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Some general techniques on linear preserver problems

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Abstract

Several general techniques on linear preserver problems are described. The first one is based on a transfer principle in Model Theoretic Algebra that allows one to extend linear preserver results on complex matrices to matrices over other algebraically closed fields of characteristic 0. The second one concerns the use of some simple geometric technique to reduce linear preserver problems to standard types so that known results can be applied. The third one is about solving linear preserver problems on more general (operator) algebras by reducing the problems to idempotent preservers. Numerous examples will be given to demonstrate the proposed techniques. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

An active research topic in matrix theory is the linear preserver problems (LPP) that deal with the characterization of linear operators on matrix spaces with some

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special properties such as leaving certain functions, subsets or relations invariant. One may see [37] for an extensive survey and see [27] for a gentle introduction of the subject. As mentioned in [27], in the study of LPP one may focus on one specific question (see [37, Chapter 3]) or a family of related questions (see [37, Chapters 2 and 4]). Also, one may focus on general techniques that can cover many different LPP (see, e.g., [27, Sections 5 and 6]). In fact, there are a number of well-developed techniques for studying LPP. To name a few examples, we have:

(i) the projective geometry technique (see [37, Chapter 4 and Section 8.5]),
(ii) the algebraic geometry technique (see [12,18,26]),
(iii) the differential geometry technique (see [27, Section 6] and references therein),
(iv) the duality technique (see [27, Section 6] and references therein),
(v) the group theory technique (see [11,13,14,17,38,43] and [37, Section 8.4]), and
(vi) the functional identity technique (see [4]).

In this paper, we describe three more general techniques for studying LPP.

First, we discuss how to use a transfer principle in Model Theoretic Algebra to extend linear preserver results on complex matrices to matrices over other algebraically closed fields of characteristic 0.

In the study of LPP, many results were first obtained for complex matrices, and then extended to matrices over other fields or rings. Sometimes it is easy to do the extension, but in some cases a great deal of effort is needed to achieve the goal. In Section 2, we show that using the transfer principle in Model Theoretic Algebra provides an efficient mean to do the job in many situations. Of course, another obvious advantage of this approach is: one can use all kinds of complex analysis techniques to prove results for the complex case and extend them to other fields whenever the transfer principle is applicable.

It is worth noting that a standard procedure of studying LPP on matrix spaces over an arbitrary field (or even ring) is to solve the corresponding problem in the algebraic closure of the field and then deduce the results for the original problem. Of course, precautions have to be taken in the processes of “going up”, i.e., extending the problem to the algebraically closed field, and “coming back”, i.e., specializing the result to the original field. Thus, having results on algebraically closed fields is useful in studying LPP on arbitrary fields.

Another common scheme for solving LPP is to reduce the given question to some well-studied LPP such as the rank preserver or nilpotent preserver problems so that known results can be applied. In Section 3, we discuss a geometric technique that can be used to do the reduction. As one can see in (i)–(iii), geometric techniques have often been used in the study of LPP. The technique we are going to introduce is linear algebraic and elementary in nature and does not require too much additional knowledge of other areas. Yet, examples will be given to show that the technique can be used to deduce some non-trivial results effectively.

Finally, we consider LPP on infinite-dimensional spaces or other general algebras. In Chapter 4, we show that an efficient way to study LPP in infinite-dimensional case is to reduce the problem to idempotent preserver problem.

The following notation will be used in our discussion:

- $M_{m,n}(\mathbf{F})$: the space of $m \times n$ matrices over the field $\mathbf{F}$,
- $M_n(\mathbf{F})$: $M_{n,n}(\mathbf{F})$,
- $\{E_{11}, E_{12}, \ldots, E_{mn}\}$: standard basis for $M_{m,n}(\mathbf{F})$,
- $\sigma(A)$: spectrum of $A \in M_n(\mathbf{F})$.

2. A transfer principle

In this section, we discuss how to use a transfer principle in Model Theoretic Algebra to study LPP. It is worth noting that there were attempts to apply the transfer principle to prove some results in Algebraic Geometry, see [39,44]. Let us begin by introducing some basic terminology. Our main references are Refs. [10,22,40].

Definition 2.1. First-order sentences in the language of fields are those mathematical statements which can be written down using only:

(a) variables denoted by $x, y, \ldots$ varying over the elements of the field;
(b) the distinguished elements "0" and "1";
(c) the quantifiers "for all" ($\forall$) and "there exists" ($\exists$);
(d) the relation symbol "$=$";
(e) the function symbols "$+$" and "$\cdot$";
(f) logical connectives: $\neg$ (negation), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), and $\leftrightarrow$ (equivalent);
(g) the separation symbols: left square bracket " [" and right square bracket "]".

First-order conditions or properties are those conditions or properties describable in first-order sentences.

Definition 2.2. Two fields $\mathbf{F}_1$ and $\mathbf{F}_2$ are elementarily equivalent if and only if the set of all first-order statements that are true in $\mathbf{F}_1$ is the same as the set of all first-order statements that are true in $\mathbf{F}_2$.

We have the following result (see [22, Theorem 1.13]).

Theorem 2.3 (Transfer principle). *Two algebraically closed fields* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *are elementarily equivalent if and only if* $\mathrm{char}(\mathbf{F}_1) = \mathrm{char}(\mathbf{F}_2)$. *Consequently, if a first-order property holds in one algebraically closed field it holds in each algebraically closed field of the same characteristic.*

Let us describe the general idea of how to apply the transfer principle to extend linear preserver results on complex field to general algebraically closed field of characteristic 0 in the following.

Suppose we want to prove that the linear preservers of a certain first-order property L on $m \times n$ matrices over $\mathbf{F}$ have a specific form describable in first-order sentences.

We formulate the following assertion concerning the field **F** as follows: "Given positive integers m and n, if a linear map $\phi : M_{m,n}(\mathbf{F}) \to M_{m,n}(\mathbf{F})$ has the preserving property P, then ϕ is of the specific form". Here, of course, the preserving property P can be expressed as: "For every $A \in M_{m,n}(\mathbf{F})$ we have: A has property L implies that $\phi(A)$ has property L". Since one can identify ϕ as a family of $(mn)^2$ elements in **F** acting on mn tuples of elements in **F** under the usual rule of linear map, that involves only multiplications and additions of the elements, it is evident that the assertion can be formalized by first-order statements in the language of fields. Therefore, if we can obtain the result for complex matrices, then the transfer principle will ensure that the same result holds for any algebraically closed field of characteristic 0.

Let us illustrate this scheme in the following. Some details will be given to the proof of the first result. Then a number of other examples with references will be mentioned with brief comments.

In [1], Beasley characterized those linear operators on $M_{m,n}(\mathbf{C})$ mapping the set of rank r matrices into itself, where $r \leqslant \min\{m, n\}$ is a fixed positive integer. His proof depends heavily on a result on rank r spaces (see Definition 2.8) on complex matrices by Westwick [45]. Meshulam (see, e.g., [28]) later extended the result of Westwick to algebraically closed fields of characteristic 0, and the rank r matrix preserver result of Beasley was then extended accordingly. In the following, we illustrate how to extend the result of Beasley to arbitrary algebraically closed field of characteristic 0 using the transfer principle.

Theorem 2.4. *Let* **F** *be an algebraically closed field of characteristic* 0. *Suppose* r, m, n *are positive integers such that* $r \leqslant \min\{m, n\}$. *If* ϕ *is a linear operator acting on* $M_{m,n}(\mathbf{F})$ *mapping the set of rank r matrices into itself, then there exist invertible* $P \in M_m(\mathbf{F})$ *and* $Q \in M_n(\mathbf{F})$ *such that* ϕ *is of the form*

$$X \mapsto PXQ \quad \text{or} \quad X \mapsto PX^{\mathrm{t}}Q \ \text{ in the case of } m = n. \tag{2.1}$$

Proof. For the complex case, see [1]. For the general algebraically closed field, we use the transfer principle. In view of the explanation given above it is enough to show that for a matrix $A \in M_{m,n}(\mathbf{F})$ (identified with an mn-tuple of elements of **F**) the property of being of rank r can be formalized as a first-order sentence and also that the forms (2.1) are describable in first-order sentences. The statement "rank $A = r$" is equivalent to:

(a) there exists an $r \times r$ submatrix with non-zero determinant, and
(b) if $r < \min\{m, n\}$, then all determinants of $(r + 1) \times (r + 1)$ submatrices are zero.

So, a finite set of expressions involving only $+$, $\cdot$, and our variables must hold true.

To see that the conclusion of the theorem is also describable in first-order sentences, one needs only to check the existence of collections of m^2 and n^2 elements in **F** corresponding to the matrices P and Q with $\det P \neq 0$ and $\det Q \neq 0$ so that

(i) $\phi(X) = PXQ$ for all $m \times n$ matrix X, or
(ii) $\phi(X) = PX^{\mathrm{t}}Q$ for all $m \times n$ matrix X in case $m = n$. $\square$

One can specialize the above theorem to the case when $m = n = r$ to get the result on linear preservers of the general linear group in $M_n(\mathbf{F})$. Alternatively, one can apply the transfer principle to LPP related to classical groups on $M_n(\mathbf{C})$, see [2,31,36], and deduce the results on more general fields. For instance, we have the following result.

Theorem 2.5. *Let $M_n(\mathbf{F})$ be the algebra of $n \times n$ matrices over an algebraically closed field $\mathbf{F}$ of characteristic 0. Suppose ϕ is a linear operator on M_n mapping the general (special) linear group into itself. Then there exist invertible $P, Q \in M_n$ (with $\det(PQ) = 1$) such that ϕ is of the form*

$$X \mapsto PXQ \quad or \quad X \mapsto PX^{\mathrm{t}}Q.$$

The transfer principle works well for linear preservers of relations. One can extend many results in [18,19] concerning linear preservers of equivalence relations on complex matrix spaces to arbitrary algebraically closed fields of characteristic 0. More precisely, we have the following result.

Theorem 2.6. *Let $\sim$ be any one of the following equivalence relations on matrices*:
(a) (Equivalence) *$A \sim B$ in $M_{m,n}(\mathbf{F})$ if $B = PAQ$ for some invertible $P \in M_m(\mathbf{F})$ and $Q \in M_n(\mathbf{F})$*;
(b) (Similarity) *$A \sim B$ in $M_n(\mathbf{F})$ if $B = S^{-1}AS$ for some invertible $S \in M_n(\mathbf{F})$*;
(c) (t-congruence or orthogonally t-congruence) *$A \sim B$ in $M_n(\mathbf{F})$ (or on symmetric or skew-symmetric matrices) if $B = S^{\mathrm{t}}AS$ for some invertible $S \in M_n(\mathbf{F})$ (with $S^{\mathrm{t}}S = I$).*

Then the corresponding linear preserver results on complex matrices are valid for matrices over any algebraically closed field of characteristic 0.

Note that sometimes we have to restate the hypotheses of the linear preserver results to see that they are indeed first-order conditions. For example, A is similar to B can be expressed as follows: there exists $T \in M_n$ such that $\det(T) \neq 0$ and $\det(T)A = \operatorname{adj}(T)BT$, which is a first-order condition, here $\operatorname{adj}(T)$ denotes the adjoint of the matrix T.

The transfer principle has been used in [34] to prove the following result on linear preservers of the commutativity relation.

Theorem 2.7. *Let $\mathbf{F}$ be an algebraically closed field of characteristic 0, and let $n \geqslant 3$. Suppose ϕ is a linear operator acting on $M_n(\mathbf{F})$ such that*

$$\phi(A)\phi(B) = \phi(B)\phi(A) \quad whenever \; AB = BA.$$

Then either the range of ϕ is commutative or there exists a scalar α, an invertible S and a linear functional f on $M_n(\mathbf{F})$ such that ϕ is of the form

$$X \mapsto \alpha S^{-1}XS + f(X)I \quad \text{or} \quad X \mapsto \alpha S^{-1}X^{\mathrm{t}}S + f(X)I. \tag{2.2}$$

Also, one may consider other LPP arising in applications. In systems theory, notion of controllability plays an important role, see [25]. Linear controllability preservers over $\mathbf{C}$ were characterized in [16] and the results can be extended to any algebraically closed field $\mathbf{F}$ with characteristic 0.

The transfer principle can also be used to extend results related to LPP. We illustrate this on the results concerning rank r spaces—an important concept and tool in the study of rank preservers.

Definition 2.8. Let r, m, n be positive integers such that $r \leqslant \min\{m, n\}$. A linear subspace $V \subseteq M_{m,n}(\mathbf{F})$ is called a rank r space if $A \in V$ implies either rank $A = r$ or $A = 0$.

One of the most interesting questions in the theory of rank r spaces important especially for LPP is what is the maximal dimension of such subspaces. In [45] one can find several estimates (depending, of course, on m, n and r) for these maximal dimensions in the complex case. One readily checks that these results can be formalized as first-order sentences. Hence, we have the following result.

Theorem 2.9. *If every rank r space in $M_{m,n}(\mathbf{C})$ has dimension at most k, then so is a rank r space in $M_{m,n}(\mathbf{F})$ for every algebraically closed field $\mathbf{F}$ of characteristic* 0.

While the transfer principle works very well with many linear preserver (and related) problems, it is not applicable to questions involving A^*—the conjugate transpose of a matrix A. Here, we discuss a slight extension of the transfer principle that allows us to get around the problem.

Definition 2.10. A field $\mathbf{F}$ is called real closed if $\mathbf{F}$ admits an ordering as an ordered field and no proper algebraic extension has this property.

We have the following result concerning real closed fields (see [42, Chapter XI, Section 81] and [22, Theorem 1.16]).

Theorem 2.11. *Real closed field is not algebraically closed, but the extension of a real closed field with the square root of* (-1) *is algebraically closed. Moreover, any two real closed fields are elementarily equivalent.*

Now, let us consider those algebraically closed fields obtained by extending a real closed field with the square root of (-1). It then follows that all $\mathbf{F}[\sqrt{-1}]$ are elementarily equivalent. In $\mathbf{F}[\sqrt{-1}]$, we can consider the involution $(a+b\sqrt{-1})^* = a - b\sqrt{-1}$ as in the complex field. Furthermore, we can define the conjugate transpose A^* of a matrix A. With this setting, many linear preserver results on properties or invariants involving complex conjugate can be transferred to such algebraically closed fields. We mention a few examples in the following, see [19].

Theorem 2.12. *Let* $\mathbf{F}$ *be an algebraically closed field obtained by extending a real closed field with the square root of* (-1). *Suppose* $\sim$ *is any one of the following equivalence relations on matrices over* $\mathbf{F}$:
(a) (Unitary equivalence) $A \sim B$ *in* $M_{m,n}(\mathbf{F})$ *if* $B = UAV$ *for some invertible* $U \in M_m(\mathbf{F})$ *and* $V \in M_n(\mathbf{F})$ *satisfying* $U^*U = I_m$ *and* $V^*V = I_n$;
(b) (*-Congruence and unitary similarity) $A \sim B$ *in* $M_n(\mathbf{F})$ *if* $B = S^*AS$ *for some invertible* $S \in M_n(\mathbf{F})$ (*satisfying* $S^*S = I_n$);
(c) (Con-similarity) $A \sim B$ *in* $M_n(\mathbf{F})$ *if* $B = S^{-1}A\overline{S}$ *for some invertible* $S \in M_n(\mathbf{F})$.
Then the corresponding linear preserver results on complex matrices are valid for matrices over $\mathbf{F}$.

Similarly, one may extend the results on linear preservers of the unitary group, see [3,29].

There are many other examples of linear preservers and related problems for which the transfer principle or the extended transfer principle is applicable. We will let the readers explore them.

3. A geometric technique

In this section, we discuss some techniques of reducing a linear preserver problem to some well-known cases. Such ideas of treating LPP have been used by many researchers. The real question is whether we can find a systematic and efficient way to do the reduction. Here, we propose a very simple linear algebraic method and show that it is indeed very useful despite its simple nature.

To describe our scheme, we need the following definition.

Definition 3.1. Suppose $\mathscr{S}$ is a set of matrices in $M_{m,n}(\mathbf{F})$. For a non-negative integer r, let $\mathscr{T}_r(\mathscr{S})$, or simply $\mathscr{T}_r$ if the meaning of $\mathscr{S}$ is clear in the context, be the set of matrices $A \in M_{m,n}(\mathbf{F})$ such that there exists $C \in \mathscr{S}$ satisfying $C + \alpha A \in \mathscr{S}$ for all but at most r scalar α.

The set $\mathscr{T}_r$ can be viewed as the set of all possible "directions" or "slopes" of "punctured lines" lying in $\mathscr{S}$ with at most r missing points.

Now, suppose we are interested in studying linear operators ϕ such that

$$\phi(\mathscr{S}) \subseteq \mathscr{S} \quad \text{or} \quad \phi(\mathscr{S}) = \mathscr{S}. \tag{3.1}$$

Evidently, such a ϕ also satisfies

$$\phi(\mathscr{T}_r) \subseteq \mathscr{T}_r$$

for any non-negative integer r. If $\mathscr{T}_r$ has a simple structure, say, it is the set of rank k matrices or a union of similarity orbits of nilpotent matrices, then we can use the well-studied results on rank preservers (see, e.g., [1] and Theorem 2.4 in Section 2) or nilpotent preservers (see, e.g., [26, Lemma 2.5]) to help solve the original problem.

In the following, we illustrate how to reduce some LPP to nilpotent preserver problems using the proposed scheme. Note that similar ideas have been used by other authors [23,26,41]. We need one more definition.

Definition 3.2. Let $\mathscr{S}$ be a union of similarity orbits in $M_n(\mathbf{F})$. We say that $\mathscr{S}$ has property (N_r) if the set $\mathscr{T}_r$ in Definition 3.1 is a subset of nilpotent matrices.

Theorem 3.3. *Let* $\mathbf{F}$ *be an algebraically closed field of characteristic* 0, *and let* $\mathscr{S} \subseteq M_n(\mathbf{F})$ *be a union of similarity orbits. Suppose*
(a) $\mathscr{S} \nsubseteq \mathbf{F}I$ *has property* (N_r) *for some positive integer* r, *or*
(b) $\mathscr{S}$ *contains a non-scalar diagonal matrix and has property* (N_0).
If ϕ *is an invertible linear operator on* $M_n(\mathbf{F})$ *satisfying* $\phi(\mathscr{S}) \subseteq \mathscr{S}$, *then there exist a non-zero* $c \in \mathbf{F}$ *and* $A, B \in M_n(\mathbf{F})$ *with* A *invertible such that* ϕ *is of the form*

$$X \mapsto cAXA^{-1} + (\operatorname{tr} X)B \quad \textit{or} \quad X \mapsto cAX^{\mathrm{t}}A^{-1} + (\operatorname{tr} X)B.$$

Proof. Suppose (a) holds. Let r be a positive integer such that $\mathscr{T}_r$ associated with $\mathscr{S}$ is a subset of nilpotent matrices. Now, suppose C is a non-scalar matrix in $\mathscr{S}$. We may assume that C is in the Jordan canonical form. Assume first that C is diagonal. Then $C = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_k \neq \lambda_{k+1}$ for some positive integer k, $1 \leqslant k \leqslant n-1$. Clearly, $C + \alpha E_{k,k+1}$ is similar to C for every scalar α. So, $E_{k,k+1} \in \mathscr{T}_r$. If C is not diagonal, then $C = D + N$ with D diagonal and $N \neq 0$ having non-zero elements only on the first upper diagonal. As $D + N + \alpha N$ is similar to $D + N$ for every $\alpha \neq -1$ we have $N \in \mathscr{T}_r$. Therefore, $\mathscr{T}_r$ contains a non-zero nilpotent. If $N \in \mathscr{T}_r$, then the similarity orbit of N is a subset of $\mathscr{T}_r$. So, $\mathscr{T}_r \neq \{0\}$ is a (finite) union of similarity classes of nilpotent matrices.

Now, if ϕ is an invertible linear operator satisfying $\phi(\mathscr{S}) \subseteq \mathscr{S}$, then we already know that $\phi(\mathscr{T}_r) \subseteq \mathscr{T}_r$, and consequently, $\phi(\overline{\mathscr{T}_r}) \subseteq \overline{\mathscr{T}_r}$. Here, $\overline{\mathscr{T}_r}$ denotes the Zariski closure of $\mathscr{T}_r$. In particular, rank one nilpotents are mapped into nilpotents. As rank one nilpotents span $M_n(\mathbf{F})'$, the subspace of all matrices with zero trace, we conclude that $M_n(\mathbf{F})'$ is invariant under ϕ. Therefore by [26, Lemma 2.5] ϕ is of the asserted form on trace zero matrices. Now, putting $B = (1/n)(\phi(I) - cI)$, we get the conclusion.

Similarly, one can prove the proposition if (b) holds. □

We will now show that the hypotheses of Theorem 3.3 are satisfied in many cases. Let $\mathscr{S}$ be a union of similarity orbits of matrices. Assume also that there are $2n$ distinct elements $\lambda_1, \dots, \lambda_{2n} \in \mathbf{F}$ such that $\sigma(A) \cap \{\lambda_1, \dots, \lambda_{2n}\} = \emptyset$ for any $A \in \mathscr{S}$. Then $\mathscr{S}$ has property (N_1). Indeed, assume that for a matrix $N \in M_n(\mathbf{F})$ there exists $C \in \mathscr{S}$ such that $C + \alpha N \notin \mathscr{S}$ for at most one scalar. If such a scalar exists, we denote it by α_0. Consider

$$\det(\lambda I - C - \mu N) = f(\lambda, \mu) = p_0(\lambda) + p_1(\lambda)\mu + \cdots + p_n(\lambda)\mu^n.$$

Observe that $p_0(\lambda)$ is a monic polynomial of degree n, and all other p_j have degree at most $n-1$. In particular, each p_j, $j = 1, 2, \dots, n$, either vanish at at most $n-1$ points from the set $\{\lambda_1, \dots, \lambda_{2n}\}$, or it is zero. We claim that $p_j(\lambda) \equiv 0$ for all $j = 1, 2, \dots, n$. If this is not true, then there exist distinct $\gamma_1, \dots, \gamma_{n+1} \in \{\lambda_1, \dots, \lambda_{2n}\}$ so that for every $m \in \{1, \dots, n+1\}$ we have $p_j(\gamma_m) \neq 0$ for some $j > 0$. Then it is possible to find μ_m so that $f(\gamma_m, \mu_m) = 0$, or equivalently, $\gamma_m \in \sigma(C + \mu_m N)$. It follows that $\mu_m = \alpha_0$, $m = 1, \dots, n+1$. This further implies that the polynomial $\lambda \mapsto f(\lambda, \alpha_0)$ has at least $n+1$ distinct zeroes which is impossible since it is of degree n. Thus, we see that

$$\det(\lambda I - C - \mu N) = p_0(\lambda) \quad \text{for all } \mu \in \mathbf{F}.$$

Hence $\sigma(C + \mu N) = \sigma(C)$ for all $\mu \in \mathbf{F}$. Suppose N has r non-zero eigenvalues with $r > 0$. Then we may put N in triangular form and see that the coefficient of $\lambda^{n-r}\mu^r$ is non-zero, contradicting the fact that $f(\lambda, \mu) = p_0(\lambda)$. Thus $r = 0$, i.e., N is a nilpotent matrix.

Recall that $A \in M_n(\mathbf{F})$ is a potent matrix if $A^k = A$ for some integer $k \geqslant 2$ and is of finite order if $A^k = I$ for some positive integer k. The above remark yields that $\mathscr{S}_1$, the set of all $n \times n$ potent matrices, as well as $\mathscr{S}_2$, the set of all $n \times n$ matrices of finite order are unions of similarity orbits with property (N_1). More general, if the field $\mathbf{F}$ is uncountable, then the union of spectra of elements of any countable family of similarity orbits is countable, and hence this family has property (N_1). In particular, if (p_k) is a sequence of polynomials, then the set of all matrices satisfying $p_k(A) = 0$ for some positive integer k is a union of similarity orbits and has property (N_1).

Let $\mathscr{S}$ denote the set of matrices in $M_n(\mathbf{C})$ having zero trace and n distinct eigenvalues. This is certainly a union of similarity orbits. We will see that it has property (N_0). In order to prove this we recall that for given matrices A and B the pencil $P(\alpha, \beta) = \alpha A + \beta B$ is said to have the L-property if the eigenvalues of $P(\alpha, \beta)$ are linear in α, β (see [32,33]). It is known (see [15, p. 103]) that, if $P(\alpha, 1)$ is diagonalizable for any complex number α, then $P(\alpha, \beta)$ possess the L-property. Assume that for a matrix $N \in M_n(\mathbf{C})$ there exists $C \in \mathscr{S}$ such that $C + \alpha N \in \mathscr{S}$ for all scalars α. Then $C + \alpha N$ is diagonalizable for every scalar α, and so, the pencil $\alpha N + \beta C$ has L-property. If N has two different eigenvalues, then by L-property it is possible to find α such that $C + \alpha N$ has an eigenvalue with algebraic multiplicity two. This contradiction shows that all eigenvalues of N are equal. Clearly, N has trace zero, and so, it must be a nilpotent.

We will now apply Theorem 3.3 and above remarks to reprove some linear preserver results and also to obtain some new ones. To simplify the description of our results, we list five types of linear operators on $M_n(\mathbf{F})$ in the following:

(1) There exist invertible $A, B \in M_n(\mathbf{F})$ such that ϕ is of the form

$$X \mapsto AXB \quad \text{or} \quad X \mapsto AX^{\mathrm{t}}B.$$

(2) There exist an invertible $A \in M_n(\mathbf{F})$ such that ϕ is of the form

$$X \mapsto AXA^{-1} \quad \text{or} \quad X \mapsto AX^{\mathrm{t}}A^{-1}.$$

(3) There exist an invertible $A \in M_n(\mathbf{F})$ and a non-zero $c \in \mathbf{F}$ such that ϕ is of the form

$$X \mapsto cAXA^{-1} \quad \text{or} \quad X \mapsto cAX^{\mathrm{t}}A^{-1}.$$

(4) There exist an invertible $A \in M_n(\mathbf{F})$, a non-zero $c \in \mathbf{F}$ and a linear functional f on $M_n(\mathbf{F})$ such that ϕ is of the form

$$X \mapsto cAXA^{-1} + f(X)I \quad \text{or} \quad X \mapsto cAX^{\mathrm{t}}A^{-1} + f(X)I.$$

(5) There exist a non-zero $c \in \mathbf{F}$ and $A, B \in M_n(\mathbf{F})$ with A invertible such that ϕ is of the form

$$X \mapsto cAXA^{-1} + (\operatorname{tr} X)B \quad \text{or} \quad X \mapsto cAX^{\mathrm{t}}A^{-1} + (\operatorname{tr} X)B.$$

Corollary 3.4. *Let* $\mathbf{F}$ *be an algebraically closed field of characteristic* 0. *Suppose* ϕ *is an invertible linear operator on* $M_n(\mathbf{F})$.

(a) *Let* K *be a proper non-empty subset of* $\mathbf{F}$. *Suppose* $\sigma(\phi(A)) \subseteq K$ *whenever* $\sigma(A) \subseteq K$. *If* $K \neq \{0\}, \mathbf{F}\backslash\{0\}$, *then* ϕ *is of the form* (4), *where* $f(X) = d \operatorname{tr} X$ *for some scalar d. If* $K = F\backslash\{0\}$, *then* ϕ *is of the form* (1). *If* $K = \{0\}$, *then* ϕ *is of the form* (5).

(b) *If* ϕ *maps the set of matrices having exactly n distinct eigenvalues into itself, then it is of the form* (4).

(c) *If* ϕ *maps the set of potent matrices into itself, then it is of the form* (3), *where c is a root of unity.*

(d) *If* ϕ *maps the set of matrices of finite order into itself, then it is of the form* (3), *where c is a root of unity.*

(e) *Suppose* $\mathbf{F}$ *is uncountable and* $\mathscr{S} \subseteq M_n(\mathbf{F})$ *is a countable union of similarity orbits such that* $\mathscr{S} \nsubseteq \mathbf{F}I$. *If* $\phi(\mathscr{S}) \subseteq \mathscr{S}$, *then* ϕ *is of the form* (5).

Several remarks are in order. The result (a) in full generality is new, although most of the special cases were known before. If we take the special case that $K = \mathbf{F}\backslash\{0\}$, we get the classical result on linear maps preserving invertibility [31]. The case where the complement of K has at least n elements follows from the results and proofs in [23], where one can find also some other results on linear maps preserving eigenvalue location. The assertions (b)–(d) were proved in [38] using deep results on overgroups of algebraic groups. When $\mathbf{F}$ is a complex field (c) was obtained in [6] without the non-singularity assumption. The statement (e) is an extension of the main theorem in [26] where only finite unions of similarity orbits were treated. The results on linear preservers of similarity orbits extend and unify a lot of known LPP results (see [26]). In particular, we can apply them to obtain results on linear maps preserving matrices annihilated by a given polynomial. We will omit the details here as we will study this problem in Section 4. Of course, the applicability of the reduction technique presented in this section is not restricted only to the above assertions.

Sometimes, one has to modify slightly the approaches presented in Theorem 3.3 to study a certain linear preserver problem and in many cases it is possible to simplify this approach considerably. For example, in the case that **F** is an uncountable algebraically closed field of characteristic 0 the problem of characterizing linear maps preserving potent matrices can be reduced to the problem of characterizing linear maps preserving nilpotents using the following short argument. Assume that *N* is a nilpotent matrix. Without loss of generality we can assume it is strictly upper triangular. Let *D* be a diagonal matrix with different roots of unity on the diagonal. Then $D + \lambda N$ is a potent matrix for every scalar λ (it has *n* different eigenvalues all of them being roots of unity). So, its image is potent. Therefore, for every λ there exists an integer $r > 1$ (depending on λ) such that

$$(\phi(D) + \lambda\phi(N))^r - \phi(D) - \lambda\phi(N) = 0.$$

There are uncountably many λ's. So there is an integer $r_0 > 1$ such that the above equation with $r = r_0$ holds for infinitely many λ's, and hence for all λ's. It follows that $\phi(N)$ is nilpotent as desired.

Now, we are ready to present the proof of Corollary 3.4.

To avoid trivial considerations, we assume that $n \geqslant 2$.

We will divide the proof of (a) into two cases. In the case that the complement of *K* has at least $2n$ elements we denote by $\mathscr{S}$ the set of all matrices *X* satisfying $\sigma(X) \subseteq K$. By Theorem 3.3 and the remark following it, we see that ϕ is of the form (5). So, we are done if $K = \{0\}$. In order to complete the proof in the first case we have to show that, if *K* contains a non-zero element, then *B* is a scalar matrix, or equivalently, $\phi(I)$ is a scalar matrix. We will use an idea similar to that in [23]. After applying similarity and going to transposes, if necessary, we may assume that ϕ is of the form $\phi(X) = cX + (\operatorname{tr} X)B$. Next, we recall the statement saying that for given scalars $\mu_1, \ldots, \mu_n$ a non-scalar matrix *T* is similar to a matrix whose diagonal entries are $\mu_1, \ldots, \mu_n$ if and only if $\operatorname{tr} T = \mu_1 + \cdots + \mu_n$ [24]. Choose a non-zero $\alpha \in K$ and assume that $\phi(\alpha I) = C$ is not a scalar matrix. Choose μ_1 from the complement of *K* and $\mu_2, \ldots, \mu_n \in \mathbf{F}$ such that $\operatorname{tr} C = \mu_1 + \cdots + \mu_n$. There exists an invertible *S* such that $S^{-1}CS$ has main diagonal $\mu_1, \ldots, \mu_n$. Let *N* be a nilpotent such that $S^{-1}NS$ is strictly upper triangular and $S^{-1}CS + S^{-1}NS$ is lower triangular. Then $\sigma(\alpha I + c^{-1}N) \subseteq K$, while $\mu_1 \in \sigma(\phi(\alpha I + c^{-1}N))$. This contradiction completes the proof in our first case.

It remains to consider the case that the complement of *K* has at most $2n$ elements, say $\lambda_1, \ldots, \lambda_k$. Clearly, ϕ^{-1} maps the algebraic set of matrices *X* satisfying

$\det((\lambda_1 I - X)\cdots(\lambda_k I - X)) = 0$ into itself. By [12, Lemma 1] ϕ^{-1} maps this set onto itself. In other words, we have $\sigma(X) \subseteq K$ if and only if $\sigma(\phi(X)) \subseteq K$.

We will now characterize rank one matrices using our geometric scheme. In particular, we prove the following lemma.

Lemma 3.5. *Let K be a proper subset of $\mathbf{F}$ with a finite complement, $K = \mathbf{F}\setminus\{\lambda_1, \ldots, \lambda_k\}$. Then for a non-zero $T \in M_n(\mathbf{F})$ the following two statements are equivalent*:

(a) $\operatorname{rank} T = 1$.

(b) *For every $X \in M_n(\mathbf{F})$ satisfying $\sigma(X) \subseteq K$ we have $\sigma(X + \alpha T) \subseteq K$ for all but at most k scalars α.*

Proof. If T has rank one then it is similar either to a scalar multiple of E_{11}, or to E_{12}. In both cases $\det(X + \alpha T - \lambda_j I)$ considered as a polynomial in α has degree at most 1. Its constant term $\det(X - \lambda_j I)$ is non-zero whenever $\sigma(X) \subseteq K$. So, it has at most one zero. Now, (b) follows easily.

Assume now that (b) holds. We want to show that $\operatorname{rank} T = 1$. Assume on the contrary that T has rank at least 2. Then up to a similarity T has the upper triangular block form

$$T = \begin{bmatrix} P & Q \\ 0 & R \end{bmatrix},$$

where P is

(i) $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with a and c (possibly equal) non-zero,

(ii) $\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ with a non-zero,

(iii) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ or

(iv) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Let $\mu_1, \mu_2, \mu_3 \in K$, $\tau, \tau_1, \tau_2 \in \mathbf{F}$, and define X by

$$X = \begin{bmatrix} Y & 0 \\ 0 & \mu_3 I \end{bmatrix},$$

where Y is

(i) $\begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$,

(ii) $\begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & \tau & \mu_1 \end{bmatrix}$,

(iii) $\begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 1 & 0 & \mu_1 \end{bmatrix}$ or

(iv) $\begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ \tau_1 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & \tau_2 & \mu_1 \end{bmatrix}$,

depending on whether P satisfies (i), (ii), (iii) and (iv), respectively. Then $\sigma(X) \subseteq K$. Clearly, for $i = 1, \ldots, k$, we have $\lambda_i \in \sigma(X + \alpha T)$ whenever one of the linear equations $\mu_1 + \alpha a = \lambda_i$, $\mu_2 + \alpha c = \lambda_i$ is satisfied; or one of the equations $\mu_1 + \alpha a = \lambda_i$, $(\mu_1 - \lambda_i)^2 = \tau\alpha$ is satisfied; or $(\mu_1 - \lambda_i)^3 = -\alpha^2$; or one of the equations $(\mu_1 - \lambda_i)^2 = \tau_1\alpha$, $(\mu_1 - \lambda_i)^2 = \tau_2\alpha$ is satisfied, respectively. It is now not difficult to show that $\mu_1, \mu_2, \mu_3, \tau, \tau_1$, and τ_2 can be chosen in such a way that $\sigma(X + \alpha T) \cap \{\lambda_1, \ldots, \lambda_k\} \neq \emptyset$ for at least $k + 1$ different scalars α. □

The consequence of this characterization of matrices of rank one is that ϕ maps the set of rank one matrices into itself. So, ϕ must be of the form (1) (see, e.g., [1,30] and Theorem 2.4 in Section 2). If $K = \mathbf{F} \setminus \{0\}$, we are done. So, assume that the complement of K contains a non-zero element, say λ_1. After going to transposes, if necessary, we may assume that $\phi(X) = AXB$, $X \in M_n(\mathbf{F})$. We want to show that BA is a scalar multiple of the identity. It is enough to prove that BAx and x are linearly dependent for every $x \in \mathbf{F}^n$. The linear map $X \mapsto XBA = A^{-1}\phi(X)A$ has the same eigenvalue location preserving property as ϕ. Assume that there exists $x \in \mathbf{F}^n$ such that x and BAx are linearly independent. Choose $\beta \in K$, $\gamma \in \mathbf{F}$ satisfying $\lambda_1\gamma \notin \{\lambda_1^2, \ldots, \lambda_k^2\}$, and a subspace $V \subseteq \mathbf{F}^n$ such that $\mathbf{F}^n = \operatorname{span}\{x, BAx\} \oplus V$. Define $X \in M_n(\mathbf{F})$ by $XBAx = \lambda_1 x$, $Xx = \gamma BAx$, and $Xv = \beta v$ for every $v \in V$. Then $\sigma(X) \subseteq K$ while $\lambda_1 \in \sigma(XBA)$. This contradiction completes the proof of (a).

In order to prove (b) we define θ by $\theta(X) = \phi(X) - (1/n)\operatorname{tr}\phi(X)I$. Obviously, θ maps the set of trace zero matrices having n distinct eigenvalues into itself. Applying Theorem 3.3 and the remark following it, we conclude that in the case that $\mathbf{F}$ is the field of complex numbers the mapping θ has to be of the form (5). After multiplying ϕ by a non-zero constant, applying a similarity transformation, adding to ϕ a transformation of the form $X \mapsto f(X)I$, where f is a linear functional on $M_n(\mathbf{C})$, and going to transposes, if necessary, we may assume that $\phi(X) = X + (\operatorname{tr} X)B$, $X \in$

$M_n(\mathbf{C})$. To complete the proof in the complex case we have to show that $B = [b_{ij}]$ is a scalar matrix. If this is not true, we may assume, after applying similarity, that $b_{11} \neq b_{22}$. It is not difficult to find an upper triangular X having n distinct eigenvalues satisfying $\operatorname{tr} X = 1$ such that $X + B$ is lower triangular and $x_{11} + b_{11} = x_{22} + b_{22}$. Then, of course, $\phi(X)$ has less than n eigenvalues. This contradiction completes the proof in the complex case. To extend this result to the general case we can apply the transfer principle in Section 2.

The remaining three statements are easy to verify. □ (Corollary 3.4)

In the above proof, we have used the proposed geometric scheme to characterize rank one matrices. In fact, we can use the same idea to characterize invertible linear maps on $M_n(\mathbf{F})$ that preserve matrices of rank k (or matrices of rank no greater than k), $1 \leqslant k < n$. The case $k = n$ is the problem of characterizing linear maps preserving invertibility and was considered in Corollary 3.4(a). The set of all matrices of rank no greater than k is the Zariski closure of the set of all matrices of rank k. So, if ϕ preserves matrices of rank k, then it preserves matrices of rank $\leqslant k$. So, we will assume that $A \in M_n^k(\mathbf{F})$ implies $\phi(A) \in M_n^k(\mathbf{F})$. Here, $M_n^k(\mathbf{F})$ denotes the set of all matrices of rank at most k. By the result of Dixon [12] ϕ maps $M_n^k(\mathbf{F})$ onto itself.

We reduce the problem to the problem of rank one preservers. All we have to do is to prove the following.

Proposition 3.6. *Let $A \in M_n(\mathbf{F})$ be non-zero, and let $1 < k < n$. The following conditions are equivalent*:

(i) $\operatorname{rank} A = 1$.

(ii) *There exists $T \in M_n^k(\mathbf{F})$ such that $T + \lambda A \in M_n^k(\mathbf{F})$ for every scalar λ and for every $T \in M_n^k(\mathbf{F})$ we have either $T + \lambda A \in M_n^k(\mathbf{F})$ for every scalar λ or $T + \lambda A \notin M_n^k(\mathbf{F})$ for every non-zero scalar λ.*

Proof. Assume first that $\operatorname{rank} A = 1$. We can choose invertible P and Q such that $PAQ = E_{12}$. Let $T = P^{-1}E_{11}Q^{-1}$. Then $T + \lambda A \in M_n^k(\mathbf{F})$ for every λ. Assume now that $T \in M_n^k(\mathbf{F})$ and $T + \lambda_0 A \notin M_n^k(\mathbf{F})$ for some λ_0. Without loss of generality, we can assume that $\lambda_0 = 1$. Then

$$k < \operatorname{rank}(T + A) \leqslant \operatorname{rank} T + \operatorname{rank} A \leqslant k + 1,$$

and so, $\operatorname{rank} T = k$ and

$$\operatorname{rank}(T + A) = \operatorname{rank} T + \operatorname{rank} A.$$

We say that T and A are rank additive and it is well known that this is equivalent to $C(T) \cap C(A) = \{0\}$ and $R(T) \cap R(A) = \{0\}$, where C and R denote the column space and the row space. But this is further equivalent to $C(T) \cap C(\mu A) = \{0\}$ and $R(T) \cap R(\mu A) = \{0\}$ for every non-zero scalar μ, and consequently,

$$\operatorname{rank}(T + \mu A) = \operatorname{rank} T + \operatorname{rank} \mu A = k + 1$$

for every non-zero μ. This completes the proof in one direction.

To prove the other direction we assume that rank $A = p > 1$. If $p > 2k$, then rank $(T + \lambda A) > k$ for every $T \in M_n^k(\mathbf{F})$ and every non-zero λ. So, (ii) does not hold. If $k + 1 \leqslant p \leqslant 2k$, then we can assume as above that A is diagonal with first p diagonal entries 1 and the other diagonal entries 0. Let T be diagonal with first $p - k$ diagonal entries -1 and the other diagonal entries 0. Then $\operatorname{rank}(T + A) = k$ and $\operatorname{rank}(T + 2A) > k$. It remains to consider the case that $2 \leqslant p \leqslant k$. Define T to be diagonal with first diagonal entry 0, the second diagonal entry -1, the next $k - 1$ diagonal entries 1 and the other diagonal entries 0. Then $\operatorname{rank}(T + A) = k$ and $\operatorname{rank}(T + 2A) > k$. This completes the proof. □

We remark that the idea of the proof of the above proposition may have been hidden in the work of other authors. Nonetheless, it helps us to illustrate how to apply the geometric technique we proposed.

4. Reduction to idempotent preservers

The aim of this section is to show that some of LPP can be reduced to the problem of characterizing linear maps preserving idempotents. The advantage of this technique is that it can be used also in the infinite-dimensional case as well as to study linear preservers from $M_n(\mathbf{F})$ into $M_m(\mathbf{F})$ with n different from m. The idea to reduce a linear preserver problem to the idempotent case has been already used when studying the classical problem of invertibility preserving maps [7,8]. The reduction techniques that we will present here are different from those in [7,8].

Let us first recall that a C^*-algebra $\mathscr{A}$ is of real rank zero if the set of all finite real linear combinations of orthogonal Hermitian idempotents is dense in the set of all Hermitian elements of $\mathscr{A}$. Equivalently, the set of Hermitian elements with finite spectrum is dense in the set of all Hermitian elements of $\mathscr{A}$. Every von Neumann algebra is a C^*-algebra of real rank zero. In particular, $\mathscr{B}(H)$, the algebra of all bounded linear operators on a complex Hilbert space, has real rank zero. There is a vast literature on such algebras. Usually they are defined in a more complicated way. We refer to [9] where the above simple definition can be found.

Let $\mathscr{A}$ and $\mathscr{B}$ be algebras over a field $\mathbf{F}$. A linear map $\phi : \mathscr{A} \to \mathscr{B}$ is called a Jordan homomorphism if $\phi(x^2) = \phi(x)^2$, $x \in \mathscr{A}$. Homomorphisms and antihomomorphisms (linear maps satisfying $\phi(xy) = \phi(y)\phi(x)$) are basic, but not the only examples of Jordan homomorphisms. Indeed, let each of $\mathscr{A}$ and $\mathscr{B}$ be a direct sum of two subalgebras, $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2$ and $\mathscr{B} = \mathscr{B}_1 \oplus \mathscr{B}_2$, with the operations defined componentwise. If $\phi_1 : \mathscr{A}_1 \to \mathscr{B}_1$ is a homomorphism and $\phi_2 : \mathscr{A}_2 \to \mathscr{B}_2$ is an antihomomorphism, then $\phi_1 \oplus \phi_2 : \mathscr{A} \to \mathscr{B}$ is a Jordan homomorphism.

The following theorem whose proof is a slight modification of an idea given in [5, Remark 2.2] and its consequences show that once we reduce a certain linear preserver problem to the idempotent case we can easily get its solution not only in the matrix case but also in the infinite-dimensional case.

Theorem 4.1. *Let $\mathscr{A}$ be a C^*-algebra of real rank zero and $\mathscr{B}$ any complex Banach algebra. Assume that a bounded linear map $\phi : \mathscr{A} \to \mathscr{B}$ preserves idempotents. Then ϕ is a Jordan homomorphism.*

Proof. Pick a Hermitian element h which is a finite real linear combination of orthogonal Hermitian idempotents, $h = \sum_{i=1}^{n} t_i p_i$, $p_i p_j = 0$ if $i \neq j$. Since $p_i + p_j$ is an idempotent if $i \neq j$, we have $(\phi(p_i) + \phi(p_j))^2 = \phi(p_i) + \phi(p_j)$. This yields $\phi(p_i)\phi(p_j) + \phi(p_j)\phi(p_i) = 0$. Using this relation we see that $\phi(h^2) = \phi(h)^2$. Now, the set of Hermitian elements h, which are finite real linear combinations of orthogonal Hermitian idempotents, is dense in the set of all Hermitian elements. Since ϕ is continuous, we have $\phi(h^2) = \phi(h)^2$ for all Hermitian elements. Replacing h by $h + k$, where h and k are both Hermitian, we get $\phi(hk + kh) = \phi(h)\phi(k) + \phi(k)\phi(h)$. Since an arbitrary $x \in \mathscr{A}$ can be written in the form $x = h + \mathrm{i}k$ with h, k Hermitian, the last two relations imply that $\phi(x^2) = \phi(x)^2$. This completes the proof. □

In the special case that $\mathscr{A} = M_n(\mathbf{C})$ we get the following result from [5].

Corollary 4.2. *Let $\mathscr{B}$ be any complex Banach algebra. Assume that a linear map $\phi : M_n(\mathbf{C}) \to \mathscr{B}$ preserves idempotents. Then ϕ is a sum of a homomorphism and an antihomomorphism.*

Proof. Since $M_n(\mathbf{C})$ is finite-dimensional ϕ must be bounded. So, by the previous theorem it is a Jordan homomorphism. According to [21, Theorem 7] ϕ is a sum of a homomorphism and an antihomomorphism. □

Corollary 4.3. *Let $\mathbf{F}$ be an algebraically closed field of characteristic 0 and m, n positive integers. Assume that a non-zero linear map $\phi : M_n(\mathbf{F}) \to M_m(\mathbf{F})$ preserves idempotents. Then $m \geqslant n$ and there exist an invertible matrix $A \in M_m(\mathbf{F})$ and non-negative integers k_1, k_2 such that $1 \leqslant k_1 + k_2$, $(k_1 + k_2)n \leqslant m$ and*

$$\phi(X) = A \operatorname{diag}(X, \ldots, X, X^{\mathrm{t}}, \ldots, X^{\mathrm{t}}, 0)A^{-1}, \quad X \in M_n(\mathbf{F}).$$

Here, $\operatorname{diag}(X, \ldots, X, X^{\mathrm{t}}, \ldots, X^{\mathrm{t}}, 0)$ *denotes the block diagonal matrix in which X appears k_1 times, X^{t} appears k_2 times, and 0 is a zero matrix of the appropriate size* (*possibly absent*).

Proof. We will prove here only the special case that $\mathbf{F} = \mathbf{C}$. The extension to the general case can be done using the result in Section 2 as follows. We extend the result (using the transfer principle) for every pair of positive integers m and n. When extending the result we do not assume that ϕ is non-zero. We fix m and n. If $m < n$, then the conclusion is that ϕ is zero. If $m \geqslant n$, say $m = 5$ and $n = 2$, we have the conclusion that ϕ is zero OR that ϕ is of the desired form with $k_1 = 1$ and $k_2 = 0$ OR ϕ is of the desired form with $k_1 = 0$ and $k_2 = 1$ OR ϕ is of the desired form with $k_1 = 1$ and $k_2 = 1$ OR ϕ is of the desired form with $k_1 = 2$ and $k_2 = 0$ OR ϕ is of the desired form with $k_1 = 0$ and $k_2 = 2$. For each possibility the desired form can be expressed as a first-order sentence.

Now, return to the proof of the complex case. By Corollary 4.2, ϕ is a sum of a homomorphism ϕ_1 and an antihomomorphism ϕ_2. Denote $P = \phi(I_n)$, $P_1 = \phi_1(I_n)$ and $P_2 = \phi_2(I_n)$. Clearly, $P = P_1 + P_2$. Moreover, all of these matrices are idempotents. So, we have up to a similarity

$$P = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where one of I_p or I_q may be zero and some border zeroes may be absent. Consequently, we have

$$\phi_1(X) = \begin{bmatrix} \varphi_1(X) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \phi_2(X) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varphi_2(X) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X \in M_n(\mathbf{C}),$$

where φ_1 is a unital homomorphism of $M_n(\mathbf{C})$ into $M_p(\mathbf{C})$ and φ_2 is a unital antihomomorphism of $M_n(\mathbf{C})$ into $M_q(\mathbf{C})$. Composing an antihomomorphism by the transposition we get a homomorphism. Thus, in order to complete the proof it is enough to prove that if φ is a unital homomorphism of $M_n(\mathbf{C})$ into $M_p(\mathbf{C})$, where n and p are positive integers, then n divides p and

$$\varphi(X) = B \operatorname{diag}(X, \ldots, X)B^{-1}, \quad X \in M_n(\mathbf{C}),$$

for some invertible $B \in M_p(\mathbf{C})$. Here, $\operatorname{diag}(X, \ldots, X)$ is a block diagonal matrix, where X appears p/n times.

First note that because φ is unital it preserves invertibility. If X and Y are of the same rank, then there exist invertible matrices T and S such that $X = TYS$. Consequently, $\varphi(X) = \varphi(T)\varphi(Y)\varphi(S)$ has the same rank as $\varphi(Y)$. Let $\varphi(E_{11})$ be of rank r. Then $\varphi(E_{11}), \ldots, \varphi(E_{nn})$ are all idempotents of rank r satisfying $\varphi(E_{ii})\varphi(E_{jj}) = 0$ whenever $i \neq j$. It follows that $\varphi(I_n) = I_p$ is of rank rn. So, n divides p. Obviously, the map

$$\tau(X) = \operatorname{diag}(X, \ldots, X) \in M_p(\mathbf{C}), \quad X \in M_n(\mathbf{C}),$$

is a unital algebra homomorphism. By a special case of the Noether–Skolem theorem [35, Lemma, p. 230] there exists an invertible $B \in M_p(\mathbf{C})$ such that $\varphi(X) = B\tau(X)B^{-1}$, $X \in M_n(\mathbf{C})$, as desired. This completes the proof. □

Let p be a polynomial. A linear map $\phi : \mathscr{A} \to \mathscr{B}$ preserves elements annihilated by p if $p(\phi(x)) = 0$ whenever $p(x) = 0$.

Corollary 4.4. *Let $\mathscr{A}$ be a unital C^*-algebra of real rank zero and $\mathscr{B}$ any complex unital Banach algebra. Let p be a complex polynomial,* $\deg p > 1$, *with simple zeroes (each zero has multiplicity one). Assume that a linear bounded unital map ϕ : $\mathscr{A} \to \mathscr{B}$ preserves elements annihilated by p. Then ϕ is a Jordan homomorphism. If $\mathscr{A} = M_n(\mathbf{C})$, then ϕ is a sum of a homomorphism and an antihomomorphism.*

Proof. Assume that $\lambda_1, \ldots, \lambda_k$ are zeroes of p. Let μ be any complex number and q the monic polynomial with simple zeroes $\lambda_1 - \mu, \ldots, \lambda_k - \mu$. Then ϕ preserves elements annihilated by q. Indeed, $q(A) = 0$ if and only if $p(A + \mu) = 0$. This implies that $p(\phi(A) + \mu) = 0$ which is equivalent to $q(\phi(A)) = 0$.

So, without loss of generality we may assume that $\lambda_1 = 0$ and either: (1) all λ_j's are in the closed left half complex plane and not all of them are on the imaginary axis, or (2) all λ_j's belong to $\{t\mathrm{i} : t \leqslant 0\}$ (negative part of the imaginary axis).

Let P be an arbitrary idempotent in $\mathscr{A}$. Then $p(\lambda_i P) = 0$, and so, $p(\lambda_i \phi(P)) = 0$. Let μ belong to the spectrum of $\phi(P)$. Then $\lambda_i \mu$ belongs to the spectrum of $\phi(\lambda_i P)$ which is contained in $\{0, \lambda_2, \ldots, \lambda_k\}$. So, for every i and every positive integer s we have

$$\lambda_i \mu^s \in \{0, \lambda_2, \ldots, \lambda_k\}. \tag{4.1}$$

It follows that $\mu = 0$ or there exists r such that $\mu^r = 1$. Let r be the smallest positive integer such that this is true. From the position of the λ_j's in the complex plane and (4.1) we conclude that $r = 1$. Therefore, the spectrum of $\phi(P)$ is contained in $\{0, 1\}$.

We know that

$$\lambda_2\phi(P)[\lambda_2\phi(P) - \lambda_2][\lambda_2\phi(P) - \lambda_3]\cdots[\lambda_2\phi(P) - \lambda_k] = 0.$$

Since $\lambda_2\phi(P) - \lambda_j$, $j \geqslant 3$, is invertible we have

$$\lambda_2\phi(P)[\lambda_2\phi(P) - \lambda_2] = 0.$$

Thus, $\phi(P)$ is an idempotent. Hence, ϕ preserves idempotents. The result now follows from Theorem 4.1 and Corollary 4.2. □

Corollary 4.5. *Let* $\mathbf{F}$ *be an algebraically closed field of characteristic* 0 *and* m, n *positive integers. Let p be a polynomial over* $\mathbf{F}$, $\deg p > 1$, *with simple zeroes. Assume that a linear unital map* $\phi : M_n(\mathbf{F}) \to M_m(\mathbf{F})$ *preserves matrices annihilated by p. Then n divides m and there exist an invertible matrix* $A \in M_m(\mathbf{F})$ *and nonnegative integers* k_1, k_2 *such that* $(k_1 + k_2)n = m$ *and*

$$\phi(X) = A\, \mathrm{diag}(X, \ldots, X, X^{\mathrm{t}}, \ldots, X^{\mathrm{t}})A^{-1}, \quad X \in M_n(\mathbf{F}).$$

Here, $\mathrm{diag}(X, \ldots, X, X^{\mathrm{t}}, \ldots, X^{\mathrm{t}})$ *denotes the block diagonal matrix in which X appears* k_1 *times while* X^{t} *appears* k_2 *times.*

Proof. Once again we will prove only the special case that $F = \mathbf{C}$. By Corollary 4.4, ϕ is a sum of a homomorphism and an antihomomorphism. The result now follows from Corollary 4.3 and the fact that ϕ is unital. □

The special case when $m = n$ was proved in [20] (see also [26]) under the additional assumption of bijectivity without assuming that ϕ is unital. In fact, in the special case that $m = n$ and ϕ is invertible Howard characterized linear maps preserving matrices annihilated by any given polynomial. Let us show that in our more general situation the assumption that p has simple zeroes is indispensable. To see this define $\phi : M_n(\mathbf{F}) \to M_{n^2(n+2)}(\mathbf{F})$ by

$$\phi(X) = \phi\left((x_{ij})\right) = (\mathrm{tr}X/n)(I - \varphi(I)) + \varphi(X),$$

where $\varphi(X)$ is a block diagonal matrix having on a diagonal n^2 blocks Y_{ij}, $i, j = 1, \ldots, n$, of the size $(n+2) \times (n+2)$. Here, the first row of Y_{ij} equals $(0, x_{i1}, \ldots, x_{in}, 0)$, the last column of Y_{ij} equals $(0, x_{1j}, \ldots, x_{nj}, 0)^{\mathrm{t}}$, and all other entries of Y_{ij} are zero. Then ϕ is a unital linear mapping which preserves square-zero matrices. Even more, it preserves them in both directions, that is, $\phi(X)$ is square-zero if and only if X is square-zero. But clearly, ϕ is not a Jordan homomorphism.

Another application of the reduction technique treated in this section is the characterization of linear maps preserving potent elements.

Corollary 4.6. *Let $\mathscr{A}$ be a unital C^*-algebra of real rank zero and $\mathscr{B}$ any complex unital Banach algebra. Assume that a linear bounded unital map $\phi : \mathscr{A} \to \mathscr{B}$ preserves potent elements. Then ϕ is a Jordan homomorphism. If $\mathscr{A} = M_n(\mathbf{C})$, then ϕ is a sum of a homomorphism and an antihomomorphism.*

Proof. We have to prove that ϕ preserves idempotents. Let p be any idempotent from $\mathscr{A}$. Then $\phi(p)$ is a potent element. So, we have to show that its spectrum is contained in $\{0, 1\}$. Let λ be any element of $\sigma(\phi(p))$. As p, $1 - p$, and $1 - 2p$ are all potent elements, the same must be true for $\phi(p)$, $1 - \phi(p)$ and $1 - 2\phi(p)$. Hence, each of the numbers λ, $1 - \lambda$ and $1 - 2\lambda$ is either 0 or a root of unity. This is possible only if $\lambda = 0$ or $\lambda = 1$ as desired. □

Corollary 4.7. *Let $\mathbf{F}$ be an algebraically closed field of characteristic 0 and m, n positive integers. Assume that a linear unital map $\phi : M_n(\mathbf{F}) \to M_m(\mathbf{F})$ preserves potent matrices. Then n divides m and there exist an invertible matrix $A \in M_m(\mathbf{F})$ and non-negative integers k_1, k_2 such that $(k_1 + k_2)n = m$ and*

$$\phi(X) = A\, \mathrm{diag}(X, \ldots, X, X^{\mathrm{t}}, \ldots, X^{\mathrm{t}})A^{-1}, \quad X \in M_n(\mathbf{F}).$$

Here, $\mathrm{diag}\,(X, \ldots, X, X^{\mathrm{t}}, \ldots, X^{\mathrm{t}})$ *denotes the block diagonal matrix in which X appears k_1 times while X^{t} appears k_2 times.*

If we compare the last two results with Corollary 3.4(c) we see that the underlying algebras here are much more general. However, we have the additional assumption that ϕ is unital. In Corollaries 4.6 and 4.7 one can replace the assumption that ϕ preservers potent elements by the assumption that it preserves elements of finite order and get the same conclusion. As the idea of the proof is similar we leave the details to the reader.

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