PC_{G}(P) \). Then \( P \leq Z_{s}(G) \), the hypercenter of \( G \). Hence \( G = P \times Q \) where \( Q \) is a Hall \( p' \)-subgroup of \( G \). It follows that \( L = \gamma_{s}(Q) \), \( LP/P \cong L \), and \( L \) is an abelian Hall subgroup of \( G \).

Now assume that \( G/C_{G}(P) \) contains non-trivial \( p' \)-elements; then \( p > 2 \). By (1) \( P = [P, G] \leq L \) and \( G/PC_{G}(P) \) is abelian. Hence \( G' \leq PC_{G}(P) \) and \( [P, G'] \leq [P, PC_{G}(P)] \leq P' \). This implies that \( L \) acts trivially in \( P/P' \).

Therefore, \( L \) induces \( p \)-automorphisms in \( P \) and \( P \leq Z_{s}(L) \). From this we conclude that \( L = P \times Q \) where \( Q \) is a Hall \( p' \)-subgroup of \( L \). Assume that \( Q \neq 1 \). By induction \( L/Q \) and \( L/P \) are abelian Hall subgroups of \( G/Q \) and \( G/P \), and hence \( L \) is an abelian Hall subgroup of \( G \).

Therefore, assume that \( Q = 1 \) and \( L = P \). Suppose that \( L \) is non-abelian. Note that \( L \) is modular, so we can write \( L = \langle x \rangle A \) where \( A \) is a normal abelian subgroup of \( L \) and \( x \) induces a power automorphism in \( A \). By (1) every subgroup of \( A \) is invariant under the \( p' \)-elements of \( G \) and hence is normal in \( G \). Thus all elements of \( G \) induce power automorphisms in \( A \), which shows that \( [A, [L, G]] = 1 \). Hence \( [A, L] = 1 \) and \( L \) is abelian.

(4) \( L \) has odd order.

By (1) elements of \( G \) induce automorphisms in \( L_{2} \) with 2-power order. Thus \( L_{2} \leq Z_{s}(G) \), and if \( L_{2} \neq 1 \), then \( L \neq [L, G] \), a contradiction.

(5) All the subgroups of \( L \) are normal in \( G \).

This follows from (1) and (3).

The sufficiency clause of Theorem 1 follows from the next more general result, which is needed later in this work.

**Lemma 1.** Let \( N \) be a normal Hall subgroup of a group \( G \) and assume that the following hold:

1. \( G/N \) is a \( PT \)-group;
2. every subnormal subgroup of \( N \) is normal in \( G \).

Then \( G \) is a \( PT \)-group.
Proof. Let $H$ be a subnormal subgroup of $G$. We show that $H$ is permutable. By (2) $H \cap N$ is normal in $G$ and $G/H \cap N$ satisfies (1) and (2). By induction on $|G|$ we can assume that $H \cap N = 1$. By the Schur-Zassenhaus theorem $N$ has a complement $M$ in $G$ and all complements are conjugate to $M$. Since $\langle |H|, |N| \rangle = 1$ and $H$ is subnormal, $H \leq M$. Also note that $[H, N] = 1$.

It is enough to show that $H$ permutes with any subgroup $T$ of $G$ of order $p^n$ where $p$ is a prime and $n$ is a positive integer. If $p$ divides $|N|$, then $T \leq N$ and $HT = TH$. Assume that $p, |N| = 1$. Then $T$ is contained in some conjugate of $M$, say $M^x$, where $x \in G$. By (1) $M^x$ is a PT-group and $H \leq M^x$, so that $TH = HT$ and the result follows.

Corollary 1. Let $G$ be a finite soluble PT-group. Then the following hold:

1. $G$ is metabelian.
2. $\text{Fit}(G)$, the Fitting subgroup of $G$, equals $\gamma_s(G) \times Z_s(G)$ where $\gamma_s(G)$ is the hypercommutator of $G$ and $Z_s(G)$ is the hypercenter of $G$.
3. If $H$ is a subgroup of $G$, then $H$ is a PT-group.
4. If $G' \cap Z(G) = 1$, then $G$ is a $T$-group. In particular, if $Z(G) = 1$, then $G$ is a $T$-group.

Proof. Put $L = \gamma_s(G)$. By Theorem 1 $L$ is an abelian Hall subgroup of $G$ and hence it has a complement, say $B$, in $G$. Then $G' = LB'$ and $G'' = [L, B'] = 1$ since $B$ induces power automorphisms in $L$. This establishes (1).

Let $\pi = \pi(G/L)$ and note that $L$ is a Hall $\pi'$-subgroup of $\text{Fit}(G)$. Hence $F = \text{Fit}(G) = L \times F_{\pi}$. Now $[F_{\pi, G}] \leq F_{\pi} \cap L = 1$, for some $i$, which means that $F_{\pi} \leq Z_s(G)$. Also note that $\gamma_s(G) \cap Z_s(G) = 1$, and so $F_{\pi} = Z_s(G)$ and (2) follows. Statement (3) follows from Theorem 1 and Lemma 1.

Finally, assume that $G' \cap Z(G) = 1$. By (1) and (2) $L \leq G' \leq L \times Z_s(G)$, so $G' = L$ and $G/L$ is abelian. Hence $G$ is a $T$-group by 13.4.5 of [8]. Thus (4) holds.

We remark that (4) of Corollary 1 is not true for insoluble PT-groups.

Example 1. There exists an insoluble PT-group $G$ with trivial center which is not a $T$-group.

Let $D = PSL_3(25)$. Then $D$ has a diagonal automorphism $\sigma$ of order 8, and a field automorphism $\alpha$ of order 2. Put $Q = \langle \sigma, \alpha \rangle$, and note that $\sigma^a = \sigma^5 = \sigma^{14}$. Hence $Q$ is a modular 2-group of order 16.

Let $G$ be the semidirect product of $D$ by $Q$. Then $G$ is semisimple. Let $H$ be a nontrivial subnormal subgroup of $G$. Then $D \leq H$, so that $H$ is permutable in $G$ since $G/D$ is a PT-group. Thus $G$ is a PT-group, but it is not a $T$-group since $Q$ is not. Also note that $Z(G) = 1$. 
3. SOLUBLE T GROUPS AND SOLUBLE PT GROUPS

As has been observed, every \( T \)-group is also a \( PT \)-group. Moreover the theorems of Zacher and Gaschütz show that the structures of soluble \( PT \)-groups and \( T \)-groups are quite similar, the only difference being that in the \( T \) case \( G/L \) is a Dedekind group.

The following theorem shows that the difference between \( T \)-groups and \( PT \)-groups occurs in the abelian factors.

**Theorem 2.** Let \( G \) be a \( PT \)-group. Then \( G \) is a \( T \)-group if and only if for each elementary abelian subnormal factor \( H/K \) of order \( p^2 \), with \( p \) a prime, \( N_G(H/K)/C_G(H/K) \) is a \( p' \)-group.

**Proof.** Assume that \( G \) is a \( T \)-group and let \( H/K \) be a subnormal factor as stated. Then \( H \) and \( K \) are normal in \( G \) and each \( p \)-element of \( G \) acts trivially on \( H/K \) since this is elementary abelian \( p \).

Conversely, assume that the condition holds in \( G \), but \( G \) is not a \( T \)-group. Further, let \( G \) be a counterexample of minimal order. By hypothesis there is a non-normal subnormal subgroup \( H \) of \( G \) with least order. The minimality of \( |G| \) shows that the core of \( H \) in \( G \) is 1. Hence, since \( H \) is permutable, \( H \leq Z_s(G) \) by the Maier–Schmid theorem [4] (see also Theorem 5.2.3 of [9]). Once again using the minimality of \( |H| \), we see that \( H \) is a cyclic \( p \)-group, say \( H = \langle u \rangle \). Since \( H \) is core-free, \( |H| = p \). Now the \( p' \)-elements of \( G \) normalize, and hence centralize, \( H \) since \( H \leq Z_s(G) \). Hence there is a \( p \)-element \( x \) such that \( v = [u, x] \neq 1 \). Also \( Z(G) \neq 1 \), so that \( HZ(G)/Z(G) \) is \( G \)-central. Hence \( v \in Z(G) \). Since \( v^p = [u^p, x] = 1 \), we have \( \langle u, v \rangle = \langle u \rangle \times \langle v \rangle \). Also \( x \) acts non-trivially on \( \langle u \rangle \times \langle v \rangle \). Note that \( \langle u, v \rangle \leq Z_s(G) \), so \( \langle u \rangle \times \langle v \rangle \) is subnormal in \( G \) with order \( p^2 \). This contradicts the hypothesis.

In the next result we show that a soluble \( PT \)-group can be embedded in the direct product of a nilpotent modular group and a \( T \)-group.

**Theorem 3.** The group \( G \) is a soluble \( PT \)-group if and only if there is a nilpotent modular group \( M \) and a soluble \( T \)-group \( W \) such that:

1. \( (|M|, \gamma_s(W)) = 1 \);
2. there is a monomorphism \( \alpha: G \to M \times W \) with \( G^\alpha \) subdirect in \( M \times W \) and \( \gamma_s(W) \leq G^\alpha \);
3. if \( p \) is a prime divisor of \( (|M|, |W|) \), then a Sylow \( p \)-subgroup of \( G \) is isomorphic to a Sylow \( p \)-subgroup of \( M \).

**Proof.** Assume that \( G \) is a soluble \( PT \)-group and put \( L = \gamma_s(G) \). By Theorem 1 \( L \) is an abelian Hall subgroup of \( G \), \( G/L \) is a nilpotent modular group and the elements of \( G \) act by conjugation on \( L \) as power au-
tomorphisms. Moreover, by Corollary 1, \( \text{Fit}(G) = L \times K \) where \( K \) is the hypercenter of \( G \).

Put \( M = G/L \) and \( W = G/K \). Then \( W \) is a \( T \)-group since \( Z(W) = 1 \) by part (4) of Corollary 1. Now \( \gamma_*(W) = LK/K \cong L \) and so \( (|\gamma_*(W)|, |M|) = 1 \). Hence (1) holds. The assignment \( g \mapsto (gL, gK) \) is a subdirect embedding \( \alpha \) of \( G \) into \( M \times W \). Moreover, \( \gamma_*(W) = \{(1, gK) | g \in L\} = \{(gL, gK) | g \in L\} \leq G^* \), so that (2) follows. Finally, let \( p \) be a prime divisor of \( (|M|, |W|) \). Then \( (p, |L|) = 1 \), and thus (3) is established.

Conversely, assume that (1), (2), and (3) hold in \( G \) with \( G \leq M \times W \). Then \( G \) is soluble. By Gaschütz's theorem we see that \( L = \gamma_*(W) = [W', W] \) is a normal abelian Hall subgroup of \( W \) with odd order. Also the elements of \( W \), and hence of \( M \times W \), induce power automorphisms in \( L \). From (1), (2), and (3) it follows that \( L \) is a normal Hall subgroup of \( G \) on which \( G \) acts as power automorphisms. Since \( G/L \) is nilpotent, it is enough by Lemma 1 to show that \( G/L \) is a modular group. Let \( P \) be a Sylow \( p \)-subgroup of \( G/L \). If \( p \) divides \( (|M|, |W|) \), then \( P \) is modular by (3). If \( p \) does not divide \( (|M|, |W|) \), then \( P \) is isomorphic to a Sylow \( p \)-subgroup subgroup of \( W \) or \( M \). In either case it is modular. Since a direct product of modular \( p \)-groups is modular, \( G/L \) is a nilpotent modular group.

**Example 2.** The group \( M \times W \) in Theorem 3 need not be a \( PT \)-group. Let \( W \) be a nonabelian group of order 21 and \( M \) an extraspecial 3-group of order 27 and exponent 3. Then \( M \) is a modular group. There are epimorphisms \( \alpha: W \to C_3 \), \( \beta: M \to C_3 \) and \( G = \{(x, y) \in M \times W | x^3 = y^3 \} \) is a subdirect subgroup of \( M \times W \). By Zacher's theorem \( G \) is a soluble \( PT \)-group, but \( M \times W \) is not. Also note that \( G \) is not a \( T \)-group.

**4. Proof of Theorems A and C**

We begin with three elementary results which are useful in the proofs of Theorems A and C.

**Lemma 2.** A group \( G \) satisfies \( X_p \) if and only if a Sylow \( p \)-subgroup \( P \) of \( G \) is modular and the \( p \)-elements of \( N_G(P) \) induce power automorphisms in \( P \).

**Proof.** Assume that \( G \) satisfies \( X_p \). Then a Sylow \( p \)-subgroup \( P \) of \( G \) is clearly modular. Let \( a \in P \) and let \( x \) be a \( p \)-element of \( N_G(P) \). Then \( a^{(x)} = a^{(x)} \cap (a \langle x \rangle) = \langle a \rangle \) since \( P \cap \langle x \rangle = 1 \). Thus \( x \) induces a power automorphism in \( P \). Conversely, these conditions clearly imply that \( G \) satisfies \( X_p \).
Corollary 2. Let $G$ be a group satisfying $X_p$ and let $P$ be a Sylow $p$-subgroup of $G$. If either $p$ is the smallest prime divisor of $|G|$ or $P$ is nonabelian, then $N_G(P) = P \times O_{p'}(N_G(P))$.

Proof. Assume that $p$ is the smallest prime divisor of $|G|$. By Lemma 2 $O_{p'}(N_G(P))$ centralizes $P$ and the result follows. Now assume that $P$ is nonabelian. By Hilfssatz 5 of [2] the group of power automorphisms of $P$ is a $p$-group. Again the result follows.

The next lemma can be established using a simple induction on $r$.

Lemma 3. Let $p$ be a prime and let $l \geq 1$, $r \geq 0$ be integers, with $l \geq 2$ if $p = 2$. If $a$ is an integer such that $a \equiv 1 \pmod{p}$, then

$$(1 + p^l a)^{p^r} = 1 + p^{r + l} d$$

where $d \equiv 1 \pmod{p}$.

Proof of Theorem C. Assume that $G$ is $p$-nilpotent and a Sylow $p$-subgroup $P$ of $G$ is modular. Then $G = PO_{p'}(G)$, and hence $N_G(P) = P \times O_{p'}(N_G(P))$ and $G$ satisfies $X_p$ by Lemma 2.

Conversely, let $G$ be a group satisfying $X_p$ with least order subject to not being $p$-nilpotent. Here $p$ is the smallest prime divisor of $|G|$. Let $P$ be a Sylow $p$-subgroup of $G$. Then by Lemma 2 the $p'$-elements of $N_G(P)$ centralize $P$. Hence by Burnside’s criterion (see 10.1.8 of [8]) $P$ is nonabelian and $P \cap G' \neq 1$.

Put $O_{p'}(G/G') = L/G'$. Then $P \cap L = P \cap G'$ is a Sylow $p$-subgroup of $L$ and $P$ is a Sylow $p$-subgroup of $N_G(P \cap G')$. Hence $N_G(P \cap G')$ inherits the condition $X_p$. If $N_G(P \cap G') = G$, then $P \cap G'$ is normal in $G$. On the other hand, if $N_G(P \cap G') \neq G$, then $N_G(P \cap G')$ is $p$-nilpotent, and so is $N_L(P \cap L)$. This means that $L$ satisfies $X_p$, by the first paragraph. If $L \neq G$, then $L$ is $p$-nilpotent, whence so is $G$. Thus $L = G$ and $P \leq G'$.

Therefore, there are two cases to consider: (1) $1 \neq P \cap G' \trianglelefteq G$ and (2) $P \leq G'$.

Case 1. $1 \neq P \cap G' \trianglelefteq G$.

Choose $N$ minimal normal in $G$ with $N \leq P \cap G'$. Notice that $G/N$ satisfies $X_p$ and so it is $p$-nilpotent. Note also that $N$ is the unique minimal normal subgroup of $G$ contained in $P \cap G'$. Moreover $O_{p'}(G) = 1$. Otherwise $G/O_{p'}(G)$, which inherits $X_p$, is $p$-nilpotent and hence so is $G$.

Let $O_{p'}(G/N) = Q_0/N$; then $Q_0 = QN$ where $Q$ is a $p'$-group and $Q \cap N = 1$. Next let $C = C_Q(N)$. Then $C \trianglelefteq CN \trianglelefteq QN = Q_0 \trianglelefteq G$, so $C$ is subnormal in $G$. Hence $C \leq O_{p'}(G)$ and $C_Q(N) = 1$.

We next show that $G$ splits over $N$. Indeed by the Schur–Zassenhaus theorem $Q_0 = QN$ splits conjugately over $N$. Hence it suffices to show that $C_N(Q) = 1$. Since $[N, Q] \neq 1$, it will follow that $C_N(Q) = 1$ if we can show
that $C_N(Q)$ is normal in $G = PQ$. Let $a \in C_N(Q)$, $b \in P$, and $x \in Q$. Then
$[a^b, x] = [a, x] = b^{-1} \in [a, QN] = 1$ and hence $C_N(Q)$ is normal.

It now follows that $P$ splits over $N$. Write $P = XN$ with $X \cap N = 1$. Let $a \in N$ and $x \in X$. Since
$P$ is modular, $a^{(x)} = a^{(x)} \cap \langle a \rangle \langle x \rangle = \langle a \rangle(a^{(x)} \cap \langle x \rangle) = \langle a \rangle$. This means that $x$ induces a power automorphism of $p$-power order in the elementary abelian $p$-group $N$. Hence $[N, \langle x \rangle] = 1$ and $N \leq Z(P)$.

Next note that $[N, [P, Q]] = 1$ since $[N, P] = 1$. But $C_N(N) = 1$ and consequently $[P, Q] \leq N$. Therefore $[P', Q] \leq [P, Q, P] \leq [N, P] = 1$. It follows that $P' \triangleleft PQ = G$, so that $N \leq P'$ since $P' \neq 1$ and $N$ is the unique minimal normal subgroup of $G$ contained in $P \cap G'$. Hence $[N, Q] \leq [P', Q] = 1$, a contradiction.

Case 2. $P \leq G'$.

Apply Grün’s First Theorem (see 10.2.1 of [8]) to obtain

$$P = \langle P \cap (N_G(P))^l, P \cap (P')^l | g \in G \rangle.$$ 

Since $p$ is the smallest prime divisor of $|G|$, we have $N_G(P) = P \times O_p(N_G(P))$, so $P \cap (N_G(P)) = P'$. Hence $P = \langle P \cap (P')^l | g \in G \rangle$. If $P$ is Dedekind, then $|P'| = 2$, so $P$ is generated by elements of order 2, which is false. Since $P$ is modular, Iwasawa’s theorem gives $P = \langle x \rangle A$ where $A$ is a normal abelian subgroup of $P$ and $a^x = a^{l+p'}$ for all $a \in A$, with $l \geq 1$ and $l \geq 2$ if $p = 2$.

Let exp($A$), the exponent of $A$, equal $p^k$. Then $l < k$ since $P$ is non-
abelian. Now $P' = [A, x] = A^{p'}$, so that exp($P'$) = $p^{k-1}$. Thus $P$ can be
generated by elements of order $\leq p^{k-1}$. In several steps we show that this is impossible.

(a) $|P : A| = p^{k-1}$

Let $|P : A| = p'$. Then $P/A$ is cyclic and can be generated by elements of order $\leq p^{k-1}$; therefore $r \leq k - l$. Now $x^{p'} \in A$, and so $[A, x^{p'}] = 1$. Hence, $A((1+p')^{p'-1}) = 1$ and $p^k$ divides $(1+p')^{p'-1} - 1$. By Lemma 3 $p^k$
divides $p^{r+l}$, whence $r \geq k - l$.

(b) We can assume $A$ is cyclic of order $p^k$.

Since exp($A$) = $p^k$, there is an element $a \in A$ whose order is $p^k$. Hence
$A = \langle a \rangle \times B$ where $B \leq A$. Now $B$ is normal in $P$ and $P/B$ is generated by
elements of order $\leq p^{k-1}$, while $A/B$ is cyclic of order $p^k$. Hence we can assume $B = 1$.

(c) $P$ splits over $A = \langle a \rangle$.

Let $\bar{x}$ denote the endomorphism $a \mapsto a^x = a^{l+p'}$. Then $H^2(P/A, A) \cong \text{Ker}(1-\bar{x})/\text{Im}(1+\bar{x}+\cdots+\bar{x}^{p'-1})$. Clearly Ker$(1-\bar{x}) = \langle a^{p^{k-1}} \rangle = \langle a^{p} \rangle$.

Also
$$a^{(1+\bar{x}+\cdots+\bar{x}^{p'-1})} = a^l.$$
where \( t = 1 + (1 + p') + \cdots + (1 + p')^{p' - 1} = [(1 + p')^{p'} - 1] / p' \). By Lemma 3 we have \( t = p'd \) where \( d \equiv 1 \pmod{p} \). It follows that \( \text{Im}(1 + \tilde{x} + \cdots + \tilde{x}^{p'} - 1) = A^{p'} = \langle a^{p'} \rangle \) and consequently \( H^2(P/A, A) = 0 \). This means that \( P \) splits over \( A \).

We can now assume that \( P = \langle x \rangle A, \langle x \rangle \cap A = 1, A = \langle a \rangle \), and \( a^t = a^{1+p'} \).

(d) Final step.

Let \( a_0 \in A \) and let \( i, m \) be integers with \( i \geq 0 \) and \( m \geq 0 \). We claim that \((x^{p'} a_0)^{p^m} = x^{p^{i+m}} a_0^{p^m e_m} \) where \( e_m \equiv 1 \pmod{p} \). This holds for \( m = 0 \) with \( e_0 = 1 \). Assume that it holds for \( m \). Then we have
\[
(x^{p'} a_0)^{p^{m+1}} = ((x^{p'} a_0)^{p^m})^p = (x^{p^{i+m} a_0^{p^m e_m}})^p = x^{p^{i+m+1}} (a_0^{p^m e_m})^p.
\]

Here
\[
t = 1 + (1 + p')^{p^{i+m}} + \cdots + ((1 + p')^{p^{i+m}})^{p-1}
= \frac{(1 + p')^{p^{i+m+1}} - 1}{(1 + p')^{p^{i+m}} - 1}
= \frac{p^{i+m+1} d}{p^{i+m} d'}
= pd / d'
\]
where \( d, d' \equiv 1 \pmod{p} \) by Lemma 3. Consequently,
\[
(x^{p'} a_0)^{p^{m+1}} = x^{p^{i+m+1}} a_0^{p^{m+1} e_{m+1}}
\]
where \( e_{m+1} = e_m (d / d') \equiv 1 \pmod{p} \). Hence our claim is established.

It now follows that \((x^{p'} a_0)^{p^m} = 1 \) implies \( a_0^{p^m} = 1 \) since \( \langle x \rangle \cap \langle a \rangle = 1 \). Therefore, every element of \( P \) of order \( p^{k-1} \) belongs to \( \langle x \rangle \langle a^{p'} \rangle \). But \( P \neq \langle x \rangle \langle a^{p'} \rangle \) since \( A \neq \langle a^{p'} \rangle \). This contradiction completes the proof.

Proof of Theorem A. A soluble \( PT \)-group satisfies \( X_p \) for all primes \( p \) by (3) of Corollary 1. Conversely, assume that \( G \) satisfies \( X_p \) for all primes \( p \), and \( G \) is of least order subject to not being a soluble \( PT \)-group.

Let \( p \) be the smallest prime divisor of \( |G| \). By Theorem C \( G \) is \( p \)-nilpotent and \( O_p(G) \neq G \). Put \( K = O_p(G) \), let \( q \) be a prime divisor of \( |K| \), and let \( Q \) be a Sylow \( q \)-subgroup of \( G \). Then Lemma 2 shows that \( Q \) is modular and the \( q \)-elements of \( N_K(Q) \) induce power automorphisms in \( Q \). Applying Lemma 2 again, we see that \( K \) satisfies \( X_p \). It follows from the minimality of \( G \) that \( K \) is a soluble \( PT \)-group, and so \( G \) is certainly soluble.
Let $L = \gamma_s(K)$. By Theorem 1 $L$ is an abelian normal Hall subgroup of $K$ in which $K$ induces power automorphisms. Let $r$ be a prime divisor of $|L|$ and let $R$ be a Sylow $r$-subgroup of $L$. Then $R$ is a normal Sylow $r$-subgroup of $G$. By $X$, the $r'$-elements of $G$ induce power automorphisms in $R$. Hence all the elements of $G$ induce power automorphisms in $L$. Suppose that $L \neq 1$. Then $G/L$ inherits the hypotheses of the theorem and so $G/L$ is a soluble $PT$-group. By Lemma 1 $G$ is a $PT$-group, a contradiction. Hence $L = 1$ and so $K$ is nilpotent.

Finally, let $T$ be a Sylow subgroup of $K$. Then $T$ is also a Sylow subgroup of $G$. As in the previous paragraph, if $T \neq 1$, then $G/T$ is a $PT$-group and $G$ induces a group of power automorphisms in $T$. Again $G$ is a $PT$-group by Lemma 1. This means that $K = 1$ so that $G$ is a modular $p$-group, a final contradiction.

5. PROOF OF THEOREMS B AND D

We are now able to prove these theorems using Theorem C.

Proof of Theorem B. Only the necessity of the conditions is in doubt. Let $G$ be a counterexample of least order. Then Theorem C shows that $p > 2$. Also a Sylow $p$-subgroup $P$ of $G$ is nonabelian and $G$ is not $p$-nilpotent. Let $J(P)$ be the Thompson subgroup of $P$ (see [8, p. 298]). Then $P \leq N_G(J(P))$ and $P \leq N_G(Z(P))$. By a result of Thompson ([8, 10.4.1]) $N_G(J(P))$ and $N_G(Z(P))$ cannot both be $p$-nilpotent. Since both of these subgroups satisfy $X_p$, one of them must be $G$. It follows that $P$ contains a minimal normal subgroup $N$ of $G$. Note that $G/N$ satisfies $X_p$, and so either $P/N$ is abelian or $G/N$ is $p$-nilpotent. Suppose that $P/N$ is abelian. Then, since by Corollary 2 $N_G(P) = P \times O_p(N_G(P))$, the Sylow $p$-subgroup $P/N$ lies in the center of its normalizer in $G/N$. Hence $G/N$ is $p$-nilpotent by Burnside's criterion. If $P/N$ is nonabelian, then $G/N$ is $p$-nilpotent by the minimality of $|G|$. Now follow the argument of Case 1 in the proof of Theorem C to obtain a contradiction.

Proof of Theorem D. Assume that $G$ satisfies $X_p$. By Theorem B either a Sylow $p$-subgroup $P$ of $G$ is abelian or else $G$ is $p$-nilpotent. Assume that $P$ is nonabelian. Then $G = PO_p(G)$. Let $P_0$ be a normal subgroup of $P$ and let $g \in O_p(G)$. Then $P_0O_p(G) = P_0^gO_p(G)$ and $P_0, P_0^g$ are Sylow $p$-subgroups of $J = \langle P_0, P_0^g \rangle$. Hence they are conjugate in $J$, and $P_0$ is pronormal in $G$.

Now assume that $P$ is abelian, let $P_0 \leq P$ and let $J = \langle P_0, P_0^x \rangle$ where $g \in G$. Let $P_1$ be a Sylow $p$-subgroup of $J$ containing $P_0$. Then $P_0^{x^{-1}} \leq P_1$ for some $x \in J$. Let $Q$ be a Sylow $p$-subgroup of $G$ containing $P_1$. Since
Q is abelian, $P_0 \triangleleft Q$ and $P_0 \triangleleft Q^{xg^{-1}}$. Hence $Q$ and $Q^{xg^{-1}}$ are conjugate in $N_G(P_0)$, that is, $Q = Q^{xg^{-1}}n$ where $n \in N_G(P_0)$. Thus $xg^{-1}n \in N_G(Q)$. Since $G$ satisfies $X_p$, $P_0$ is normal in $N_G(Q)$ by Lemma 2. Hence $P_0^n = P_0^x$, so that $P_0$ is pronormal in $G$.

Conversely, assume that $G$ satisfies the condition of Theorem D. Let $P$ be a Sylow $p$-subgroup of $G$. Then $P$ is modular. By Lemma 2 it is enough to show that the $p'$-elements of $N_G(P)$ induce power automorphisms in $P$. Let $P_0 \leq P$. If $P_0$ is normal in $P$, then $P_0$ is pronormal in $G$, and this is easily seen to imply $P_0$ is normal in $N_G(P)$. Thus if $P$ is abelian, the result follows.

Now assume that $P$ is nonabelian. Then $P = \langle x \rangle A$ where $A$ is abelian and $a^i = a^{1+p^i}$ for all $a \in A$, $i > 0$. If $N \triangleleft P$, then by hypothesis $N$ is pronormal in $G$ and so $N \triangleleft N_G(P)$. Let $g$ be a $p'$-element of $N_G(P)$. Then $g$ induces a power automorphism in $P/P'$ and in $A$. If $P = P'[P, g]$, then $[P, A] \leq [A, [x, g]] = 1$ since power automorphisms commute; this gives the contradiction $P' = 1$. Hence $P \neq P'[P, g]$. Since $g$ is a $p'$-element, it follows that $g$ centralizes $P/P'$ and hence $[P, g] = 1$. This shows that $G$ has $X_p$.

**Corollary 3.** The property $X_p$ is inherited by subgroups.

**Proof.** Let $H$ be a subgroup of a group of $G$ with $X_p$. If $G$ has nonabelian Sylow $p$-subgroups, then it is $p$-nilpotent by Theorem B. Clearly $H$ has $X_p$ in this case. Assume that $G$ has abelian Sylow $p$-subgroups. Let $Q$ be a Sylow $p$-subgroup of $H$ and let $Q_0 \leq Q \leq P$ where $P$ is a Sylow $p$-subgroup of $G$. Then $Q_0$ is normal in $P$, so it is pronormal in $G$ and therefore in $H$. Hence $H$ has $X_p$ by Theorem D.

**REFERENCES**