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Approximate solutions of fractional Zakharov–Kuznetsov equations by VIM

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A B S T R A C T

This paper presents the approximate analytical solution of a fractional Zakharov–Kuznetsov equation with the help of the powerful variational iteration method. The fractional derivatives are described in the Caputo sense. Several examples are given and the results are compared to exact solutions. The results show that the variational iteration method is very effective, convenient and simple to use.

1. Introduction

Fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. Hence, great attention has been given to finding numerical/exact/approximate solutions of FDEs. Our concern in this paper is on approximate numerical methods for FDEs.

Some of the recent analytical methods for FDEs include the Adomian decomposition method (ADM), the homotopy perturbation method (HPM), the variational iteration method (VIM) and homotopy analysis method (HAM). The ADM was applied to fractional diffusion equations in [1] and fractional modified KdV equations in [2]. Hosseinnia et al. [3] presented an enhanced HPM for FDEs and Abdulaziz et al. [4] extended the application of HPM to systems of FDEs. In [5], Abdulaziz et al. solved the fractional IVPs by the HPM. The HAM was applied to fractional KDV–Burgers–Kuromoto equations [6], fractional IVPs [7], time-fractional PDEs [8], linear and nonlinear FDEs [9], and systems of nonlinear FDEs [10].

The VIM was proposed in [11–17]. It is one of the methods which has received much attention. It is based on the Lagrange multiplier and the correction functional. The VIM is a powerful tool to searching for approximate solution of nonlinear problems without the requirement of linearization or perturbation. He [11] was the first to solve FDEs by VIM with great success. Later, Draganescu [18] and Odibat and Momani [19] applied the VIM to more complex FDEs, showing the effectiveness and accuracy of the method. Furthermore, Inc [20] applied VIM to solve the space- and time-fractional Burgers equations. Song et al. [21] used VIM to obtain approximate solution of the fractional Sharma–Tasso–Olever equations. Yulita et al. [22] used VIM to solve fractional heat- and wave-like equations. Recently, Das [23] found the exact solution of fractional diffusion equations using VIM.

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This paper considers the fractional version of the Zakharov–Kuznetsov equations as studied in [24]. The fractional Zakharov–Kuznetsov equations (shortly called FZK(p, q, r)) considered are of the form:

$$D_t^{\alpha} u + a(u^p)_x + b(u^q)_{xxx} + c(u^r)_{yyx} = 0, \tag{1}$$

where $u = u(x, y, t)$, α is a parameter describing the order of the fractional derivative ($0 < \alpha \leq 1$), a, b and c are arbitrary constants and p, q, and r are integers and $p, q, r \neq 0$ governs the behavior of weakly nonlinear ion acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [25,26].

The purpose of this paper is to obtain approximate solutions of the fractional Zakharov–Kuznetsov equations by VIM, and to determine the accuracy of VIM in solving these kinds of problems.

2. Basic definitions

Fractional calculus unifies and generalizes the notions of integer-order differentiation and n-fold integration [27,28]. We give some basic definitions and properties of fractional calculus theory which shall be used in this paper:

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space $\mathcal{C}_{\mu}$, $\mu \in \mathbf{R}$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in \mathcal{C}[0, \infty)$, and it is said to be in the space $\mathcal{C}_{\mu}^{m}$ iff $f^{(m)} \in \mathcal{C}_m$, $m \in \mathbf{N}$.

The Riemann–Liouville fractional integral operator is defined as follows:

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{\mu}$, $\mu \geq -1$, is defined as

$$\begin{aligned} &J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \quad \alpha > 0, x > 0, \\ &J^0 f(x) = f(x). \end{aligned} \tag{2}$$

In this paper only real and positive values of α will be considered.

Properties of the operator J^{α} can be found in [28] and we mention only the following: For $f \in C_{\mu}$, $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma \geq -1$:

1. $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x)$,
2. $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$,
3. $J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Reimann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with FDEs. Therefore, we shall introduce a modified fractional differential operator D_*^{α} proposed by Caputo in his work on the theory of viscoelasticity [29]:

Definition 2.3. The fractional derivative of $f(x)$ in Caputo sense is defined as

$$D_*^{\alpha} f(x) = J^{m-\alpha} D_*^m f(x) = \frac{1}{\Gamma(m)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s)\,\mathrm{d}s, \quad \text{for } m-1 < \alpha \leq m, m \in \mathcal{N}, x > 0, f \in C_{-1}^m. \tag{3}$$

In addition, we also need the following property:

Lemma 2.4. *If $m - 1 < \alpha \leq m$, $m \in \mathcal{N}$ and $f \in C_{\mu}^m$, $\mu \geq -1$, then*

$$D_*^{\alpha} J^{\alpha} f(x) = f(x),$$

and,

$$J^{\alpha} D_*^{\alpha} f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{x^i}{i!}, \quad x > 0. \tag{4}$$

The Caputo differential derivative is considered here because the initial and boundary conditions can be included in the formulation of the problems [27]. The fractional derivative is taken in the Caputo sense as follows:

Definition 2.5. For m to be the smallest integer that exceeds α, the Caputo fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^{\alpha} u(x, y, t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-s)^{m-\alpha-1} \frac{\partial^m u(x, y, s)}{\partial s^m}\,\mathrm{d}s, & \text{for } m-1 < \alpha \leq m, \\ \dfrac{\partial^m u(x, y, t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbf{N}. \end{cases} \tag{5}$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult [27,28].

3. Variational iteration method

To illustrate the basic concepts of VIM [14], we consider the following general nonlinear functional equation:

$$Lu(x,y,t)+Nu(x,y,t)=g(x,y,t), \tag{6}$$

where L is a linear operator and N is a nonlinear operator, and $g(x,y,t)$ is an inhomogeneous term.

VIM is based on the general Lagrange multiplier method [30]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. According to VIM, we can construct a correction functional for Eq. (6) as follows:

$$u_{k+1}(x,y,t)=u_k(x,y,t)+\int_0^t \lambda(\xi)\left[Lu_k(x,y,s)+N\widetilde{u}_k(x,y,s)-g(x,y,s)\right]\mathrm{d}s, \tag{7}$$

where λ, a general Lagrange multiplier, can be identified optimally via the variational theory. The subscript k indicates the kth approximation and $\widetilde{u}_k$ is considered as a restricted variations [11,12], i.e. $\delta\widetilde{u}_k=0$.

4. Illustrative examples

In this section, the applicability of VIM shall be demonstrated by two test examples.

4.1. Example 1

First, we consider the time-fractional FZK(2, 2, 2) in the form:

$$D_t^{\alpha}u+(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{xyy}=0, \tag{8}$$

where $0<\alpha\le 1$ is a parameter describing the order of the fractional time derivative. The exact solution to Eq. (8) when $\alpha=1$ and subject to the initial condition

$$u(x,y,0)=\frac{4}{3}\rho\sinh^2(x+y), \tag{9}$$

where ρ is an arbitrary constant, was derived in [31] and is given as:

$$u(x,y,t)=\frac{4}{3}\rho\sinh^2(x+y-\rho t). \tag{10}$$

To apply VIM to (8), we construct the correction functional as follows:

$$u_{k+1}=u_k+\int_0^t \lambda(s)\left[\frac{\partial^{\alpha}u_k}{\partial s^{\alpha}}+\left(\frac{\partial\widetilde{u_k^2}}{\partial x}\right)+\frac{1}{8}\left(\frac{\partial^3\widetilde{u_k^2}}{\partial x^3}\right)+\frac{1}{8}\left(\frac{\partial^3\widetilde{u_k^2}}{\partial y^2\partial x}\right)\right]\mathrm{d}s. \tag{11}$$

For $\alpha=1$, we have

$$\delta u_{k+1}=\delta u_k+\delta\int_0^t \lambda(s)\left(\frac{\partial u_k}{\partial s}\right)\mathrm{d}s. \tag{12}$$

Thus, we obtain the following stationary conditions:

$$1+\lambda(t)|_{s=t}=0,$$
$$\lambda'(s)=0.$$

The general Lagrange multiplier can be identified as:

$$\lambda(s)=-1. \tag{13}$$

Substituting (13) into the correction functional (11), we obtain the following iteration formula:

$$u_{k+1}=u_k-\int_0^t\left[\frac{\partial^{\alpha}u_k}{\partial s^{\alpha}}+\left(\frac{\partial u_k^2}{\partial x}\right)+\frac{1}{8}\left(\frac{\partial^3u_k^2}{\partial x^3}\right)+\frac{1}{8}\left(\frac{\partial^3u_k^2}{\partial y^2\partial x}\right)\right]\mathrm{d}s. \tag{14}$$

Table 1
Solutions using the 3-iteration of VIM for different values of α when $\rho = 0.001$ and $y = 0.9$.

VIM						
x	y	t	$\alpha = 0.67$	$\alpha = 0.75$	$\alpha = 1$	Exact ($\alpha = 1$)
0.1	0.1	0.2	5.318536379E−5	5.327473579E−5	5.355355975E−5	5.393877159E−5
		0.3	5.286311739E−5	5.297566848E−5	5.330816448E−5	5.388407669E−5
		0.4	5.257767969E−5	5.270397800E−5	5.306406852E−5	5.382941057E−5
0.6	0.6	0.2	2.954927772E−3	2.963560066E−3	2.989873669E−3	3.036507411E−3
		0.3	2.926620960E−3	2.937172119E−3	2.967173317E−3	3.035778955E−3
		0.4	2.903065355E−3	2.914474969E−3	2.945226366E−3	3.035050641E−3
0.9	0.9	0.2	1.068216255E−2	1.077158824E−2	1.102484681E−2	1.153697757E−2
		0.3	1.044865933E−2	1.054878201E−2	1.079635470E−2	1.153454074E−2
		0.4	1.027770138E−2	1.037358328E−2	1.057416210E−2	1.153210438E−2

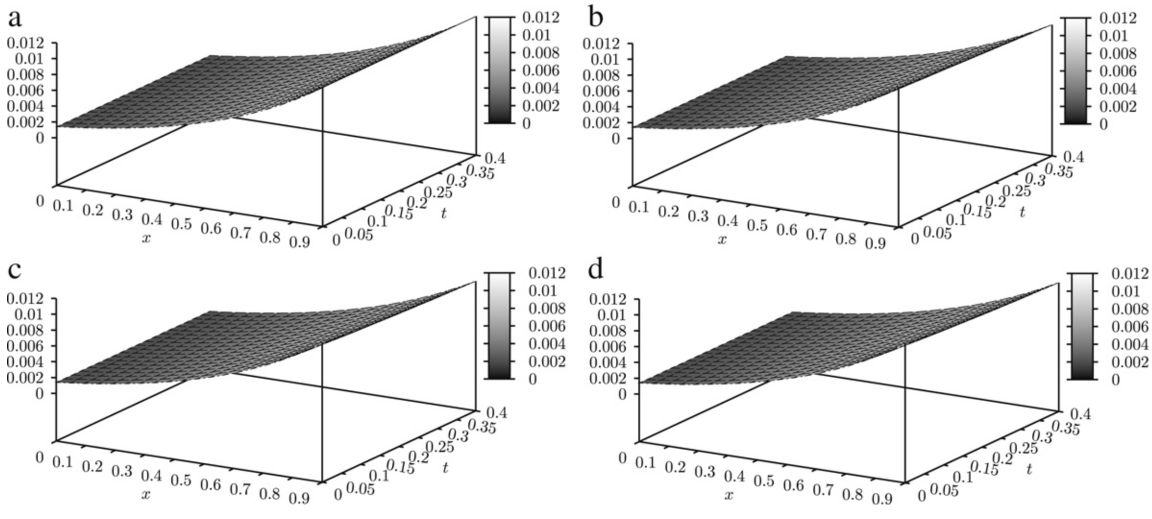

Fig. 1. Solutions using the 3-iteration of VIM for different values of α when $y = 0.9$ and $\rho = 0.001$: (a) exact ($\alpha = 1$), (b) $\alpha = 1$, (c) $\alpha = 0.75$ and (d) $\alpha = 0.67$.

The iteration starts with an initial approximation as given in (9). The iteration formula (14) now yields

$$u_1(x, y, t) = \frac{4}{3}\rho \sinh^2 w - \frac{(4\rho)^2}{9} \sinh w \cosh w \left[14 \sinh^2 w + \frac{2}{3} \cosh^2 w\right] t, \tag{15}$$

$$\begin{aligned} u_2(x, y, t) = {} & \frac{4\rho}{3} \sinh^2 w - \frac{(4\rho)^2}{9} \sinh w \cosh w[40 \cosh^2 w - 28]t \\ & - \frac{(4\rho)^3}{27}[79 - 968 \cosh^2 w + 2080 \cosh^4 w - 1200 \cosh^6 w]t^2 \\ & - \frac{(4\rho)^4}{81} \sinh w \cosh w \left[-\frac{5320}{3} + 14640 \cosh^2 w - 30400 \cosh^4 w - \frac{54400}{3} \cosh^6 w\right] t^3 \\ & - \frac{(4\rho)^2}{9} \cosh w \sinh w[14 - 135 \cosh^2 w] \frac{t^{2-\alpha}}{\Gamma(3-\alpha)}, \end{aligned} \tag{16}$$

and so on, where $w = x + y$. The remaining components of $u_k(x, y, t)$ can be completely determined such that each term is determined by using (14). Table 1 shows the approximate solutions of Eq. (8) for different values of α: $\alpha = 0.67$, $\alpha = 0.75$ and $\alpha = 1.0$ using only three iterations of the VIM solution (see also Fig. 1).

4.2. Example 2

Now, we consider FZK(3,3,3) in the form:

$$D_t^\alpha u + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{xyy} = 0, \tag{17}$$

where $0 < \alpha \leq 1$.

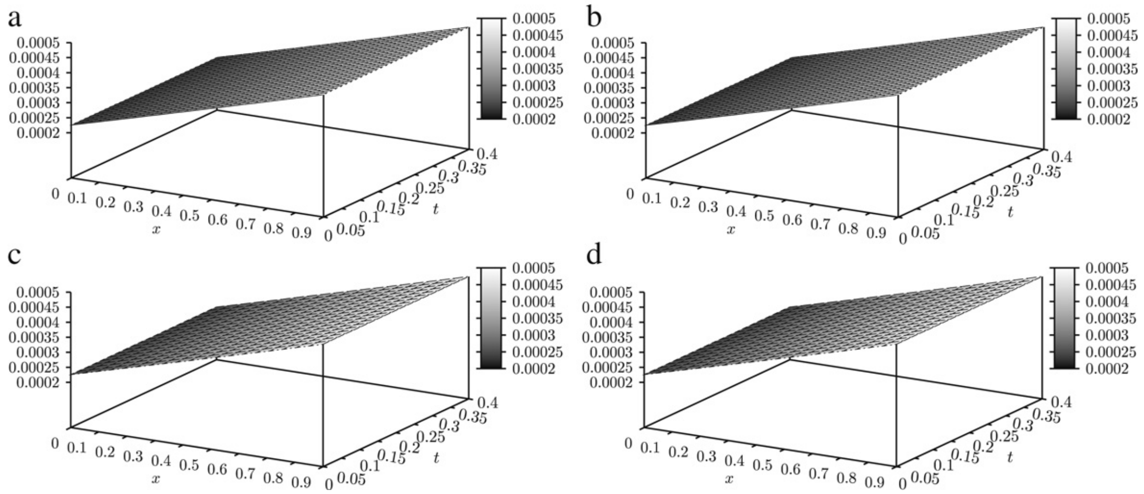

Fig. 2. Solutions using the 3-iteration of VIM for different values of α when $y = 0.9$ and $\rho = 0.001$: (a) exact ($\alpha = 1$), (b) $\alpha = 1$, (c) $\alpha = 0.75$ and (d) $\alpha = 0.67$.

Table 2
Solutions using the 3-iteration of VIM for different values of α when $\rho = 0.001$ and $y = 0.9$.

VIM						
x	y	t	$\alpha = 0.67$	$\alpha = 0.75$	$\alpha = 1$	Exact ($\alpha = 1$)
0.1	0.1	0.2	5.000912783E−5	5.000914105E−5	5.000914112E−5	4.995923204E−5
		0.3	5.000907777E−5	5.000909430E−5	5.000915456E−5	4.993421817E−5
		0.4	5.000903250E−5	5.000905153E−5	5.000910586E−5	4.990920434E−5
0.6	0.6	0.2	3.020038194E−4	3.020038425E−4	3.020039162E−4	3.019530008E−4
		0.3	3.020037516E−4	3.020037779E−4	3.020038551E−4	3.019274992E−4
		0.4	3.020036895E−4	3.020037195E−4	3.020037937E−4	3.019019978E−4
0.9	0.9	0.2	4.567801885E−4	4.567802187E−4	4.567802934E−4	4.567281735E−4
		0.3	4.567800915E−4	4.567801293E−4	4.567802556E−4	4.567020404E−4
		0.4	4.567800089E−4	4.567800482E−4	4.567801785E−4	4.566759074E−4

The exact solution to Eq. (17) when $\alpha = 1$ and subject to the initial condition

$$u(x, y, 0) = \frac{3}{2}\rho \sinh\left[\frac{1}{6}(x + y)\right] \tag{18}$$

where ρ is an arbitrary constant, was derived in [31] and is given by

$$u(x, y, t) = \frac{3}{2}\rho \sinh\left[\frac{1}{6}(x + y - \rho t)\right]. \tag{19}$$

To apply VIM, we construct the following correction functional for Eq. (17):

$$u_{k+1} = u_k + \int_0^t \lambda(s)\left[\frac{\partial^\alpha u_k}{\partial s^\alpha} + \left(\frac{\partial \widetilde{u_k^3}}{\partial x}\right) + 2\left(\frac{\partial^3 \widetilde{u_k^3}}{\partial x^3}\right) + 2\left(\frac{\partial^3 \widetilde{u_k^3}}{\partial y^2 \partial x}\right)\right] \mathrm{d}s. \tag{20}$$

The general Lagrange multiplier for this example is exactly the same as in (13). Hence we obtain the following iteration formula:

$$u_{k+1} = u_k - \int_0^t \left[\frac{\partial^\alpha u_k}{\partial s^\alpha} + \left(\frac{\partial u_k^3}{\partial x}\right) + 2\left(\frac{\partial^3 u_k^3}{\partial x^3}\right) + \left(\frac{\partial^3 u_k^3}{\partial y^2 \partial x}\right)\right] \mathrm{d}s. \tag{21}$$

Using (18) as an initial condition yields the following:

$$u_1(x, y, t) = 3\left(\frac{\rho}{2}\right)\sinh w - 3\left(\frac{\rho}{2}\right)^3 \cosh w[-8 + 4\cosh^2 w]t, \tag{22}$$

$$\begin{aligned} u_2(x,y,t) &= 3\left(\frac{\rho}{2}\right)\sinh w + 3\left(\frac{\rho}{2}\right)^3\cosh w[16-18\cosh^2 w]t + 3\left(\frac{\rho}{2}\right)^5\sinh w[182-1458\cosh^2 w \\ &\quad + 1530\cosh^4 w]t^2 + 3\left(\frac{\rho}{2}\right)^7\cosh w\left[\frac{2944}{3} - 6324\cosh^2 w + 10803\cosh^4 w - 5481\cosh^6 w\right]t^3 \\ &\quad + 3\left(\frac{\rho}{2}\right)^9\sinh w\left[128 - 5472\cosh^2 w + 29250\cosh^4 w - 48195\cosh^6 w + \frac{98415}{4}\cosh^8 w\right]t^4 \\ &\quad + 3\left(\frac{\rho}{2}\right)^3\cosh w[9\cosh^2 w - 8]\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}, \end{aligned} \tag{23}$$

and so on, where $w = \frac{1}{6}(x+y)$.

Table 2 and Fig. 2 show the solutions obtained using the 3-iterates of VIM for different values of α when $\rho = 0.001$ and $y = 0.9$.

5. Conclusion

In this paper, we presented the application of VIM to fractional Zakharov–Kuznetsov equations. The VIM gives series solutions of the equation. Numerical experiments show that a few iterations of the VIM recursive formula can yield good solutions.

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