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$\mathbb{Z}_2$-graded cocharacters for superalgebras of triangular matrices☆

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Abstract

Let $\mathbb{K}$ be a field of characteristic zero, let A, B be $\mathbb{K}$-algebras with polynomial identity and let M be a free (A,B)-bimodule. The algebra $R=\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ can be endowed with a natural $\mathbb{Z}_2$-grading. In this paper, we compute the graded cocharacter sequence, the graded codimension sequence and the superexponent of R. As a consequence of these results, we also study the above PI-invariants in the setting of upper triangular matrices. In particular, we completely classify the algebra of 3×3 upper triangular matrices endowed with all possible $\mathbb{Z}_2$-gradings.
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1. Introduction

In the theory of algebras with polynomial identity a prominent role is played by the superalgebras and their identities. In fact, within the results of Kemer about the structure of varieties of associative algebras, the superalgebras come into play in a very natural way (see the monograph [17]). For instance, in case $char\,\mathbb{K}=0$, any variety $\mathfrak{V}$ of algebras is generated by the Grassmann envelope $G(B)$ of a suitable finite dimensional superalgebra B. In terms of T-ideals of the free algebra, this means that given any PI-algebra A, $T(A) = T(G(B))$ (see [16]), that is, A satisfies the same polynomial

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identities as the Grassmann envelope $G(B)$ of some finite dimensional superalgebra B. At the light of this, it seems an interesting problem to investigate the graded polynomial identities of a superalgebra and more generally, of a G-graded algebra, in case G is an arbitrary group ([2,6,12]).

In this context the matrix algebras with their gradings and their graded polynomial identities are a central object of study ([1]). For instance, the description of all $\mathbb{Z}_2$-gradings on matrix algebras is an important step in the study of verbally prime varieties, which are the building blocks of Kemer's theory.

As a natural extension of matrix algebras, one can consider the so-called block-triangular matrices. These algebras play an exceptional role in the investigation on the codimension growth of varieties (see [13–15]). The simplest block-triangular matrix algebras are the full matrix algebras $M_n(\mathbb{K})$ and the upper-triangular matrix algebras $UT_n(\mathbb{K})$. In [3] the authors classified all possible gradings on the algebra $M_n(\mathbb{K})$ when $\mathbb{K}$ is algebraically closed, and in [20] the authors described all possible gradings of upper-triangular matrix algebras when the group is finite abelian and the field is algebraically closed.

Some of the block-triangular matrix algebras are of type $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ where A, B are PI-algebras and M is a free (A,B)-bimodule. In [9] the authors described a generating set for the $\mathbb{Z}_2$-graded polynomial identities for these block-triangular matrices endowed with the natural grading

$$\begin{pmatrix} A & M \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \oplus \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix},$$

in terms of the ordinary polynomial identities of A and B. They obtained also a description of the relatively free superalgebra generated by $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ in the spirit of a result of Lewin ([19]).

In this paper, we describe the graded cocharacter sequence, the graded codimension sequence and we determine the so-called superexponent of these superalgebras, in terms of the ordinary polynomial identities of A and B. From these results, we derive a description of these PI-invariants for upper-triangular matrix superalgebras, in case the $\mathbb{Z}_2$-grading is of type $(0,\ldots,0,1,\ldots,1)$.

Of course, not all possible $\mathbb{Z}_2$-gradings on $UT_n(\mathbb{K})$ are of this kind for $n \geqslant 3$. In the last section of this paper we determine explicitly the sequence of graded cocharacters of $UT_3(\mathbb{K})$ endowed with all possible non-equivalent gradings and from this we derive the superexponent and the sequence of graded codimensions of the superalgebra.

2. General notions and tools

Let $\mathbb{K}$ be a field of characteristic zero. A unitary associative $\mathbb{K}$-algebra A is a *superalgebra*, or a $\mathbb{Z}_2$-*graded algebra*, if it is the direct sum of two vector subspaces A_0, A_1 satisfying the property $A_iA_j \subseteq A_{i+j}$ where $i, j \in \mathbb{Z}_2$. A classical (and very important) example of superalgebra is the so called *Grassmann algebra* of an infinite dimensional vector space.

When studying superalgebras, one is interested in homomorphisms preserving the superalgebra structure. Namely, if $A = A_0 \oplus A_1$ and $B = B_1 \oplus B_2$ are superalgebras, the chosen homomorphisms are the algebra homomorphisms $\varphi\colon A \to B$ such that $\varphi(A_i) \subseteq B_i$ for $i = 0, 1$.

One defines a free object in the class of superalgebras by considering the free $\mathbb{K}$-algebra over the disjoint union of two countable sets of variables, Y and Z, whose elements are regarded as *even* (i.e. with $\mathbb{Z}_2$-degree 0) and *odd* (their $\mathbb{Z}_2$-degree is 1) respectively. We shall denote this *free superalgebra* by $\mathbb{K}\langle Y, Z\rangle$. Its even part is the space spanned by those monomials in which the elements of Z occur in even number. The remaining monomials span the odd component of $\mathbb{K}\langle Y, Z\rangle$.

A polynomial $f(y_1, \ldots, y_n, z_1, \ldots, z_m)$ in $\mathbb{K}\langle Y, Z\rangle$ is called a $\mathbb{Z}_2$-graded polynomial identity for a superalgebra A if it is in the kernel of all $\mathbb{Z}_2$-graded homomorphisms $\varphi\colon \mathbb{K}\langle Y, Z\rangle \to A$. In other words, f is a graded polynomial identity for A if it vanishes under all possible substitutions of the variables by elements of A with the same parity: the y_i's are replaced by $a_i \in A_0$ and the z_i's by $b_i \in A_1$. One often calls these substitutions *admissible* (or *graded*) *substitutions*. If $\mathscr{E}$ is an admissible substitution, the evaluation of f at $\mathscr{E}$ will be denoted by $f|_{\mathscr{E}}$.

The set $T_2(A)$ of all graded polynomial identities of A is an ideal of the free superalgebra invariant under all graded endomorphisms of $\mathbb{K}\langle Y, Z\rangle$. It is called the T_2-ideal of (the graded polynomial identities of) A. The factor algebra $\mathbb{K}\langle Y, Z\rangle / T_2(A)$ inherits the superalgebra structure of the free superalgebra, and is a free object for the class of the superalgebras B such that $T_2(A) \subseteq T_2(B)$. This factor algebra is called the *relatively free superalgebra* associated to A.

The T_2-ideal of a superalgebra is very large in general, and it is more convenient to study the $\mathbb{Z}_2$-*graded multilinear* polynomials lying in it. A natural way of defining $\mathbb{Z}_2$-graded multilinear polynomials is the following:

Definition 2.1. For $n \in \mathbb{N}$, the vector space

$$V_n^{\mathbb{Z}_2} := \mathrm{span}\langle x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} | \sigma \in S_n,\ x_i \in \{y_i, z_i\}\rangle$$

is called the space of $\mathbb{Z}_2$-graded multilinear polynomials.

Since the characteristic of the ground field $\mathbb{K}$ is zero, a standard process of multilinearization shows that $T_2(A)$ is generated, as a T_2-ideal, by the subspaces $V_n^{\mathbb{Z}_2} \cap T_2(A)$. Actually, it is more efficient to study the factor space

$$V_n^{\mathbb{Z}_2}(A) := \frac{V_n^{\mathbb{Z}_2}}{V_n^{\mathbb{Z}_2} \cap T_2(A)}.$$

An effective tool to this end is provided by the representation theory of the symmetric group.

Indeed, one can notice that $V_n^{\mathbb{Z}_2}$ is an S_n-module with respect to the natural left action, and $V_n^{\mathbb{Z}_2} \cap T_2(A)$ is an S_n-submodule, hence the factor space $V_n^{\mathbb{Z}_2}(A)$ is an S_n-module, as well. We shall denote by $\chi_n^{\mathbb{Z}_2}(A)$ its character (the nth $\mathbb{Z}_2$-*graded cocharacter* of A) and by $c_n^{\mathbb{Z}_2}(A)$ its dimension (the nth $\mathbb{Z}_2$-graded codimension of A).

Actually, the study of the structure of $V_n^{\mathbb{Z}_2}(A)$ can be furthermore simplified by considering "smaller" spaces of multilinear polynomials. To be more precise, for fixed h, k, set

$$V_{h,k} := \mathrm{span}\langle m \text{ monomials of } V_{h+k}^{\mathbb{Z}_2} | y_1,\dots,y_h,z_{h+1},\dots,z_{h+k} \text{ occur in } m\rangle.$$

Setting $n := h+k$, and $\mathscr{H}_{h,k} := Sym(\{1,\dots,h\}) \times Sym(\{h+1,\dots,n\}) \leqslant S_n$, the space $V_{h,k}$ is an $\mathscr{H}_{h,k}$-module, and the subspace $V_{h,k} \cap T_2(A)$ is a submodule. Therefore one can form the factor $\mathscr{H}_{h,k}$-module

$$V_{h,k}(A) := \frac{V_{h,k}}{V_{h,k} \cap T_2(A)}.$$

We shall denote by $\chi_{h,k}(A)$ its $\mathscr{H}_{h,k}$-character, and by $c_{h,k}(A)$ its dimension.

We briefly recall that if H is a subgroup of a group G and M is an H-module, we can turn M into a G-module by considering the induced G-module structure. In other words, one sets $M^G := \mathbb{K}G \bigotimes_{\mathbb{K}H} M$. This is the so-called G-module induced by M. The relation between the S_n-structure of $V_n^{\mathbb{Z}_2}(A)$ and the $\mathscr{H}_{h,k}$-structure of $V_{h,k}(A)$ is then displayed by the following result (see [4,7]):

Theorem 2.2. *Let A be a superalgebra. Then for all $n \in \mathbb{N}$*

$$V_n^{\mathbb{Z}_2}(A) \cong \sum_{k=0}^{n} (V_{n-k,k}(A))^{S_n}$$

as S_n-modules. In particular,

$$c_n^{\mathbb{Z}_2}(A) = \sum_{k=0}^{n} \binom{n}{k} c_{n-k,k}(A).$$

In this way the study of the S_n-structure of $V_n^{\mathbb{Z}_2}(A)$ is reduced to the study of the modules $V_{n-k,k}(A)$.

We give a small account on the representation theory of the groups $\mathscr{H}_{h,k} = S_h \times S_k$ ($h+k := n$). The irreducible $\mathscr{H}_{h,k}$-characters are in one-to-one correspondence with the pairs of partitions (λ,μ) where $\lambda \vdash h$ and $\mu \vdash k$. More precisely, if χ_v denotes the irreducible $S_{|v|}$-character associated to the partition v, then the irreducible $\mathscr{H}_{h,k}$-character associated to (λ,μ) is $\chi_{\lambda\mu} = \chi_\lambda \otimes \chi_\mu$.

In order to simplify the notation, we shall often identify the irreducible character χ_v of the symmetric group with the corresponding partition $v = (v_1,\dots,v_r)$. So, for instance, we shall write

$$\chi_{n-k,k}(A) = \sum_{\substack{\lambda \vdash n-k \\ \mu \vdash k}} m_{\lambda\mu} \lambda \otimes \mu$$

for some multiplicities $m_{\lambda\mu} = m_{\lambda\mu}(A) \geqslant 0$.

There is another useful approach, involving the notion of *Y-proper polynomials*. If R is an associative algebra, the *commutator* of $a,b \in R$ is the Lie product

$$[a,b] := ab - ba.$$

One defines inductively higher (left-normed) commutators by setting

$$[a_1,\dots,a_n] := [[a_1,\dots,a_{n-1}],a_n].$$

By the Poincarè–Birkhoff–Witt theorem $\mathbb{K}\langle Y,Z\rangle$ has a basis

$$\{y_1^{s_1}\cdots y_p^{s_p}z_1^{t_1}\cdots z_q^{t_q}u_1^{r_1}\cdots u_n^{r_n}|s_h,t_i,r_j\geqslant 0,\quad p,q,n\in\mathbb{N}\},$$

where $u_1,u_2,\dots$ are higher commutators.

We denote by B the vector subspace of $\mathbb{K}\langle Y,Z\rangle$ spanned by all products

$$\{z_1^{r_1}\dots z_m^{r_m}u_1^{s_1}\dots u_n^{s_n}|r_i,s_j\geqslant 0,\quad m,n\in\mathbb{N}\}.$$

The Y-proper polynomials are the elements of B. $B^{(n)}$ denotes its homogeneous component of degree n.

An alternative definition of Y-proper polynomials is the following: f is Y-proper if all formal partial derivatives $\partial f/\partial y_i$, defined by $\partial y_j/\partial y_i := \delta_{i,j}$ (Kronecker delta), are zero for all $i=1,\dots,m$.

It is well known (see, for instance, Lemma 1, Section 2 in [10]) that all graded polynomial identities of a superalgebra A follow from the Y-proper ones. This means that the set $T_2(A)\cap B$ generates the whole $T_2(A)$ as a T_2-ideal. Let us denote $B(A) := B/(T_2(A)\cap B)$.

We shall denote $\Gamma_{h,k}$ the set of multilinear polynomials of $V_{h,k}$ which are Y-proper. It is not difficult to see that $\Gamma_{h,k}$ is a left $\mathscr{H}_{h,k}$-submodule of $V_{h,k}$ and the same holds for $\Gamma_{h,k}\cap T_2(A)$. Hence the factor module

$$\Gamma_{h,k}(A) := \frac{\Gamma_{h,k}}{\Gamma_{h,k}\cap T_2(A)}$$

is an $\mathscr{H}_{h,k}$-submodule of $V_{h,k}(A)$. We shall denote by $\xi_{h,k}$ its character (a $\mathbb{Z}_2$-*graded proper cocharacter of* A) and by $\gamma_{h,k}(A)$ its dimension.

The following result relates the structure of $V_{h,k}(A)$ with the structure of $\Gamma_{h,k}(A)$ (see [10] Proposition 1, Section 2):

Proposition 2.3. *Let A be a unitary superalgebra, and let $\xi_{i,k}(A)=\sum m_{\lambda,\mu}\lambda\otimes\mu$ be the sequence of proper cocharacters of A, Then*

$$\chi_{h,k}(A)=\sum_{\mu\vdash k}\left(\sum_{i=0}^{h}\sum_{\lambda\vdash i}m_{\lambda,\mu}(\chi_{(h-i)}\otimes\chi_\lambda)^{S_h}\right)\otimes\chi_\mu,$$

where $\chi_{(h-i)}$ is the S_{h-i}-irreducible character associated to the partition $(h-i)$. Moreover,

$$c_{h,k}(A)=\sum_{i=0}^{h}\binom{h}{i}\gamma_{i,k}(A).$$

Since in the sequel also ordinary PI-algebras occur, we recall that with obvious meaning one can consider the ordinary S_n-modules $V_n(A)$, for a PI-algebra A. We shall denote by $\chi_n(A)$ its character, and by $c_n(A)$ its dimension. In this ordinary case, it has been proved that the limit

$$\lim_n \sqrt[n]{c_n(A)}$$

does exist for any nontrivial PI-algebra (see [13] and [14]) and it is a non-negative integer, called the *PI-exponent* of A, and denoted by $\exp(A)$. This PI-invariant can be used in order to classify the varieties of PI-algebras, as suggested by the mentioned papers. In the case of superalgebras, one can define a "*superexponent*" by setting

$$\exp^{\mathbb{Z}_2}(A) := \lim_n \sqrt[n]{c_n^{\mathbb{Z}_2}(A)},$$

if this limit does exist.

3. The main result

The main result is the following.

Theorem 3.1. *Let A,B be PI-algebras, and let M be a free (A,B)-bimodule. Let the matrix algebra $R := \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be endowed with the following $\mathbb{Z}_2$-grading*

$$R_0 := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad R_1 := \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}.$$

Then the $\mathbb{Z}_2$-graded cocharacter sequence for R is the following

$$\chi_{n,0}(R) = \chi_n(A \oplus B),$$

$$\chi_{n,1}(R) = \sum_{p=0}^{n} (\chi_p(A) \otimes \chi_{n-p}(B))^{S_n} \otimes \chi_{(1)},$$

$$\chi_{n,k}(R) = 0 \quad \mathit{for}\ k \geqslant 2$$

$$(n \in \mathbb{N}).$$

As a Corollary, we shall obtain that $\exp^{\mathbb{Z}_2}(R) = \exp(A) + \exp(B)$.

In order to prove Theorem 3.1, let us decompose $V_{n,1}^{\mathbb{Z}_2}$ into the sum of linear subspaces in the following way.

Definition 3.2. For every $J \subseteq \hat{n} := \{1,\ldots,n\}$ let us denote by $V_{n,J}$ the subspace of $V_{n,1}$

$$V_{n,J} := \operatorname{span}\langle y_{i_1} \ldots y_{i_p} z y_{j_1} \ldots y_{j_q} \,|\, \{i_1,\ldots,i_p\} = J, \{j_1,\ldots,j_q\} = \hat{n} \setminus J\rangle.$$

Remark 3.3. Clearly, $V_{n,1} = \bigoplus_{J \subseteq \hat{n}} V_{n,J}$.

For a generic element of $V_{n,1}$ the following holds.

Lemma 3.4. *Let* $f=\sum_{J\subseteq\hat{n}} f_J \in V_{n,1}$ $(f_J\in V_{n,J})$. *Then*

$$f\in T_2(R) \text{ if and only if } f_J\in T_2(R) \text{ for all } J\subseteq\hat{n}.$$

Proof. If all the components of f are in $T_2(R)$, it is clear that $f\in T_2(R)$ as well. So assume that $f\in T_2(R)$ and let J be a subset of $\hat{n}$. If $|J|=p$ we may assume without loss of generality that $J=\{1,\dots,p\}$. We want to prove that f_J is zero under all graded substitutions $\mathscr{E}$ of the variables $y_1,\dots,y_n,z$. Actually, since f_J is multilinear, what needs to be checked is just that f_J vanishes under all substitutions $\mathscr{E}$ of type $z\mapsto m\mathbf{e}_{12}$ and $y\mapsto a\mathbf{e}_{11}$ or $y\mapsto b\mathbf{e}_{22}$, where $m\in M$, $a\in A$ and $b\in B$.

It is easy to see that f_J vanishes under the substitution $\mathscr{E}$ in case $y_i\mapsto b_i\mathbf{e}_{22}$ for some $i\leqslant p$. The same happens if $y_i\mapsto a_i\mathbf{e}_{11}$ for some $i>p$. So just the substitutions $\mathscr{E}$ of type

$$y_i\mapsto\begin{cases} a_i\mathbf{e}_{11} & i\leqslant p\\ b_i\mathbf{e}_{22} & i>p\end{cases}\qquad z\mapsto m\mathbf{e}_{12}$$

for $a\in A$, $b\in B$ and $m\in M$ need to be checked.

On the other hand, for each such substitution, the remaining components f_T, for $T\neq J$, of f vanish. Hence

$$0=f|_{\mathscr{E}}=\sum_{T\subseteq\hat{n}} f_T|_{\mathscr{E}}=f_J|_{\mathscr{E}}$$

and f_J vanishes under these substitutions, as well. Therefore, f_J vanishes under all possible substitutions, that is $f_J\in T_2(R)$. □

By the previous argument, we obtain

$$V_{n,1}\cap T_2(R)=\underset{J\subseteq\hat{n}}{\oplus}(V_{n,J}\cap T_2(R)).$$

We may notice that $V_{n,J}\cap T_2(R)$ is not an S_n-module. However, for all $p\leqslant n$ the sum

$$W_{n,p}:=\bigoplus_{\substack{J\subseteq\hat{n}\\ |J|=p}} V_{n,J}$$

is an S_n-module, and we are going to show that the following decomposition of S_n-modules holds:

$$\frac{V_{n,1}}{V_{n,1}\cap T_2(R)}\cong\bigoplus_{p=0}^{n}\frac{W_{n,p}}{W_{n,p}\cap T_2(R)}. \tag{1}$$

It should be clear that this is a vector space isomorphism, under a canonical isomorphism φ. Since the induced action of S_n over the summands $\frac{W_{n,p}}{W_{n,p}\cap T_2(R)}$ is

$$\sigma(f_p+(W_{n,p}\cap T_2(R)))=\sigma(f_p)+(W_{n,p}\cap T_2(R))$$

and this action commutes with the isomorphism φ, the latter is an isomorphism between S_n-modules, as well.

Let us denote the factor module $\frac{W_{n,p}}{W_{n,p}\cap T_2(R)}$ by $W_{n,p}(R)$. By Eq. (1), we are led to study the S_n-module structure of the $W_{n,p}(R)$ in order to obtain the structure of $V_{n,1}(R)$.

Remark 3.5. Notice that if J, $I \subseteq \hat{n}$ have the same cardinality, then the factor vector spaces $V_{n,J}/(V_{n,J} \cap T_2(R))$ and $V_{n,I}/(V_{n,I} \cap T_2(R))$ are isomorphic. In particular, all the vector spaces decomposing $W_{n,p}(R)$ have the same dimension, which we shall denote by $d_{n,p}(R)$. From this it follows that

$$\dim_{\mathbb{K}} W_{n,p}(R) = \binom{n}{p} d_{n,p}(R).$$

Furthermore, let $p \leqslant n$, $J := \{1, \ldots, p\}$, and denote by S_{n-p} the group $Sym\{\hat{n} \setminus J\}$. Notice that there is a canonical $S_p \times S_{n-p}$-action on $V_{n,J}/(V_{n,J} \cap T_2(R))$.

Lemma 3.6. *The following isomorphism of S_n-module holds*:

$$W_{n,p}(R) \cong \mathbb{K}S_n \bigotimes_{\mathbb{K}(S_p \times S_{n-p})} \frac{V_{n,J}}{V_{n,J} \cap T_2(R)}.$$

Proof. As a first remark, let us notice that the vector spaces in the statement have the same dimension over $\mathbb{K}$, as a consequence of general results about induced representations (cf. [5], 12.27).

Now we are going to build up an S_n-module isomorphism between them. Let us define

$$\zeta : \mathbb{K}S_n \times \frac{V_{n,J}}{V_{n,J} \cap T_2(R)} \to \bigoplus_{\substack{I \subseteq \hat{n} \\ |I|=p}} \frac{V_{n,I}}{V_{n,I} \cap T_2(R)}$$

by

$$(\sigma, \bar{f}) \mapsto \overline{\sigma(f)}$$

for all $\sigma \in S_n$ and $\bar{f} \in V_{n,J}/(V_{n,J} \cap T_2(R))$. It is clear that ζ is well-defined. Moreover, it is clearly $\mathbb{K}$-linear.

In order to show that ζ is $S_p \times S_{n-p}$-balanced, it is enough to consider the monomials in place of the whole f. This is clearly true, as well, i.e. for all $\lambda\mu \in S_p \times S_q$, ($q := n - p$), $\sigma \in S_n$ and monomials $\overline{y_{i_1} \ldots y_{i_p} z y_{j_1} \ldots y_{j_q}}$

$$\begin{aligned}\zeta(\sigma(\lambda\mu), \overline{y_{i_1} \ldots y_{i_p} z y_{j_1} \ldots y_{j_q}}) &= \overline{y_{\sigma\lambda(i_1)} \ldots y_{\sigma\lambda(i_p)} z y_{\sigma\mu(j_1)} \ldots y_{\sigma\mu(j_q)}} \\ &= \zeta(\sigma, (\lambda\mu)(\overline{y_{i_1} \ldots y_{i_p} z y_{j_1} \ldots y_{j_q}})).\end{aligned}$$

Therefore ζ induces a homomorphism

$$\bar{\zeta}\colon \mathbb{K}S_n \bigotimes_{\mathbb{K}(S_p \times S_{n-p})} \frac{V_{n,J}}{V_{n,J} \cap T_2(R)} \to \bigoplus_{\substack{I \subseteq \hat{n} \\ |I|=p}} \frac{V_{n,I}}{V_{n,I} \cap T_2(R)}.$$

Since the dimensions of the vector spaces are equal, it is sufficient to prove that $\bar{\zeta}$ is surjective. Indeed if $\overline{y_{i_1} \ldots y_{i_p} z y_{j_1} \ldots y_{j_q}}$ is a monomial in $\bigoplus_{\substack{I \subseteq \hat{n} \\ |I|=p}} \frac{V_{n,I}}{V_{n,I} \cap T_2(R)}$, it is the image of $(\sigma, \overline{y_1 \ldots y_p z y_{p+1} \ldots y_n})$ via $\bar{\zeta}$ where

$$\sigma = \begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & n \\ i_1 & \cdots & i_p & j_1 & \cdots & j_q \end{pmatrix}.$$

Finally, this linear isomorphism clearly commutes with the S_n-action, hence it is an isomorphism of S_n-modules. □

Remark 3.7. Although we wrote the S_n-module $\mathbb{K}S_n \bigotimes_{\mathbb{K}(S_p \times S_{n-p})} \frac{V_{n,J}}{V_{n,J} \cap T_2(R)}$ explicitly, in the standard notation adopted in the representation theory of groups it should be denoted shortly as $\left(\frac{V_{n,J}}{V_{n,J} \cap T_2(R)}\right)^{S_n}$. In other words, it is the S_n-module induced by the $(S_p \times S_{n-p})$-module $\frac{V_{n,J}}{V_{n,J} \cap T_2(R)}$. Therefore, we may rewrite the statement of the previous Lemma as

$$W_{n,p}(R) \cong_{S_n} \left(\frac{V_{n,J}}{V_{n,J} \cap T_2(R)}\right)^{S_n}.$$

The next step is therefore to study, for all $p \leqslant n$, the structure of the $(S_p \times S_{n-p})$-module

$$V_{n,J}(R) := \frac{V_{n,J}}{V_{n,J} \cap T_2(R)},$$

where $J := \{1, \ldots, p\}$.

Remark 3.8. With the same notation, notice that if $f(x_1, \ldots, x_n) \in T(A)$ and $g(x_1, \ldots, x_m) \in T(B)$ then $f(y_1, \ldots, y_n)z \in T_2(R)$ and $zg(y_1, \ldots, y_m) \in T_2(R)$.

Lemma 3.9. *Let $p \leqslant n \in \mathbb{N}$. The following isomorphism of $(S_p \times S_{n-p})$-modules holds*

$$V_{n,J}(R) \cong \frac{V_p}{V_p \cap T(A)} \otimes \frac{V_{n-p}}{V_{n-p} \cap T(B)}.$$

Proof. Let us start by defining

$$\zeta\colon V_p(A) \times V_{n-p}(B) \to V_{n,J}(R)$$

mapping $(f + T(A), g + T(B)) \mapsto (fzg + (V_{n,J} \cap T_2(R)))$. This map is well defined by Remark 3.8.

Actually, the map ζ is $\mathbb{K}$-linear hence induces an homomorphism of linear spaces $\bar{\zeta} : V_p(A) \otimes V_{n-p}(B) \to V_{n,J}(R)$ which commutes with the $(S_p \times S_{n-p})$-action.

Now we have to show that $\bar{\zeta}$ is bijective. To this aim we shall exhibit an inverse for $\bar{\zeta}$.

Let ψ: $V_{n,J} \to V_p(A) \otimes V_{n-p}(B)$ the map defined by

$$m_p z m_{n-p} \mapsto \overline{m_p} \otimes \overline{m_{n-p}},$$

where m_p is a multilinear monomial in the variables $\{y_1, \dots, y_p\}$, m_{n-p} a multilinear monomial in the remaining variables, and by $\overline{m_p}$, $\overline{m_{n-p}}$ we denote $m_p + (V_p \cap T(A))$, $m_{n-p} + (V_{n-p} \cap T(B))$.

In order to induce a $(S_p \times S_{n-p})$-module homomorphism from $V_{n,J}(R)$, we need to check that $\ker\psi \supseteq (V_{n,J} \cap T_2(R))$. Actually, if this inclusion is not true, then there should be a polynomial $f \in (V_{n,J} \cap T_2(R))$ with $\psi(f) \neq 0$. Let us choose $f = \sum_{i=1}^{k} f_i z g_i$ for a minimal k.

We may assume that the g_i are $\mathbb{K}$-linearly independent modulo $T(B)$. Indeed, if $g_1 = \sum_{i \geqslant 2}^{k} \alpha_i g_i$, then $f = \sum_{i=2}^{k} (\alpha_i f_1 + f_i) z g_i$, contradicting the minimality of k.

Moreover, we may assume that $f_i \notin T(A)$ for all $i = 1, \dots, k$. Indeed, let f' be the sum of the $f_i z g_i$ such that $f_i \in T(A)$ (if any) and let f'' be the sum of the remaining ones. Then $f = f' + f''$. We may notice that $f_i \in T(A)$ implies that $f_i z \in T_2(R)$ (by Remark 3.8) and therefore $f' \in T_2(R)$. On the other hand, since the f_i occurring in f' are in $T(A)$ we get $\psi(f') = 0$. Therefore $f'' = f - f'$ satisfies $f'' \in V_{n,J} \cap T_2(R)$ and $\psi(f'') = \psi(f) \neq 0$. Hence, if $f' \neq 0$, we obtain a contradiction to the minimality of k once again.

Finally, these conditions on f, the g_i's and the f_i's imply that the polynomial f cannot be in $T_2(R)$, as showed in the proof of Theorem 1, p. 731 in [9], which is a contradiction. Therefore $\ker\psi \supseteq V_{n,J} \cap T_2(R)$, and we get an induced $S_p \times S_{n-p}$-module homomorphism $\bar{\psi} : V_{n,J}(R) \to V_p(A) \otimes V_{n-p}(B)$ which inverts $\bar{\zeta}$. $\square$

Now the proof of Theorem 3.1 follows easily by collecting the obtained S_n-module isomorphisms.

Proof of Theorem 3.1. The first statement of Theorem 3.1 is trivial: $V_{n,0} \cap T_2(R) = V_n \cap (T(A) \cap T(B)) = V_n \cap T(A \oplus B)$.

In order to prove the second statement, one writes the $S_n \times S_1$-module isomorphisms

$$\begin{aligned} V_{n,1}(R) &\cong \left(\bigoplus_{p=0}^{n} W_{n,p}(R) \right) \otimes \mathbb{K} \\ &\cong \left(\bigoplus_{p=0}^{n} (V_{n,\{1,\dots,p\}}(R))^{S_n} \right) \otimes \mathbb{K} \\ &\cong \left(\bigoplus_{p=0}^{n} (V_p(A) \otimes V_{n-p}(B))^{S_n} \right) \otimes \mathbb{K} \end{aligned}$$

by Lemmas 3.4, 3.6 and 3.9. $\square$

Corollary 3.10. *The graded codimension sequence of R is related to the ordinary codimension sequences of A, B and $A \oplus B$ by the following formula*:

$$c_n^{\mathbb{Z}_2}(R) = c_n(A \oplus B) + n \sum_{p+q=n-1} \binom{n-1}{p} c_p(A) c_q(B). \tag{2}$$

Proof. By Theorem 2.2 one has

$$c_n^{\mathbb{Z}_2}(R) = \sum_{i=0}^{n} \binom{n}{i} c_{n-i,i}(R).$$

By [9], Theorem 1, it follows that $c_{n-i,i}(R) = 0$ if $i \geqslant 2$, hence

$$c_n^{\mathbb{Z}_2}(R) = c_{n,0}(R) + n c_{n-1,1}(R).$$

The explicit formula follows then as a consequence of Theorem 3.1. □

Corollary 3.11. *The $\mathbb{Z}_2$-graded PI-exponent of R is*

$$\exp^{\mathbb{Z}_2}(R) := \lim_n \sqrt[n]{c_n^{\mathbb{Z}_2}(R)} = \exp(A) + \exp(B).$$

Proof. By the results of Giambruno and Zaicev [13,14], the exponent of the algebra A does exist and it is an integer, e_A. Moreover, there exist constants $a_1, \alpha_1, b_1, \beta_1$ such that

$$a_1 n^{b_1} e_A^n \leqslant c_n(A) \leqslant \alpha_1 n^{\beta_1} e_A^n.$$

Similarly, there exist constants a_2, α_2, b_2, β_2 such that

$$a_2 n^{b_2} e_B^n \leqslant c_n(B) \leqslant \alpha_2 n^{\beta_2} e_B^n$$

and constants a_3, α_3, b_3, β_3 such that

$$a_3 n^{b_3} e_{A\oplus B}^n \leqslant c_n(A \oplus B) \leqslant \alpha_3 n^{\beta_3} e_{A\oplus B}^n.$$

Recall that $\exp(A \oplus B) = \max\{\exp(A), \exp(B)\}$. Without loss of generality, let us assume that the maximum is e_A.

Now we are going to find an upper bound for the sequence $c_n^{\mathbb{Z}_2}(R)$. By formula 2 one has

$$c_n^{\mathbb{Z}_2}(R) \leqslant \alpha_3 n^{\beta_3} e_A^n + n \sum_{p+q=n-1} \alpha_1 p^{\beta_1} \alpha_2 q^{\beta_2} \binom{n-1}{p} e_A^p e_B^q.$$

Now, setting $\alpha := \max\{\alpha_1, \alpha_2, \alpha_3\}$ and $\beta := \max\{\beta_1, \beta_2, \beta_3\}$, and noticing that $p, q \leqslant n$, the latter expression satisfies

$$\leqslant \alpha n^{\beta} e_A^n + \alpha^2 n^{2\beta+1} \sum_{p+q=n-1} \binom{n-1}{p} e_A^p e_B^q$$

$$\leqslant \alpha^2 n^{2\beta+1} (e_A^n + (e_A + e_B)^{n-1})$$

$$\leqslant \alpha^2 n^{2\beta+1} (e_A^n + (e_A + e_B)^n).$$

Similarly, we may find a lower bound using formula 2 once again:

$$c_n^{\mathbb{Z}_2}(R) \geqslant a_3 n^{b_3}(e_{A\oplus B})^n + n \sum_{p+q=n-1} a_1 p^{b_1} a_2 q^{b_2} \binom{n-1}{p} e_A^p e_B^q$$

$$\geqslant a_1 a_2 \sum_{p+q=n-1} p^{b_1} q^{b_2} \binom{n-1}{p} e_A^p e_B^q.$$

Set $b := \min\{b_1, b_2\}$ and notice that $p^{b_1} q^{b_2} \geqslant (pq)^b$. If $b \geqslant 0$ then

$$c_n^{\mathbb{Z}_2}(R) \geqslant a_1 a_2 \sum_{\substack{p+q=n-1\\ pq\neq 0}} (pq)^b \binom{n-1}{p} e_A^p e_B^q$$

$$\geqslant a_1 a_2 \sum_{\substack{p+q=n-1\\ pq\neq 0}} \binom{n-1}{p} e_A^p e_B^q$$

$$\geqslant a_1 a_2((e_A + e_B)^{n-1} - e_A^{n-1} - e_B^{n-1}) \geqslant a_1 a_2((e_A + e_B)^{n-1} - 2e_B^{n-1}).$$

If on the contrary $b < 0$ then notice that $(pq)^b \geqslant (n-1)^{2b}$. Therefore

$$c_n^{\mathbb{Z}_2}(R) \geqslant a_1 a_2 \sum_{\substack{p+q=n-1\\ pq\neq 0}} (pq)^b \binom{n-1}{p} e_A^p e_B^q$$

$$\geqslant a_1 a_2 (n-1)^{2b}((e_A + e_B)^{n-1} - 2e_B^{n-1}).$$

Now, if we consider the corresponding nth root sequences, we obtain

$$e_A + e_B \leqslant \lim_n \sqrt[n]{c_n^{\mathbb{Z}_2}(R)} \leqslant e_A + e_B,$$

hence $\exp^{\mathbb{Z}_2}(R) = e_A + e_B$. $\square$

A description of the proper graded cocharacter sequence of R in the spirit of Theorem 3.1 is also possible.

Theorem 3.12. *Let A, B PI-algebras, and let M be a free (A,B)-bimodule. Let the matrix algebra $R := \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be endowed with the following $\mathbb{Z}_2$-grading*

$$R_0 := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad R_1 := \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}.$$

Then the Y-proper $\mathbb{Z}_2$-graded cocharacter sequence for R is the following

$$\xi_{m,0}(R) = \xi_m(A \oplus B)$$

$$\xi_{m,1}(R) = \sum_{r+s+t=m} (\xi_r(A) \otimes (t) \otimes \xi_s(B))^{S_m} \otimes (1)$$

$$(n \in \mathbb{N}),$$

where $\xi_i(A)$ and $\xi_i(B)$ are the ordinary ith proper cocharacters of A and B and (t) denotes the irreducible S_t-character corresponding to the partition (t).

Proof. The relation is clear for $\xi_{n,0}(R)$, Indeed, by the equalities

$$\chi_{n,0}(R) = \chi_n(A \oplus B)$$

and

$$\chi_n(A \oplus B) = \sum_{m=0}^{n} ((n-m) \otimes \xi_m(A \oplus B))^{S_n}$$

the statement follows a fortiori.

For $\xi_{m,1}(R)$ we need to show that

$$\chi_{n,1}(R) = \sum_{m=0}^{n} \left((n-m) \otimes \left(\sum_{r+s+t=m} \xi_r(A) \otimes (t) \otimes \xi_s(B) \right) \right)^{S_n} \otimes (1).$$

We know by Theorem 3.1 that

$$\begin{aligned}
\chi_{n,1}(R) &= \sum_{p+q=n} (\chi_p(A) \otimes \chi(B))^{S_n} \otimes (1) \\
&= \sum_{p+q=n} \left(\sum_{r+t=p} ((t) \otimes \xi_r(A))^{S_p} \otimes \sum_{s+u=q} ((u) \otimes \xi_s(B))^{S_q} \right)^{S_n} \otimes (1) \\
&= \sum_{p+q=n} \left(\sum_{\substack{r+t=p \\ s+u=q}} ((t) \otimes \xi_r(A))^{S_p} \otimes ((u) \otimes \xi_s(B))^{S_q} \right)^{S_n} \otimes (1).
\end{aligned}$$

Now note that the following equality holds:

$$\begin{aligned}
&((t) \otimes \xi_r(A))^{S_p} \otimes ((u) \otimes \xi_s(B))^{S_q} \\
&\quad = ((t) \otimes \xi_r(A) \otimes (u) \otimes \xi_s(B))^{S_p \times S_q}
\end{aligned}$$

(see [5], Theorem 43.2). Hence

$$\begin{aligned}
\chi_{n,1}(R) &= \sum_{p+q=n} \left(\sum_{\substack{r+t=p \\ s+u=q}} ((t) \otimes \xi_r(A) \otimes (u) \otimes \xi_s(B))^{S_q \times S_q} \right)^{S_n} \otimes (1) \\
&= \sum_{p+q=n} \left(\sum_{\substack{r+t=p \\ s+u=q}} (t) \otimes \xi_r(A) \otimes (u) \otimes \xi_s(B) \right)^{S_n} \otimes (1)
\end{aligned}$$

$$= \sum_{m=0}^{n} \left(\sum_{r+s+t=m} (n-m) \otimes \xi_r(A) \otimes (t) \otimes \xi_s(B) \otimes \right)^{S_n} \otimes (1)$$

and the second equality follows from [5], Theorem 38.4. Therefore, a fortiori, the Y-proper graded cocharacter sequence of R is as stated. □

4. Graded cocharacter sequences for $UT_n(\mathbb{K})$

An immediate application of Theorem 3.1 (and of Theorem 3.12) concerns the algebras of upper triangular matrices with entries from $\mathbb{K}$. It is known (see [20]) that if G is an abelian group then all possible G-gradings for $UT_n(\mathbb{K})$ are elementary, i.e. the unit matrices are all G-homogeneous. Here we consider the $\mathbb{Z}_2$-gradings only.

A convenient way to describe a fixed grading is to display a vector $\mathbf{g} \in \mathbb{Z}_2^n$

$$\mathbf{g} = (g_1, \ldots, g_n):$$

the homogeneous G-degree of the unit matrix $\mathbf{e}_{ij}$ is then $g_j - g_i$. Notice that in an elementary grading all diagonal matrix units, $\mathbf{e}_{ii}$, are therefore in the 0-component, hence $\mathbf{1} \in R_0$.

In [9], Corollary, it has been proved that if $UT_n(\mathbb{K})$ is endowed with the $\mathbb{Z}_2$-grading

$$\mathbf{g} := (\underbrace{0, \ldots, 0}_{k}, 1, \ldots, 1),$$

then

$$T_2(UT_n(\mathbb{K})) = T_2(R) \text{ for } R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix},$$

where A, B are PI-algebras satisfying $T(A) = T(UT_k(\mathbb{K}))$, $T(B) = T(UT_{n-k}(\mathbb{K}))$ and M is a free (A, B)-bimodule.

Proposition 4.1. *Let $UT_n(\mathbb{K})$ be endowed by the $\mathbb{Z}_2$-grading $(\underbrace{0, \ldots, 0}_{k}, 1, \ldots, 1)$. Then its Y-proper graded cocharacter sequence is*

$$\xi_{m,0}^{\mathbb{Z}_2}(UT_n(\mathbb{K})) = \sum_{r=0}^{\max\{k, n-k\}} \sum_{\substack{p_i \geqslant 2 \\ p_1 + \cdots + p_r = m}} ((p_1 - 1, 1) \otimes \cdots \otimes (p_r - 1, 1))^{S_m}$$

$$\xi_{m,1}^{\mathbb{Z}_2}(UT_n(\mathbb{K})) = \sum_{r=0}^{n-2} \sum_{\substack{p_i \geqslant 2 \\ p_1 + \cdots + p_r + t = m}} ((p_1 - 1, 1) \otimes \cdots \otimes (p_r - 1, 1) \otimes (t))^{S_m} \otimes (1)$$

for $m \geqslant 2$.

Proof. We recall as a key step that the proper cocharacter sequence for the algebra $UT_k(\mathbb{K})$ has been obtained by Drensky and Kasparian in [11], Theorem 2.7, and is the following (up to the notation)

$$\xi_m(UT_k(\mathbb{K})) = \sum_{r=0}^{k-1} \sum_{\substack{p_i \geqslant 2 \\ p_1+\cdots+p_r=m}} ((p_1-1,1)\otimes\cdots\otimes(p_r-1,1))^{S_m}. \tag{3}$$

In order to show the first equality of the proposition, we recall that $T(A\oplus B) = T(A)\cap T(B)$ and that if $k \geqslant h$ then $T(UT_k(\mathbb{K})) \subseteq T(UT_h(\mathbb{K}))$. Hence, $T(UT_k(\mathbb{K}) \oplus UT_{n-k}(\mathbb{K})) = T(UT_j(\mathbb{K}))$ where $j = \max\{k, n-k\}$. The formula of the proposition follows then by Theorem 3.12 and by formula 3.

Now we are going to prove the second equality of the proposition. By Theorem 3.12 it holds that

$$\xi^{\mathbb{Z}_2}_{m,1}(UT_n(\mathbb{K})) = \sum_{r+s+t=m} (\xi_r(UT_k(\mathbb{K}))\otimes(t)\otimes\xi_s(UT_{n-k}(\mathbb{K}))^{S_m}\otimes(1).$$

Then applying Eq. (3) to the ξ_r and ξ_s, and rearranging the order of the factors in the tensor, we may rewrite

$$\begin{aligned}
&\xi^{\mathbb{Z}_2}_{m,1}(UT_n(\mathbb{K}))\\
&\quad = \sum_{r+s+t=m} \left(\left(\sum_{a=0}^{k-1} \sum_{\substack{p_i \geqslant 2 \\ p_1+\cdots+p_a=r}} (p_1-1,1)\otimes\cdots\otimes(p_a-1,1) \right)^{S_r} \right.\\
&\qquad \left. \otimes \left(\sum_{b=0}^{n-k-1} \sum_{\substack{q_i \geqslant 2 \\ q_1+\cdots+q_b=s}} (q_1-1,1)\otimes\cdots\otimes(q_b-1,1) \right)^{S_s} \otimes (t) \right)^{S_m} \otimes (1).
\end{aligned}$$

By [5], Theorem 43.2, it follows that

$$\begin{aligned}
\xi^{\mathbb{Z}_2}_{m,1}(UT_n(\mathbb{K})) &= \sum_{r+s+t=m} \sum_{a=0}^{k-1} \sum_{b=0}^{n-k-1} \sum_{\substack{p_i,q_j \geqslant 2 \\ p_1+\cdots+p_a=r \\ q_1+\cdots+q_b=s}} ((p_1-1,1)\otimes\cdots\otimes(p_a-1,1)\\
&\quad \otimes(q_1-1,1)\otimes\cdots\otimes(q_b-1,1)\otimes(t))^{S_m}\otimes(1)\\
&= \sum_{l=0}^{n-2} \sum_{\substack{h_i \geqslant 2 \\ h_1+\cdots+h_l+t=m}} ((h_1-1,1)\otimes\cdots\otimes(h_l-1,1)\otimes(t))^{S_m}\otimes(1)
\end{aligned}$$

by rewriting suitably the summation. □

It is possible to give the graded cocharacter sequences for the algebra $UT_3(\mathbb{K})$ with respect to all possible nonequivalent $\mathbb{Z}_2$-gradings. These are the following:

(1) $(0,0,0)$. The even part is then $UT_3(\mathbb{K})$ and the odd part is simply 0. The Y-proper graded cocharacter sequence is then the proper cocharacter sequence of the full algebra, and it is a particular case of formula 3 obtained by Drensky and Kasparian in [11].

(2) $(0,0,1)$. In this case the even and odd parts of the algebra are

$$R_0 = \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad R_1 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix},$$

where the $*$ denote entries from $\mathbb{K}$. Its Y-proper graded cocharacter sequence is described by Theorem 3.12 and the result of Drensky and Kasparian expressed by formula 3. Explicitly, it is the following

$$\xi^{\mathbb{Z}_2}_{m,0} = \xi_m(UT_2(\mathbb{K})) = (m-1,1)$$

$$\xi^{\mathbb{Z}_2}_{m,1} = \sum_{p+q=m} ((p-1,1)\otimes(q))^{S_m} \otimes (1).$$

Its graded codimension sequence can be obtained by Corollary 3.10. Recall that the codimension sequence for $UT_2(K)$ is $c_n(UT_2(\mathbb{K})) = 2 + 2^{n-1}(n-2)$ (see [18]). Therefore, one has

$$\begin{aligned} c_n^{\mathbb{Z}_2}(R) &= c^{\mathbb{Z}_2}_{n,0}(R) + n \sum_{p+q=n-1} \binom{n-1}{p} c_p(UT_2(\mathbb{K}))c_q(\mathbb{K}) \\ &= c_n(UT_2(\mathbb{K})) + n \sum_{p+q=n-1} \binom{n-1}{p} c_p(UT_2(\mathbb{K}))c_q(\mathbb{K}) \\ &= 2 + 2^{n-1}(n-2) + n \sum_{p+q=n-1} \binom{n-1}{p} (2 + 2^{p-1}(p-2)) \\ &= 3^{n-2}n(n-4) + 2^{n-2}(3n-2) + 2 \end{aligned}$$

as one can easily verify using the formula

$$\sum_{p=0}^{n} \binom{n}{p} p2^p = 2n3^{n-1},$$

since the equality $p\binom{n}{p} = n\binom{n-1}{p-1}$ holds.

From this it follows that the superexponent of R is

$$\exp^{\mathbb{Z}_2}(R) = \lim_n \sqrt[n]{c_n^{\mathbb{Z}_2}(R)} = 3.$$

(3) $(0,1,1)$. In this case the grading is

$$R_0 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the relatively free superalgebra is the opposite of the relatively free superalgebra of the previous case. It is immediate that the quantitative information are the same.

One grading is not included in the previous list. It is the grading $(0,1,0)$. Explicitly, it is the following

$$R_0 = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

In the rest of this section we are going to study the graded cocharacter sequence for this algebra.

From the results in [8] it follows that

Proposition 4.2. *Let* $R := \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \oplus \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$ *the* $\mathbb{Z}_2$*-grading for the algebra* $UT_3(\mathbb{K})$). *Then*

$$\mathscr{B} := \{z_1z_2z_3, z_1[y_1,y_2], [y_1,y_2]z_1, [y_1y_2][y_3,y_4]\}$$

is a basis for the graded polynomial identities of R. *Moreover, the spaces of* Y*-proper multilinear polynomials* $\Gamma_{n,0}(R)$, $\Gamma_{n,1}(R)$, $\Gamma_{n,2}(R)$ (*for* $n \geqslant 0$) *have the following* $\mathbb{K}$*-basis*:

$\Gamma_{n,0}(R)$: $[y_{i_1}, y_1, y_{i_2}, \dots, y_{i_{n-1}}]$ *with* $i_2 < i_3 < \cdots < i_{n-1}$ *and* $\{i_1,\dots,i_{n-1}\} = \{2,\dots,n\}$.
$\Gamma_{n,1}(R)$: $[z_1, y_1, \dots, y_n]$.
$\Gamma_{n,2}(R)$: $[z_1, y_{i_1}, \dots, y_{i_k}][z_2, y_{i_{k+1}}, \dots, y_{i_n}]$, $[z_2, y_{i_1}, \dots, y_{i_k}][z_1, y_{i_{k+1}}, \dots, y_{i_n}]$, *where* $k = 0,1,\dots,n$, $i_1 < i_2 < \cdots < i_k$ *and* $i_{k+1} < i_{k+2} < \cdots < i_n$.

Corollary 4.3. *With the same notation of the previous Proposition*, *the* Y*-proper graded codimension sequences for* R *are the following*:

$$\gamma_{n,0}(R) = n-1$$

$$\gamma_{n,1}(R) = 1$$

$$\gamma_{n,2}(R) = 2^{n+1}.$$

Now we will list the proper cocharacter sequences.

Proposition 4.4. *With the same notation of Proposition* 4.2, *the* Y*-proper graded cocharacter sequences of* R *are the following*:

$$\xi_{n,0}(R) = (n-1,1)$$

$$\xi_{n,1}(R) = (n)\otimes(1)$$

$$\xi_{n,2}(R) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m_k(R)((n-k,k)\otimes(2))$$

$$\oplus \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m_k(R)((n-k,k)\otimes(1,1)),$$

where $\lfloor \frac{n}{2} \rfloor$ *denotes the largest integer not greater than* $n/2$ *and* $m_k(R){:} = n-2k+1$.

Proof. It should be clear that the S_n-submodule generated by $[y_2, y_1, y_3, \ldots, y_n]$ in $\Gamma_{n,0}(R)$ is isomorphic to the irreducible submodule associated to the partition $(n-1,1)$. By counting the dimension, it is the whole $\Gamma_{n,0}(R)$. Hence the first equality holds. The same argument, even more easily shows that the second equality is true. A little more tricky is to show the last equality.

First, notice that $[z_1 y_1, \ldots, y_k]$ generates an $S_k \times S_1$-submodule isomorphic to the irreducible module associated to the "double partition" $(k)\otimes(1)$. The same arguments show that $[z_2, y_{k+1}, \ldots, y_n]$ generates an $S_{n-k} \times S_1$-module isomorphic to the one corresponding to $(n-k)\otimes(1)$. Therefore the polynomial $[z_1, y_1, \ldots, y_k][z_2, y_{k+1}, \ldots, y_n]$ generates an $(S_k \times S_{n-k}) \times (S_1 \times S_1)$-module M isomorphic to

$$((k)\otimes(n-k))\otimes((1)\otimes(1)).$$

By Proposition 4.2, the $S_n \times S_2$-submodule of $\Gamma_{n,2}(R)$ generated by that polynomial is isomorphic to the $S_n \times S_2$-induced submodule $M^{S_n \times S_2}$. Its dimension over $\mathbb{K}$ can be computed noticing that

$$\dim_{\mathbb{K}}((k)\otimes(n-k))^{S_n} = [S_n{:}(S_k \times S_{n-k})] = \binom{n}{k}.$$

(see for instance [5], 12.27). By "moving" the square brackets correspondingly to the possible $k = 0, 1, \ldots, n$, one obtains in $\Gamma_{n,2}(R)$ the $S_n \times S_2$-submodule

$$\sum_{k=0}^{n} ((k)\otimes(n-k))^{S_n} \otimes ((1)\otimes(1))^{S_2},$$

whose dimension over $\mathbb{K}$ is exactly 2^{n+1}. Therefore, it is the whole $\Gamma_{n,2}(R)$.

Finally, by applying the Young-Pieri rule, one gets the decomposition into $S_n \times S_2$-irreducible modules:

$$\left(\sum_{k=0}^{\lfloor n/2 \rfloor} m_k(R)(n-k,k)\right) \otimes ((2) \oplus (1,1)),$$

where $m_k(R) = n - 2k + 1$. Therefore, its character follows accordingly. $\square$

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