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ORIGINAL ARTICLE

On a fractional difference operator

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Abstract In the present article, a set of new difference sequence spaces of fractional order has been introduced and subsequently, an application of these spaces, the notion of the derivatives and the integrals of a function to the case of non-integer order have been generalized. Certain results involving the unusual and non-uniform behavior of the corresponding difference operator have been investigated and also been verified by using some counter examples. We also verify these unusual and non-uniform behaviors by studying the geometry of fractional calculus.
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1. Introduction, preliminaries and definitions

The study of sequence spaces occupies a very prominent position in Functional analysis, which has vast range of applications in several branches of mathematics. It provides major links among various branches of mathematics including operator theory, linear algebra, calculus.

In fact, the theory of difference sequence spaces has made a significant contribution in enveloping the classical theory of fractional calculus and numerical analysis. The theory of fractional calculus deals with the investigation of derivatives and integrations of a function with arbitrary orders. Fractional derivative provides an extensive knowledge for description of memory and hereditary properties of various material and processes including certain natural and physical phenomenon. The application of fractional derivatives becomes more apparent in modeling mechanical and electrical properties of real materials as well as in the description of rheological properties of rocks and in many other fields. In particular, the theory of fractional derivatives has been extensively used in the study of fractal theory, theory of control of dynamic systems, theory of viscoelasticity, electrochemistry, diffusion processes and many others (see [1–3]).

Let w be the space of all real valued sequences and $\mathbb{N}$ be the set of all natural numbers including zero. Any subspace of w is called a *sequence space* and by ℓ_∞, c and c_0, we denote the spaces of all bounded, convergent and null sequences, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. Initially, Kızmaz [7] introduced the idea of difference sequence space associated with basic sequences ℓ_∞, c and c_0 by defining the difference operator Δ of order one, where

$$(\Delta x)_k = x_k - x_{k+1}, (k \in \mathbb{N}).$$

Later on, these sequence spaces have been generalized to the case of integral order m by Et and Colak [10] using operator Δ^m and

$$(\Delta^m x)_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}, (k \in \mathbb{N}).$$

Recently, Baliarsingh [13] generalized the above difference operator by introducing fractional difference operator Δ^α, where

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$$(\Delta^{\alpha}x)_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}x_{k+i}, (k\in\mathbb{N}).$$

Many other sequence spaces have also been studied by combining the notion of difference operators and operators derived from different means such as Cesàro mean, weighted mean, and Euler mean (see [16,17,19]). In the other hand, the spectral properties of difference operators Δ and Δ^m have been studied by Altay and Başar [9] and Dutta and Baliarsingh [14] and many others.

As already discussed, the study of fractional calculus has a direct impact on the theory involving the solution of diverse problems in mathematics, science, and engineering and in order to stimulate more interest in the subject and to show its utility, several definitions of fractional derivatives have been developed. The most popular definitions of fractional derivatives and integrations have been introduced by Riemann–Liouville and Grunwald–Letnikov. Caputo reformulated the classical definition of the Riemann–Liouville fractional derivative in order to solve fractional differential equations with certain initial conditions. The definition of fractional calculus introduced by Leibniz has also been reformulated by Grunwald–Letnikov. Recently, Kilbas et al. [1] reformulated again, the Riemann–Liouville fractional derivatives using Euler gamma function, hyper geometric functions and Mittag–Leffler function, etc. Some numerous applications and physical manifestations of fractional calculus have been found in electro chemistry and mechanics which were studied by Blutzer and Torvik [2] and Dreisigmeyer and Young [3]. More investigations on fractional calculus and its several applications to real world problems including ordinary and partial differential equations in applied mathematics and fluid mechanics are found in [4–6].

Now, we give a brief review of fractional calculus. Several definitions of fractional derivatives and fractional integral have been introduced, among which Riemann–Liouville, Caputo, Grunwald–Letnikov formulas are being extensively used and stated below:

Fractional derivatives and integrals:

- The left and right *Riemann–Liouville* fractional derivatives of order α, respectively are given by

$$ {}_aD_x^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt,$$
$$ {}_xD_b^{\alpha}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (x-t)^{n-\alpha-1}f(t)dt.$$

- The left and right *Riemann–Liouville* fractional integral of order α, respectively are given by

$$ {}_aI_x^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,$$
$$ {}_xI_b^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (x-t)^{\alpha-1}f(t)dt.$$

- The left and right *Caputo* fractional integral of order α, respectively are given by

$$ {}_a^cD_x^{\alpha}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt,$$
$$ {}_x^cD_b^{\alpha}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b (x-t)^{n-\alpha-1}f^{(n)}(t)dt.$$

- *Grunwald–Letnikov* fractional derivative of order α, is given by

$$ {}^aD_x^{\alpha}f(x) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{\left[\frac{x-a}{h}\right]}(-1)^j\binom{\alpha}{j}f(x-jh).$$

The main objective of the present study is to generalize the concept of fractional calculus via difference sequence spaces. In fact, the generalized finite difference scheme is one of the convenient and powerful tools to approximate these fractional differentiations and integrations.

2. Fractional difference operator

Let $x=(x_k)$ be any sequence in w, the space of all scalar sequences of real numbers and h be a positive constant. For real numbers a, b and c, we define a generalized difference sequence via difference operator $\Delta_h^{a,b,c}: w\to w$, defined by

$$(\Delta_h^{a,b,c}x)_k = \sum_{i=0}^{\infty}\frac{(-a)_i(-b)_i}{i!(-c)_i h^{a+b-c}}x_{k-i}, (k\in\mathbb{N}), \tag{2.1}$$

where $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $(\alpha)_k$ denotes the *Pochhammer* symbol or *shifted factorial* of a real number α which is being defined using familiar Euler gamma function as

$$(\alpha)_k = \begin{cases} 1, & (\alpha=0 \text{ or } k=0) \\ \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+k-1), & (k\in\mathbb{N}) \end{cases}$$

Note that the series defined in (2.1) converges for all $c>a+b$ (see [18]) which may be presumed throughout the text. In particular, the difference operator $\Delta_h^{a,b,c}$ includes following special cases:

(i) The difference operator $\Delta^{(1)}$ for $a=1,\ b=c$ and $h=1$ (see [8,9]).
(ii) The difference operator $\Delta^{(m)}$ for $a=m\in\mathbb{N}_0,\ b=c$ and $h=1$ (see [10,11]).
(iii) The difference operator $\Delta^{(\alpha)}$ for $a=\alpha\in\mathbb{R}$, (the set of all real numbers), $b=c$ and $h=1$ (see [12,13]).
(iv) The difference operator Δ_v^r for $a=r,\ b=c,\ h=1$ and $v=(1,1,1,\dots)$ (see [14]).

Since study of sequence spaces has been used as a powerful and effective tool in matrix, spectral and operator theories (see [14–17]), we define the following sequence space via difference operator of fractional order which not only generalizes the previous works but contributes significantly to the theory of fractional calculus.

Now, using the difference operator defined in (2.1), we define the following new classes of difference sequence spaces:

$$X(\Delta_h^{a,b,c}) = \{x=(x_k)\in w : \Delta_h^{a,b,c}(x)\in X\},$$

where $X=c,\ c_0$ and ℓ_∞. It is noticed that each element of the above classes can be obtained by taking the $\Delta_h^{a,b,c}$-transform of the sequence x and $\Delta_h^{a,b,c}$ represents a triangle as

$$(\Delta_h^{a,b,c})_{nk} = \begin{cases} 1, & (n=k) \\ (-1)^{n-k}\frac{a(a-1)\dots(a-(n-k-1))b(b-1)\dots(b-(n-k-1))}{(n-k)!c(c-1)\dots(c-(n-k-1))h^{a+b-c}}, & (0\leqslant k<n) \\ 0, & (k>n) \end{cases}$$

Theorem 2.1. *Let $X = c,\ c_0,\ \ell_\infty$ and $a \geqslant 1$ with $b = c$, the classes $X(\Delta_h^{a,b,c})$ are complete normed linear spaces, norm defined by*

$$\|x\|_{\Delta_h^{a,b,c}} = \sum_{i=0}^{[a]} |x_k| + \sup_k \left|\Delta_h^{a,b,c}(x_k)\right|. \tag{2.2}$$

where $[a]$ indicates the integral part of a.

Proof. Due to elementary verification, the proof is omitted. □

Theorem 2.2. *Let $X = c,\ c_0,\ \ell_\infty$ and $0 \leqslant a < 1$ with $b = c$, the classes $X(\Delta_h^{a,b,c})$ are semi-normed spaces, semi-norm defined by*

$$g(x) = \sup_k \left|\Delta_h^{a,b,c}(x_k)\right|. \tag{2.3}$$

Proof. For $0 \leqslant a < 1$ with $b = c$, let us consider

$$g(x) = \sup_k \left|\sum_{i=0}^{\infty} \frac{(-a)_i}{i!h^a} x_{k-i}\right| = 0,$$

Then it does not necessarily implies that $x = \theta = (0, 0, 0, \ldots)$. This follows from the following example:

Let us take $x = (1, 1, 1, \ldots)$, a constant sequence, then it is clear that $x \neq \theta$, whereas $g(x) = 0$. □

Remark 2.1 (*[7,8,10,11]*). It is suggested that if a is an integer, then the norm defined in Theorem 2.1 reduces to the norm

$$\|x\|_{\Delta_h^{a,b,c}} = \sum_{i=0}^{a} |x_k| + \sup_k \left|\Delta_h^{a,b,c}(x_k)\right|.$$

Now, we state some applications of the difference operator $\Delta_h^{a,b,c}$ in the field of fractional calculus.

Let $h \to 0$ and $f(x)$ be a differentiable(with fractional order) function. Associated with this function $f(x)$, define the sequence $f_h(\cdot) = (f(x - kh))_{k\in\mathbb{N}_0}$, and the sequence spaces $\Delta_{h,x}^{a,b,c}(f_h(\cdot))$ via the difference operator $\Delta_{h,x}^{a,b,c}$ as

$$\Delta_{h,x}^{a,b,c} f(x) = \sum_{i=0}^{\infty} \frac{(-a)_i(-b)_i}{i!(-c)_i h^{a+b-c}} f(x - ih). \tag{2.4}$$

Clearly, the difference operator defined in Eq. (2.4) is a linear operator and it generalizes the concept of integral and fractional order derivatives. For instances,

- For $a = 1, 2,\ b = c$, we have $\Delta_{h,x}^{1,b,b} \equiv \frac{d}{dx}$, and $\Delta_{h,x}^{2,b,b} \equiv \left(\frac{d}{dx}\right)^2$, more specifically,

$$\Delta_{h,x}^{1,b,b} f(x) = \frac{f(x) - f(x-h)}{h} \text{ and also } \Delta_{h,x}^{2,b,b} f(x) = \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}$$

- For $a = \alpha \in \mathbb{R},\ b = c$, the operator $\Delta_{h,x}^{\alpha,b,b}$ reduces to fractional difference operator $\left(\frac{d}{dx}\right)^\alpha$, where

$$\Delta_{h,x}^{\alpha,b,b} f(x) = \sum_{i=0}^{\infty} \frac{(-\alpha)_i}{i!h^\alpha} f(x - ih)$$

- For $\alpha \in \mathbb{R}(\notin \mathbb{N}),\ b = c$, the operator $\Delta_{h,x}^{-\alpha,b,b}$ reduces to fractional integro operator $\left(\frac{d}{dx}\right)^{-\alpha}$, where

$$\Delta_{h,x}^{-\alpha,b,b} f(x) = \sum_{i=0}^{\infty} \frac{(\alpha)_i}{i!h^{-\alpha}} f(x - ih)$$

However, due to restriction of Euler gamma function, for $a = \alpha \in \mathbb{N},\ b = c$, the operator $\Delta_{h,x}^{-\alpha,b,b}$ reduces to integral operator $\left(\frac{d}{dx}\right)^{-\alpha}$ by splitting the integer α to a sum of finite number of proper fractions.

Theorem 2.3. *If $\alpha, \beta > 0$, then*

(i) *The difference operator $\Delta_{h,x}^{\alpha,b,c}$ is a linear operator over $\mathbb{R}$.*
(ii) $\Delta_{h,x}^{\alpha,b,b}\left(\Delta_{h,x}^{\beta,b,b} f(x)\right) = \Delta_{h,x}^{\alpha,b,b}\left(\sum_{i=0}^{\infty} \frac{(-\beta)_i}{i!h^{-\alpha}} f(x - ih)\right) = \Delta_{h,x}^{\alpha+\beta,b,b} f(x).$
(iii) $\Delta_{h,x}^{\alpha,b,b}\left(\Delta_{h,x}^{-\alpha,b,b} f(x)\right) = \Delta_{h,x}^{-\alpha,b,b}\left(\Delta_{h,x}^{\alpha,b,b} f(x)\right) = f(x).$

Proof.

(i) Proof is a routine work, hence omitted.
(ii)

$$\begin{aligned}
&\Delta_{h,x}^{\alpha,b,b}\left(\Delta_{h,x}^{\beta,b,b} f(x)\right]\\
&= \Delta_{h,x}^{\alpha,b,b}\left[\sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\beta+1)}{i!\Gamma(\beta-i+1)h^\beta} f(x-ih)\right]\\
&= \Delta_{h,x}^{\alpha,b,b} h^{-\beta}\Big[f(x) - \beta f(x-h) + \frac{\beta(\beta-1)}{2!} f(x-2h)\\
&\quad - \frac{\beta(\beta-1)(\beta-2)}{3!} f(x-3h) + \ldots\Big]\\
&= h^{-\beta-\alpha}\Big[\sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} f(x-ih)\\
&\quad -\beta\sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} f(x-(i+1)h)\\
&\quad +\frac{\beta(\beta-1)}{2!}\sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} f(x-(i+2)h) - \ldots\Big]\\
&= h^{-\beta-\alpha}\Big[\Big\{f(x) - \alpha f(x-h) + \frac{\alpha(\alpha-1)}{2!} f(x-2h)\\
&\quad - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} f(x-3h) + \ldots\Big\}\\
&\quad - \beta\Big\{f(x-h) - \alpha f(x-2h) + \frac{\alpha(\alpha-1)}{2!} f(x-3h)\\
&\quad - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} f(x-4h) + \ldots\Big\}\\
&\quad + \frac{\beta(\beta-1)}{2!}\Big\{f(x-2h) - \alpha f(x-3h) + \frac{\alpha(\alpha-1)}{2!} f(x-4h) - \ldots\Big\} + \ldots\Big]\\
&= h^{-(\alpha+\beta)}\Big[f(x) - (\alpha+\beta) f(x-h) + \left(\frac{\alpha(\alpha-1)}{2!} + \alpha\beta + \frac{\beta(\beta-1)}{2!}\right) f(x-2h)\\
&\quad - \left(\frac{\alpha(\alpha-1)(\alpha-2)}{3!} + \frac{\beta\alpha(\alpha-1)}{2!} + \frac{\alpha\beta(\beta-1)}{2!} + \frac{\beta(\beta-1)(\beta-2)}{3!}\right) f(x-3h) + \ldots\Big]\\
&= h^{-(\alpha+\beta)}\Big[f(x) - (\alpha+\beta) f(x-h) + \left(\frac{(\alpha+\beta)(\alpha+\beta-1)}{2!}\right) f(x-2h)\\
&\quad - \left(\frac{(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)}{3!}\right) f(x-3h) + \ldots\Big] = \Delta_{h,x}^{\alpha+\beta,b,b} f(x).
\end{aligned}$$

(iii) Proof is similar to that of (ii). □

Theorem 2.4. *Let α and $0 \neq \beta \in \mathbb{R}$, then*

$$\Delta_{h,x}^{\alpha,b,b} x^\beta = x^{\beta-\alpha} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}.$$

Proof. From Eq. (2.4), if $h \to 0$, it is seen that

$$
\begin{aligned}
\Delta^{\alpha,b,b}_{h,x} x^\beta &= \sum_{i=0}^{\infty} \frac{(-\alpha)_i (x-ih)^\beta}{i! h^\alpha} \\
&= h^{-\alpha}\Big[x^\beta - \alpha(x-h)^\beta + \frac{\alpha(\alpha-1)}{2!}(x-2h)^\beta \\
&\quad - \frac{\alpha(\alpha-1)(\alpha-2)}{3!}(x-3h)^\beta + \dots\Big] \\
&= x^\beta h^{-\alpha}\left[1 - \alpha\left(1-\frac{h}{x}\right)^\beta + \frac{\alpha(\alpha-1)}{2!}\left(1-\frac{2h}{x}\right)^\beta \right.\\
&\quad \left. - \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\left(1-\frac{3h}{x}\right)^\beta + \dots\right] \\
&= x^{\beta-\alpha}\delta^{-\alpha}\Big[1 - \alpha(1-\delta)^\beta + \frac{\alpha(\alpha-1)}{2!}(1-2\delta)^\beta \\
&\quad - \frac{\alpha(\alpha-1)(\alpha-2)}{3!}(1-3\delta)^\beta + \dots\Big] \\
&= x^{\beta-\alpha}\delta^{-\alpha}\left[1 - \alpha\left(1 - \beta\delta + \frac{\beta(\beta-1)}{2!}\delta^2 - \frac{\beta(\beta-1)(\beta-2)}{3!}\delta^3 + \dots\right)\right. \\
&\quad + \frac{\alpha(\alpha-1)}{2!}\left(1 - 2\delta + \frac{\beta(\beta-1)}{2!}(2\delta)^2 - \frac{\beta(\beta-1)(\beta-2)}{3!}(2\delta)^3 + \dots\right) \\
&\quad \left. - \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\left(1 - 3\delta + \frac{\beta(\beta-1)}{2!}(3\delta)^2 - \frac{\beta(\beta-1)(\beta-2)}{3!}(3\delta)^3 + \dots\right) + \dots\right] \\
&= x^{\beta-\alpha}\delta^{-\alpha}\left[\left(1 - \alpha + \frac{\alpha(\alpha-1)}{2!} + \dots\right) + \beta\delta\left(\alpha - 2\frac{\alpha(\alpha-1)}{2!} + 3\frac{\alpha(\alpha-1)(\alpha-2)}{3!} - \dots\right)\right. \\
&\quad - \frac{\beta(\beta-1)}{2!}\delta^2\left(\alpha - 2^2\frac{\alpha(\alpha-1)}{2!} + 3^2\frac{\alpha(\alpha-1)(\alpha-2)}{3!} - \dots\right) + \frac{\beta(\beta-1)(\beta-2)}{3!}\delta^3 \\
&\quad \left. \times\left(\alpha - 2^3\frac{\alpha(\alpha-1)}{2!} + 3^3\frac{\alpha(\alpha-1)(\alpha-2)}{3!} - \dots\right) + \dots\right] \\
&= x^{\beta-\alpha}\left[\beta\delta^{1-\alpha}\left(\alpha - 2\frac{\alpha(\alpha-1)}{2!} + 3\frac{\alpha(\alpha-1)(\alpha-2)}{3!} - \dots\right) - \frac{\beta(\beta-1)}{2!}\delta^{2-\alpha}\right. \\
&\quad \times\left(\alpha - 2^2\frac{\alpha(\alpha-1)}{2!} + 3^2\frac{\alpha(\alpha-1)(\alpha-2)}{3!} - \dots\right) + \dots \\
&\quad \left. + (-1)^{k-1}\frac{\beta(\beta-1)\dots(\beta-k+1)}{k!}\delta^{k-\alpha}\left(\alpha - 2^k\frac{\alpha(\alpha-1)}{2!} + 3^k\frac{\alpha(\alpha-1)(\alpha-2)}{3!} - \dots\right)\right] \\
&= x^{\beta-\alpha}\frac{\beta(\beta-1)\dots(\beta-k+1)}{\Gamma(\alpha+1)}\Gamma(\alpha+1), \quad = x^{\beta-\alpha}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}.
\end{aligned}
$$

It is noticed that $\delta^{k-\alpha} \to 0$, as $h \to 0$ for all $k > \alpha$, where $\delta = h/x$.

It is being suggested that the above formula is valid for $\beta \neq 0$. For $\beta = 0, f(x) = x^\beta = 1$, which is a constant function. As a result, we have $\Delta^{\alpha,b,b}_{h,x} x^\beta = 0$, but using this formula, one may claim that $\Delta^{\alpha,b,b}_{h,x} x^\beta = x^{-\alpha}\frac{1}{\Gamma(1-\alpha)}$ which is a contradiction. □

Now, we illustrate some counter examples involving certain functions and their fractional derivatives:

- Let $f(x) = x$ and $\alpha = 1/2$, then
  $\Delta^{\alpha,b,b}_{h,x} f(x) = \dfrac{2}{\sqrt{\pi}} x^{1/2}.$
- Let $f(x) = x^{1/2}$ and $\alpha = 1/2$, then
  $\Delta^{\alpha,b,b}_{h,x} f(x) = \sqrt{\pi}/2.$
- Let $f(x) = x^2$ and $\alpha = 1/2$, then
  $\Delta^{\alpha,b,b}_{h,x} f(x) = \dfrac{8}{3\sqrt{\pi}} x^{3/2}.$
- Let $f(x) = \sin wx$ and $w, \alpha \in \mathbb{R}$, then
  $\Delta^{\alpha,b,b}_{h,x} f(x) = \displaystyle\sum_{i=0}^{\infty} (-1)^i \frac{(wx)^{2i+1-\alpha}}{\Gamma(2i+2-\alpha)}.$
- Let $f(x) = \cos wx$ and $w, \alpha \in \mathbb{R}$, then
  $\Delta^{\alpha,b,b}_{h,x} f(x) = \displaystyle\sum_{i=0}^{\infty} (-1)^i \frac{(wx)^{2i-\alpha}}{\Gamma(2i+1-\alpha))}.$
- Let $f(x) = e^{wx}$ and $w, \alpha \in \mathbb{R}$, then
  $\Delta^{\alpha,b,b}_{h,x} f(x) = \displaystyle\sum_{i=0}^{\infty} \frac{(wx)^{i-\alpha}}{\Gamma(i+1-\alpha)}.$

Basically, non-integral calculus involves more complicated and comprehensive results due to their unusual and violating behaviors. Applying Theorem 2.4, we discuss following results on integral calculus which may violet for non integral cases:

Remark 2.5. For any integer α, $\Delta^{\alpha,b,b}_{h,x}(e^x) = e^x$, but it may not hold for a proper fraction α.

In particular, for $\alpha = 1/2$, using Theorem 2.4, it can be verified that

$$\Delta^{1/2,b,b}_{h,x}(e^x) = \sum_{i=0}^{\infty} \frac{x^{i-1/2}}{\Gamma(i+1/2)} \neq e^x.$$

Remark 2.6. Let $f = f(x)$ and $g = g(x)$ be two functions. For any integer n,

$$\Delta^{n,b,b}_{h,x}(fg) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{k,b,b}_{h,x}(f)\Delta^{n-k,b,b}_{h,x}(g),$$

but it does not hold for a proper fraction α. For example, we take $f(x) = x^p$, $(p > 0)$ and $g(x) = x^q$, $(q > 0)$ and using Theorem 2.4, we have

$$
\begin{aligned}
\Delta^{\alpha,b,b}_{h,x}(x^{p+q}) &= x^{p+q-\alpha}\frac{\Gamma(p+q+1)}{\Gamma(p+q+1-\alpha)} \\
&\neq \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} \Delta^{\alpha,b,b}_{h,x}(f)\Delta^{\alpha-k,b,b}_{h,x}(g) \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} \frac{\Gamma(q+1)}{\Gamma(q-\alpha+k+1)} x^{p+q-2\alpha+k}
\end{aligned}
$$

Remark 2.7. Let $f = f(x)$ and $g = g(x)$ be two functions. For any integer n, the Chain rule

$$\Delta^{n,b,b}_{h,x}(f(g(x)) = \Delta^{n,b,b}_{h,g(x)}(f(g)) \cdot \Delta^{1,b,b}_{h,x} g(x),$$

is true but, it does not hold for a proper fraction α. For example, we take $f(x)$ and $g(x)$ as considered in Remark 2.6 and using Theorem 2.4 on $f(g(x))$, we derive that

$$\Delta^{\alpha,b,b}_{h,x} f(g(x)) = \Delta^{\alpha,b,b}_{h,x}(x^{pq}) = x^{pq-\alpha}\frac{\Gamma(pq+1)}{\Gamma(pq+1-\alpha)}. \tag{2.5}$$

But, applying Chain rule and Theorem 2.4, one can easily calculate

$$
\begin{aligned}
\Delta^{\alpha,b,b}_{h,g(x)} f(g(x)) \cdot \Delta^{1,b,b}_{h,x} g(x) &= \Delta^{\alpha,b,b}_{h,x^q}(x^{pq}) \cdot \Delta^{1,b,b}_{h,x} x^q \\
&= \Delta^{\alpha,b,b}_{h,t}(t^p) \cdot \Delta^{1,b,b}_{h,x} x^q \quad (\textit{where } t = x^q) \\
&= t^{p-\alpha}\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{q-1}\frac{\Gamma(q+1)}{\Gamma(q+1-1)} \\
&= x^{qp-q\alpha}\frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} \cdot q x^{q-1} \\
&= x^{qp-q\alpha+q-1} q \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}
\end{aligned}
\tag{2.6}
$$

Combining Eqs. (2.5) and (2.6), we complete the proof.

3. Geometrical approach

In this section, we provide the geometrical interpretation of fractional derivatives of certain functions and mention their non-uniform and unusual behaviors. In fact, in Fig. 1, we have taken fractional derivatives of different orders starting from 0.1 to 1.6 of the function $f(x) = x$ and plotted them with respect to different values of $x (0 \leqslant x \leqslant 5)$ and in addition to that we have included its fractional integration of order α in 2nd figure.

In Fig. 1, set of all green curves represent the fractional derivatives of $f(x) = x$ of order α, where $0 < \alpha \leqslant 1$. They all meet each other before they intersect the original curve $f(x) = x$. Another set of curves (blue curves) belong to the family of fractional derivatives of the given function with $1 < \alpha \leqslant 1.6$. It is seen that all the curves approach the x-axis and the line $y = x$ if the value of $\alpha \geqslant 2$ and $\alpha \to 0$, respectively.

In Fig. 2, we have mentioned fractional integrations of the given function with different orders such as 0.1–1.6 which constitutes the family of red curves. It is also noticed that the curves of this family meet each other after intersecting the original function and for $\alpha < -1$, all the curves approach to the parabola $y = x^2/2$.

Furthermore, fractional derivatives of the functions $\cos x$, $\sin x$, e^x, $\cosh x$ and $\sinh x$ have been plotted for the order α varying from 0 to 1.3 in Figs. 3–7, respectively. In Fig. 8, fractional derivatives of the function $f(x) = 1/(1-x)$, with $|x| < 1$, have been mentioned for different order from -1.5 to 3.

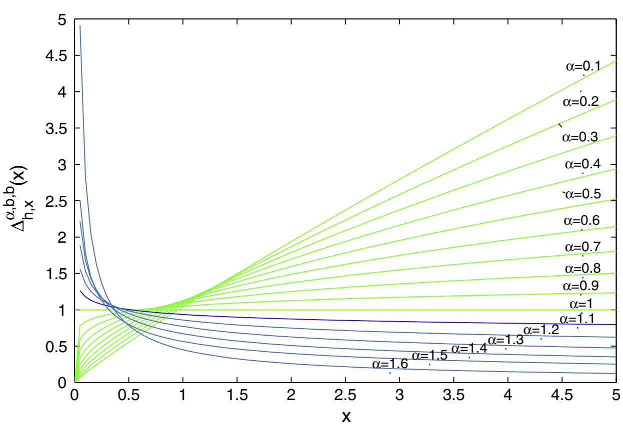

Figure 1 Fractional derivatives of the function $f(x) = x$ with order α, where $0 < \alpha \leqslant 1.6$.

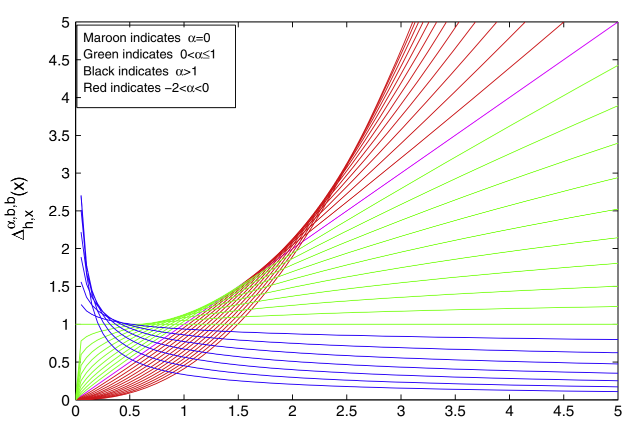

Figure 2 Fractional derivatives of the function $f(x) = x$ with order α, where $-1.6 < \alpha \leqslant 1.6$.

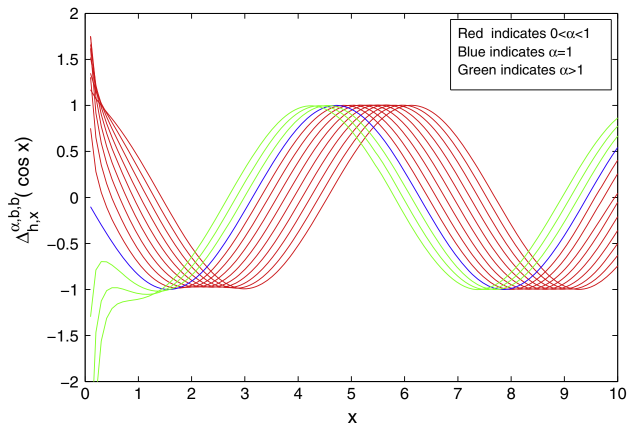

Figure 3 Fractional derivatives of the function $f(x) = \cos x$ with order α, where $0 < \alpha \leqslant 1.3$.

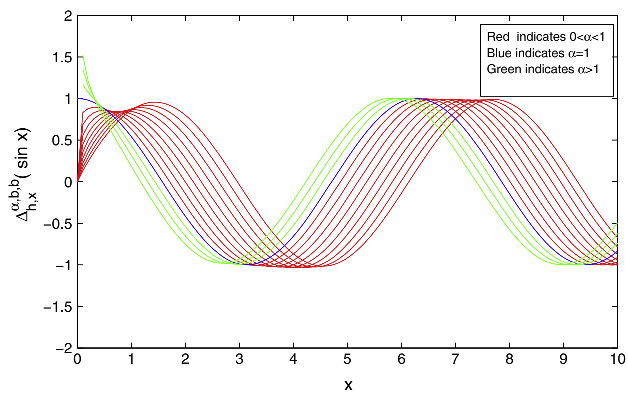

Figure 4 Fractional derivatives of the function $f(x) = \sin x$ with order α, where $0 < \alpha \leqslant 1.3$.

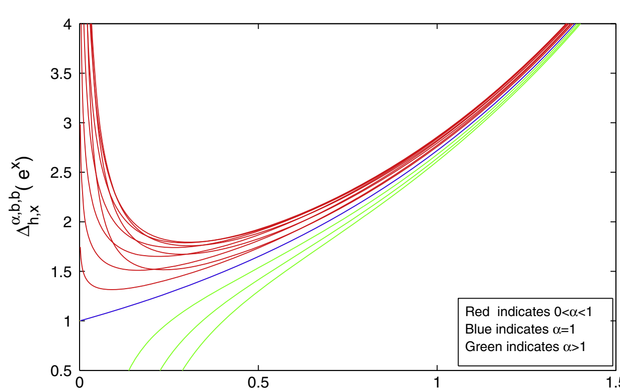

Figure 5 Fractional derivatives of the function $f(x) = e^x$ with order α, where $0 < \alpha \leqslant 1.3$.

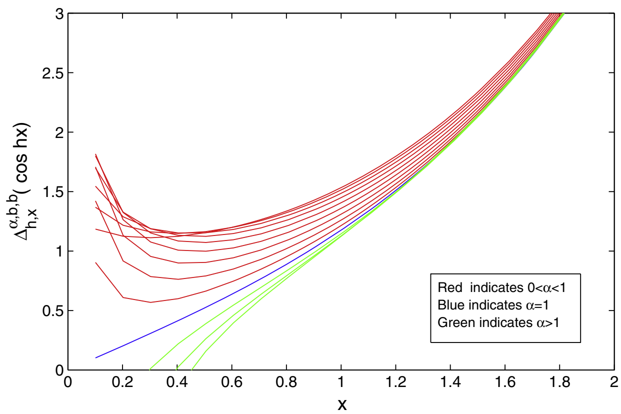

Figure 6 Fractional derivatives of the function $f(x) = \cosh x$ with order α, where $0 < \alpha \leqslant 1.3$.

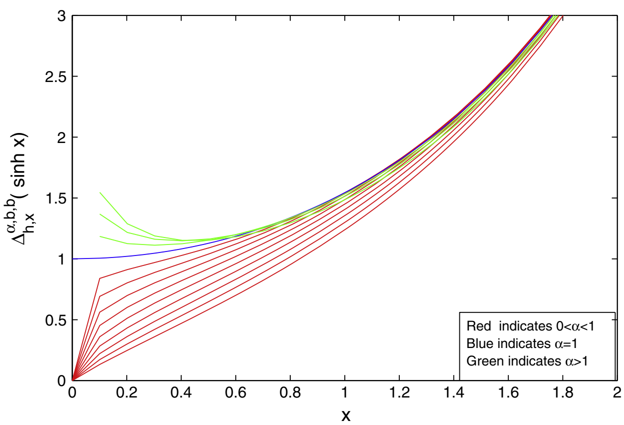

Figure 7 Fractional derivatives of the function $f(x) = \sinh x$ with order α, where $0 < \alpha \leqslant 1.3$.

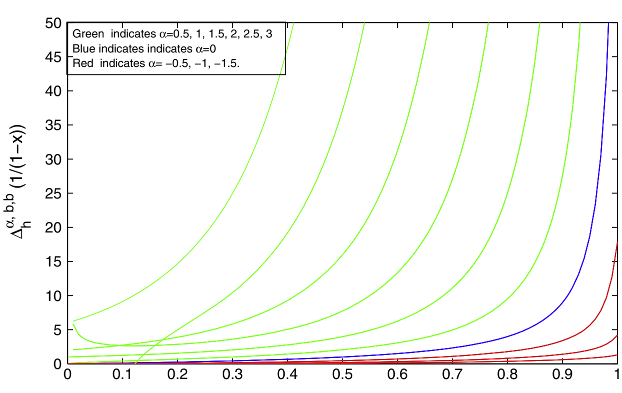

Figure 8 Fractional derivatives of the function $f(x) = \frac{1}{1-x}$ with order α, where $-1.5 < \alpha \leqslant 3$.

4. Conclusion

New difference sequence spaces have been introduced using fractional difference operator and their topological properties are studied. Applying this operator, we provide an alternative approach to study the notion of derivatives and integrations of non-integral order. Finally, with the help of some counter examples and graphs, we have shown that fractional calculus deviates most of the rules which are based on the theory of calculus in classical sense.

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