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Iterated antiderivative extensions

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ABSTRACT

Let F be a characteristic zero differential field with an algebraically closed field of constants and let E be a no new constants extension of F. We say that E is an *iterated antiderivative extension* of F if E is a liouvillian extension of F obtained by adjoining antiderivatives alone. In this article, we will show that if E is an iterated antiderivative extension of F and K is a differential subfield of E that contains F then K is an iterated antiderivative extension of F.

1. Introduction

Let $F := \mathbb{C}(z)$ be the differential field of rational functions in one complex variable z with the usual derivation d/dz. Consider the liouvillian extensions $E_1 := F(e^{z^2}, \int e^{z^2})$ and $E_2 := F(\sqrt{1-z^2}, \sin^{-1} z)$ of F. In [5], M. Rosenlicht and M. Singer show that the differential subfield $F((\int e^{z^2})/e^{z^2})$ of E_1 and the differential subfield $F(\sqrt{1-z^2}\sin^{-1} z)$ of E_2 are not liouvillian extensions of F. Thus, differential subfields of liouvillian extensions, in general, need not be liouvillian. However, if $L := \mathbb{C}(z, \log z, \log(\log z))$ then one can list all the differential subfields of L that contains $\mathbb{C}$ and they are $\mathbb{C}$, $\mathbb{C}(z)$, $\mathbb{C}(z, \log z)$ and L, see Example 4.1. Clearly, in this case, all the differential subfields are liouvillian. Thus, it is of considerable interest to know when differential subfields of a liouvillian extension are liouvillian. In this article, we will show that if a liouvillian extension is obtained by adjoining antiderivatives alone then its differential subfields can also be obtained by adjoining antiderivatives alone. This is the main result of this article and it appears as Theorem 5.3. An analogue of Theorem 5.3 for generalized elementary extensions can be found in [5] and [6].

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1.1. Differential fields

Let F be a field of characteristic zero. A *derivation* on a field F, denoted by $'$, is an additive map $' : F \to F$ that satisfies the Leibniz law $(xy)' = x'y + xy'$ for every $x, y \in F$. A field equipped with a derivation map is called a *differential field*. The set of *constants* of a differential field is the kernel of the map $'$ and it can be seen that the set of constants is a differential subfield of F. Let E and F be differential fields. We say that E is a *differential field extension* of F if E is a field extension of F and the restriction of the derivation of E to F coincides with the derivation of F. A differential field extension E of F is called a *no new constants* extension if the constants of E are the same as the constants of F.

Throughout this article, we fix a ground differential field F of characteristic zero. All the differential fields considered henceforth are either differential subfields of F or a differential field extension of F. We reserve the notation $'$ to denote the derivation map of any given differential field. We do not require the field of constants of F to be algebraically closed until Section 4.

Let E be a no new constants extension of F. An element $\zeta \in E$ is called an *antiderivative* (of an element) of F if $\zeta' \in F$. We say that E is an *antiderivative extension* of F if $E = F(\zeta_1, \zeta_2, \dots, \zeta_n)$, where $\zeta_1, \zeta_2, \dots, \zeta_n$ are antiderivatives of F. Elements $\zeta_1, \zeta_2, \dots, \zeta_n \in E$ are called *iterated antiderivatives* of F if $\zeta_1' \in F$ and for $i \geqslant 2$, $\zeta_i' \in F(\zeta_1, \zeta_2, \dots, \zeta_{i-1})$. We call E an *iterated antiderivative extension* of F if $E = F(\zeta_1, \zeta_2, \dots, \zeta_n)$, where $\zeta_1, \zeta_2, \dots, \zeta_n$ are iterated antiderivatives of F. And, if $E = F(\zeta_1, \zeta_2, \dots, \zeta_n)$ and for each i, $\zeta_i' \in F(\zeta_1, \dots, \zeta_{i-1})$ or $\zeta_i'/\zeta_i \in F(\zeta_1, \dots, \zeta_{i-1})$ or ζ_i is algebraic over $F(\zeta_1, \dots, \zeta_{i-1})$ then we call E a *liouvillian* extension of F. Now it is clear that the differential fields E_1, E_2 and L, mentioned in the beginning of this article, are examples of liouvillian extensions of $\mathbb{C}$ and that L is an iterated antiderivative extension of $\mathbb{C}$. A field automorphism of E that fixes the elements of F and commutes with the derivation is called a differential field automorphism and the group of all such automorphisms will be denoted by $G(E|F)$. That is, $G(E|F) = \{\sigma \in Aut(E|F) \mid \sigma(y)' = \sigma(y') \text{ for all } y \in E\}$.

Every antiderivative extension of F is an iterated antiderivative extension of F. But the converse is not true: for example, consider the differential field $\mathbb{C}(z, \log z)$ with the usual derivation d/dz, where $\mathbb{C}$ is the field of complex numbers. Clearly, $\mathbb{C}(z, \log z)$ is an iterated antiderivative extension of $\mathbb{C}$. Observe that all the antiderivatives of the field $\mathbb{C}$ are of the form $cz + d$ where $c, d \in \mathbb{C}$ and since $\log z \notin \mathbb{C}(z)$, we see that $\mathbb{C}(z, \log z)$ is not an antiderivative extension of $\mathbb{C}$.

2. Preliminary results

It is a well-known fact that if E is a no new constants extension of F and if $\zeta \in E$ is an antiderivative of an element of F then either ζ is transcendental over F or $\zeta \in F$. Please see [3, p. 7], or [5, p. 329] for a proof. Using this fact, we will now show that every iterated antiderivative extension of F is a purely transcendental extension of F.

Theorem 2.1. *Let E and K be differential subfields of some no new constants extension of F. Suppose that $E = F(\zeta_1, \zeta_2, \dots, \zeta_n)$ is an iterated antiderivative extension of F and that $K \supseteq F$. Then $KE := K(\zeta_1, \zeta_2, \dots, \zeta_n)$ is an iterated antiderivative extension of K. Furthermore, if $KE \neq K$ then the set $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ contains algebraically independent iterated antiderivatives $\eta_1, \eta_2, \dots, \eta_t$ of K such that $KE = K(\eta_1, \eta_2, \dots, \eta_t)$.*

Proof. Since K contains F, it is easy to see that $\zeta_i' \in K(\zeta_1, \zeta_2, \dots, \zeta_{i-1})$ and thus KE is an iterated antiderivative extension of K. Assume that $K(E) \neq K$. To find a transcendence base for KE, consisting of iterated antiderivatives of K, we use an induction on n. Case $n = 1$: Since $KE = K(\zeta_1) \neq K$, we have $\zeta_1 \notin K$. And since $\zeta_1' \in F \subseteq K$, as noted earlier, ζ_1 is transcendental over K. Set $\eta_1 := \zeta_1$ to prove the theorem. Assume the theorem for $n - 1$ iterated antiderivatives. Induction step: Choose l smallest such that $\zeta_l \notin K$ and set $\eta_1 := \zeta_l$. Since $\zeta_1, \dots, \zeta_{l-1} \in K$, we see that η_1 is an antiderivative of K and since $\eta_1 \notin K$, η_1 is transcendental over K. Note that KE is generated as a field by $n - l$ iterated antiderivatives, namely $\zeta_{l+1}, \dots, \zeta_n$, and the differential field $K(\eta_1)$. Now we may apply induction to the iterated antiderivative extension KE of $K(\eta_1)$ and obtain iterated antiderivatives $\eta_2, \dots, \eta_t \in \{\zeta_{l+1}, \dots, \zeta_n\}$ of $K(\eta_1)$ such that $\eta_2, \dots, \eta_t$ are algebraically independent over $K(\eta_1)$ and that $KE = K(\eta_1)(\eta_2, \dots, \eta_t)$. $\square$

In Theorem 2.1, if we choose $K = F$, we obtain that E is a purely transcendental extension of F with a transcendence base consisting of iterated antiderivatives of F. Note that Theorem 2.1 is valid for antiderivative extensions as well. Thus, hereafter, when we say $E = F(\zeta_1, \zeta_2, \dots, \zeta_t)$ is an antiderivative extension or an iterated antiderivative extension of F, it is understood that $\zeta_1, \zeta_2, \dots, \zeta_t$ are algebraically independent over F. We will use the notation $\mathrm{tr.d.}(E|F)$ to denote the transcendence degree of any field extension E over F.

Corollary 2.1.1. *Let E be an iterated antiderivative extension of F and let K_1 and K be differential subfields of E. If $K_1 \supset K \supseteq F$ then* $\mathrm{tr.d.}(K_1|F) > \mathrm{tr.d.}(K|F)$.

Proof. Suppose that $K_1 \supset K$. Then we have $E \supset K$ and therefore from Theorem 2.1, we know that $KE = E$ is a purely transcendental extension of K. Thus if $u \in K_1 - K$ then $u \in E - K$ and therefore u is transcendental over K. Thus $\mathrm{tr.d.}(K_1|K) \geqslant 1$. Note that $\mathrm{tr.d.}(K_1|F) = \mathrm{tr.d.}(K_1|K) + \mathrm{tr.d.}(K|F)$ and that $\mathrm{tr.d.}\, K_1|F < \infty$ since $\mathrm{tr.d.}(E|F) < \infty$. Hence $\mathrm{tr.d.}(K_1|F) > \mathrm{tr.d.}(K|F)$. $\square$

Let M be a differential field extension of F. We call M a *minimal* differential field extension of F if $M \supset F$ and if K is a differential subfield of M such that $M \supseteq K \supseteq F$ then $K = M$ or $K = F$.

Corollary 2.1.2. *Let E, K and K_1 be as in Corollary* 2.1.1. *Then K_1 contains a minimal differential field extension of K.*

Proof. If K_1 is not a minimal differential field extension of K then it contains a proper subfield $K_1 \supset M \supset K$. And, from Corollary 2.1.1, we know that $\mathrm{tr.d.}(K_1|K) > \mathrm{tr.d.}(M|K)$. Since $\mathrm{tr.d.}(K_1|K) < \infty$, the rest of the proof follows by an induction on $\mathrm{tr.d.}(K_1|F)$. $\square$

Theorem 2.2. *Let E be an iterated antiderivative extension of F and $K \supseteq F$ be a differential subfield of E. If there is an element $u \in E$ such that $u'/u \in K$ then $u \in K$.*

Proof. To avoid triviality, we may assume $E \neq K$. We observe from Theorem 2.1 that $E = K(\eta_1, \eta_2, \dots, \eta_t)$ is an iterated antiderivative extension of K. Let $u \in E$ and $u'/u \in K$. We will use an induction on t to prove our proposition. Assume that if $u \in K(\eta_1, \eta_2, \dots, \eta_{t-1})$ then $u \in K$. Write $u = P/Q$, where $P, Q \in K(\eta_1, \eta_2, \dots, \eta_{t-1})[\eta_t]$ are relatively prime polynomials and Q is monic. Then $u' = (P'Q - Q'P)/Q^2$ and since $f := u'/u \in K$, we obtain

$$QPf = P'Q - Q'P.$$

Since P and Q are relatively prime, we then obtain P divides P' and Q divides Q'. Now the facts that Q is monic, $\deg Q' \lneqq \deg Q$ and Q divides Q', all together, will force $Q = 1$. Thus $u = P$ and $P' = fP$. Write $P = \sum_{i=0}^{n} a_i \eta_t^i$ with $a_n \neq 0$ and observe that

$$a_n' \eta_t^n + \left(a_{n-1}' + n a_n \eta_t'\right)\eta_t^{n-1} + \cdots + a_1 \eta_t' + a_0' = f\left(\sum_{i=0}^{n} a_i \eta_t^i\right),$$

and comparing the leading coefficients, we obtain $a_n' = f a_n$. Thus $(u/a_n)' = 0$. Since E is a no new constants extension of F, there is a $c \in C$ such that $u = c a_n$. Now $a_n \in K(\eta_1, \eta_2, \dots, \eta_{t-1})$ will imply $u \in K(\eta_1, \eta_2, \dots, \eta_{t-1})$. $\square$

Remark. Consider the differential field $K := \mathbb{C}(z, \log z)$ with the derivation d/dz, $\overline{K}$ being its algebraic closure and let $u \in \overline{K} - K$. We claim that for any iterated antiderivative extension E of $\mathbb{C}$, the element $u \notin E$. First we note that if $E \neq \mathbb{C}$ is an iterated antiderivative extension of $\mathbb{C}$ with the derivation d/dz then $z \in E$. Now, suppose that the claim is false. Then by applying 2.1 to the iterated antiderivative extension $E(\log z)$ of $\mathbb{C}$ we obtain a contradiction to the assumption that $u \notin K$. Thus, if $u = \sqrt{z} + \sqrt[5]{\log z}$, then there are no polynomials P, $Q \in \mathbb{C}[x_1, x_2, x_3]$ such that $\sqrt{z} + \sqrt[5]{\log z} = \frac{P(z, \log z, \log(\log z))}{Q(z, \log(z+1), Li_2(z))}$, where $Li_2(z)$ is the dilogarithm $-\int_0^z \frac{\log(1-w)}{w}\,dw$.

Similarly, as an application of Theorem 2.2, one can obtain that $e^{\alpha z}$, where $\alpha \in \mathbb{C} - \{0\}$ and e^{-z^2} are not in any iterated antiderivative extension of $\mathbb{C}$. In particular, $\int e^{-z^2}$ is not in any iterated antiderivative extension of $\mathbb{C}$, and thus cannot be expressed in terms of logarithms or polylogarithms.

3. Structure of antiderivative extensions

The following theorem characterizes the algebraic dependence of antiderivatives and will be used in numerous occasions in this article. In this section we will use this theorem to describe the structure of differential subfields of antiderivative extensions.

Theorem 3.1. *Let $E \supset F$ be a no new constants extension and for $i = 1, 2, \ldots, n$, let $\zeta_i \in E$ be antiderivatives of F. Then either ζ_i's are algebraically independent over F or there is a tuple $(\alpha_1, \ldots, \alpha_n) \in C^n - \{\vec{0}\}$ such that $\sum_{i=1}^n \alpha_i \zeta_i \in F$.*

Proof. See [1, p. 260] or [7, p. 9]. □

Proposition 3.2. *Let $E = F(\zeta_1, \zeta_2, \ldots, \zeta_t)$ be an antiderivative extension of F. An element $\zeta \in E$ is an antiderivative of F if and only if there are a tuple $(\alpha_1, \ldots, \alpha_t) \in C^t - \{\vec{0}\}$ and an element $a_\zeta \in F$ such that $\zeta = \sum_{i=1}^t \alpha_i \zeta_i + a_\zeta$.*

Proof. Let $\zeta \in E$ be an antiderivative of F. The set $\{\zeta, \zeta_1, \zeta_2, \ldots, \zeta_t\}$ contains $t+1$ antiderivatives of F and therefore has to be algebraically dependent over F. We apply Theorem 3.1 and obtain constants $\beta_i, \gamma \in C$ such that $\gamma\zeta + \sum_{i=1}^t \beta_i \zeta_i \in L$. Since $\{\zeta_1, \zeta_2, \ldots, \zeta_t\}$ is algebraically independent over L, we know that $\gamma \neq 0$. Therefore

$$\zeta - \sum_{i=1}^{t} \alpha_i \zeta_i \in L, \quad \text{where } \alpha_i := \frac{-\beta_i}{\gamma} \tag{3.1}$$

and thus there is an $a_\zeta \in F$ such that $\zeta = \sum_{i=1}^t \alpha_i \zeta_i + a_\zeta$. Note that every element of the form $\sum_{i=1}^t \alpha_i \zeta_i + a$, where $(\alpha_1, \ldots, \alpha_t) \in C^t$ and $a \in F$, is clearly an antiderivative of F. □

Theorem 3.3. *Let $E = F(\zeta_1, \zeta_2, \ldots, \zeta_t)$ be an antiderivative extension of F and let K be a differential subfield of E containing F. Then K is an antiderivative extension of F.*

Proof. Let $W := \operatorname{span}_C\{\zeta_1, \ldots, \zeta_t\}$ denote the vector space generated by the elements $\zeta_1, \ldots, \zeta_t$ over the field of constants C of F. Let $V := K \cap W$ and note that V is a subspace of W. Let $S_1 \subset W$ be a C-basis for V. We claim that $K = F(S_1)$. Choose a set $S_2 \subset W$ so that $S_1 \cup S_2$ is a C-basis for W. Clearly, $S_1 \cup S_2$ is a finite set consisting of antiderivatives of F, the field $F(S_1)$ is a differential field and $K \supseteq F(S_1) \supset V$. Also note that $F(S_1 \cup S_2) = F(W) = E$. If elements of S_2 are algebraically dependent over K then by Theorem 3.1, K contains a non-zero C-linear combination of elements of S_2. But then, such a linear combination should be in V, a contradiction to the fact that $S_1 \cup S_2$ is linearly independent over C. Thus S_2 is algebraically independent over K. Therefore, tr.d.$(E|K) =$ tr.d.$(E|F(S_1))$ and since $K \supseteq F(S_1)$, we see that K is algebraic over $F(S_1)$. Now by Theorem 2.1, we obtain $K = F(S_1)$. Hence our claim. Now since $S_1 \subset W$, we see that S_1 consists of antiderivatives of F and thus K is an antiderivative extension of F. □

3.1. Differential automorphisms of antiderivative extensions

Let $E = F(\zeta_1, \ldots, \zeta_t)$ be an antiderivative extension of F. By definition, E is a no new constants extension of F. In light of Theorem 3.2, we may assume $\zeta_1, \zeta_2, \ldots, \zeta_t$ are algebraically independent over F. Let $R := F[\zeta_1, \ldots, \zeta_t] \subset E$ and note that R is a differential ring. Let $\sigma \in G := G(E|F)$. Then since $\zeta_i' \in F$, we have $\sigma(\zeta_i)' = \sigma(\zeta_i') = \zeta_i'$. That is, $(\sigma(\zeta_i) - \zeta_i)' = 0$. Since E is a no new constants extension of F, there is an element $\alpha_{i\sigma} \in C$ such that $\sigma(\zeta_i) - \zeta_i = \alpha_{i\sigma}$ and therefore, $\sigma(\zeta_i) = \zeta_i + \alpha_{i\sigma}$. Also note that $\sigma(\phi(\zeta_i)) = \zeta_i + \alpha_{i\sigma} + \alpha_{i\phi} = \zeta_i + \alpha_{i\phi} + \alpha_{i\sigma} = \phi(\sigma(\zeta_i))$. Since any automorphism of E fixing F is completely determined by its action on $\zeta_1, \zeta_2, \ldots, \zeta_t$, we see that the group G is commutative and that there is an injective group homomorphism from G to $(C^t, +)$ given by $\sigma \hookrightarrow (\alpha_{1\sigma}, \ldots, \alpha_{t\sigma})$. To prove surjectivity, let $\vec{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_n) \in C^t$. Define a ring ($F$-algebra) homomorphism $\sigma_{\vec{\alpha}} : R \to R$ by setting $\sigma_{\vec{\alpha}}(\zeta_i) = \zeta_i + \alpha_i$ and $\sigma_{\vec{\alpha}}(f) = f$ for all $f \in F$. The ring homomorphism obtained by mapping $\zeta_i \mapsto \zeta_i - \alpha_i$ and fixing elements of F is the inverse of $\sigma_{\vec{\alpha}}$ and therefore $\sigma_{\vec{\alpha}}$ is a ring automorphism. Since $\sigma_{\vec{\alpha}}(\zeta_i)' = \sigma_{\vec{\alpha}}(\zeta_i')$, we see that $\sigma_{\vec{\alpha}}$ is a differential ring automorphism. Now we extend $\sigma_{\vec{\alpha}}$ to the field of fractions E of R to obtain a differential field automorphism. Thus G is isomorphic to the commutative group $(C^t, +)$. We refer the reader to [3] and [4] for a thorough treatment of differential fields and Picard–Vessiot theory.

Proposition 3.4. *Let $E = F(\zeta_1, \zeta_2, \ldots, \zeta_t)$ be an antiderivative extension of F. Then the fixed field $E^{G(E|F)} := \{y \in E \mid \sigma(y) = y \text{ for all } \sigma \in G(E|F)\}$ equals F.*

Proof. Let $u \in E - F$ and consider $F\langle u\rangle$, the differential field generated by F and u. Then by Theorem 3.3, $F\langle u\rangle$ contains an element of the form $\sum_{i=1}^{t} \alpha_i \zeta_i$, where at least one of the α_i is non-zero, say $\alpha_1 \neq 0$. Let $\vec{e}_1 := (1, 0, \ldots, 0) \in C^t$. The differential automorphism $\sigma_{\vec{e}_1}$ induced by $\vec{e}_1$ fixes all ζ_i when $i \geqslant 2$ and maps ζ_1 to $\zeta_1 + 1$. Therefore $\sigma_{\vec{e}_1}(\sum_{i=1}^{t} \alpha_i \zeta_i) \neq \sum_{i=1}^{t} \alpha_i \zeta_i$. And since $\sum_{i=1}^{t} \alpha_i \zeta_i \in F\langle u\rangle$, we obtain $\sigma_{\vec{e}_1}(u) \neq u$. Thus $E^{G(E|F)} = F$. □

4. Preparation for a structure theorem

Hereafter, we will assume that the field of constants C of F is an algebraically closed field.

4.1. Normal tower

Let N be a no new constants extension of F. We say that K is the *antiderivative closure of F in N* if K is generated over F by all antiderivatives of F that are in N. Let $E = F(\zeta_1, \zeta_2, \ldots, \zeta_t)$ be an iterated antiderivative extension of F and for every integer $i \geqslant 1$, let E_i denote the antiderivative closure of E_{i-1} in E, where $E_0 := F$. Since $\zeta_i \in E_i$, we see that $E_t = E$. Choose the smallest integer m such that $E_m = E_{m+1}$. Clearly such an m exists, $E_i \supset E_{i-1}$ for all $1 \leqslant i \leqslant m$ and $E = E_m$. We will call the tower

$$E = E_m \supset E_{m-1} \supset \cdots \supset E_1 \supset E_0 = F \tag{4.1}$$

the normal tower of E.

We will now show that the normal tower of E is kept invariant under the action of $G := G(E|F)$. We use the notation GK to denote the differential field $\{\sigma(y) \mid \sigma \in G \text{ and } y \in K\}$. Since G fixes K and $K \supseteq F$, G fixes $E_0 := F$ and thus $GE_0 \subseteq E_0$. Assume that $GE_{i-1} \subseteq E_{i-1}$ for some i and let $\eta \in E_i$ be an antiderivative of E_{i-1}. Observe that $\sigma(\eta)' = \sigma(\eta')$ and since $\eta' \in E_{i-1}$, by our assumption, $\sigma(\eta') \in E_{i-1}$. Thus, for each $\sigma \in G$, $\sigma(\eta)$ is an antiderivative of E_{i-1} and therefore $\sigma(\eta) \in E_i$. Since E_i is generated as a field by antiderivatives of E_{i-1}, $GE_i \subseteq E_i$. Hence by induction, $GE_i \subseteq E_i$ for all i.

Let N be a no new constants extension of F. Let $\eta_1, \eta_2, \ldots, \eta_n \in N$ be iterated antiderivatives (respectively, antiderivatives) of F and let $H \subseteq G(N|F)$ be a set consisting of commuting differential automorphisms. We say the $\eta_1, \eta_2, \ldots, \eta_n \in N$ are H-invariant iterated antiderivatives (respectively, H-invariant antiderivatives) of F if $\eta_1, \eta_2, \ldots, \eta_n$ are algebraically independent iterated antiderivatives (respectively, antiderivatives) of F and for each i, $HF_i \subseteq F_i$, where $F_i := F(\eta_1, \eta_2, \ldots, \eta_{i-1})$ and $F_0 := F$.

Example 4.1. Consider the fields $L := \mathbb{C}(z, \log z, \log(\log z))$ and $\mathfrak{L} := \mathbb{C}(z, S, \mathfrak{S})$, where $S := \{\log(z + \alpha) \mid \alpha \in \mathbb{C}\}$ and $\mathfrak{S} := \{\log(\beta + \log(z + \alpha)) \mid \alpha, \beta \in C\}$. It can be shown that $\mathfrak{L}$ is a no new constants extension of $\mathbb{C}$ with respect to the usual derivation d/dz and that the set $\{z\} \cup S \cup \mathfrak{S}$ consists of elements algebraically independent over $\mathbb{C}$, see [7].

For convenience, we will use $'$ to denote d/dz. Let $K \neq \mathbb{C}$ be a differential subfield of L. If $\text{tr.d.}(K|\mathbb{C}) = 3$ then since $\text{tr.d.}(L|\mathbb{C}) = 3$, by Theorem 2.1 we have $K = L$. Assume $\text{tr.d.}(K|\mathbb{C}) = 2$. We claim that $K = \mathbb{C}(z, \log z)$. It is enough to show that $z, \log z \in K$. Suppose that $z \notin K$. Then $\text{tr.d.}(K(z)|\mathbb{C}) = 3$ and thus $K(z) = L$. Now let $\sigma_1 \in G(K(z)|K)$ be a differential automorphism that sends z to $z + 1$. Since $\log z \in K(z)$ and $(\log z)' = \frac{1}{z}$, we see that $(\sigma_1^n(\log z))' = \frac{1}{z+n}$, for any integer $n \geqslant 1$. Since $\mathfrak{L}$ is a no new constants extension of $\mathbb{C}$ and $(\log(z + n))' = \frac{1}{z+n}$, we obtain that $\log(z + n) = \sigma_1^n(\log z) + c_n \in L$ for some constants $c_n \in \mathbb{C}$. Since the set S is algebraically independent over $\mathbb{C}$, we obtain a contradiction to the fact that L has a finite transcendence degree over $\mathbb{C}$. Thus $z \in K$.

Note that if $\log z \notin K$ then $K(\log z) = L$ and there is a $\sigma_1 \in G(K(\log z)|K)$ that sends $\log z$ to $1 + \log z$. Then $\log(n + \log z) = \sigma_1^n(\log(\log z)) + c_n \in L$ for some $c_n \in \mathbb{C}$, which again contradicts the fact that L has a finite transcendence degree over $\mathbb{C}$. Hence the claim follows. Similarly, one proves that if $\text{tr.d.}\, K|\mathbb{C} = 1$ then $K = \mathbb{C}(z)$. Thus we have shown that the differential subfields of L that contains $\mathbb{C}$ are L, $\mathbb{C}(z, \log z)$, $\mathbb{C}(z)$ and $\mathbb{C}$. Indeed, the normal tower of L is

$$L \supset \mathbb{C}(z, \log z) \supset \mathbb{C}(z) \supset \mathbb{C}.$$

Remark. From the above discussion, we note that L cannot be a subfield of (or not imbeddable in) any Picard–Vessiot extension of $\mathbb{C}(z)$. Otherwise, there is an automorphism $\sigma \in G(L|\mathbb{C}(z))$ such that $\sigma(\log z) = c + \log z$ for some $c \in C - \{0\}$. Then $\log(nc + \log z) = \sigma^n(\log(\log z)) + c_n \in L$ for some $c_n \in C$ and for all non-negative integers n, which contradicts the fact that L is of finite transcendence degree over $\mathbb{C}(z)$. One can list all the finitely differentially generated subfields of $\mathfrak{L}$, see [7]. The rest of this section discusses the action of differential automorphisms on iterated antiderivatives.

Lemma 4.1. *Let F be a differential field with an algebraically closed field of constants C and let N be a no new constants extension of F. Let E and L be differential fields such that $N \supseteq E \supset L \supseteq F$ and let H be a commutative subset of $G(N|F)$ such that $HE \subseteq E$ and $HL \subseteq L$. If E is an antiderivative extension of L then there are H-invariant antiderivatives $\eta_1, \eta_2, \ldots, \eta_t$ of L such that $E = L(\eta_1, \eta_2, \ldots, \eta_t)$. Moreover, for each i and for each $\sigma \in H$,*

$$\sigma(\eta_i) = \delta_{i\sigma}\eta_i + \sum_{j=1}^{i-1} \gamma_{ij\sigma}\eta_j + a_{i\sigma},$$

for some $\delta_{i\sigma}, \gamma_{ij\sigma} \in C$ and $a_{i\sigma} \in L$. In particular, $\sigma(\eta_i) - \delta_{i\sigma}\eta_i \in L_{i-1}$.

Proof. Suppose that $E = L(\zeta_1, \zeta_2, \ldots, \zeta_t)$ is an antiderivative extension of L. Since H keeps L and E invariant, for each $\sigma \in G$, $\sigma(\zeta_i) \in E$ is an antiderivative of L. For each i, we apply Proposition 3.2 and obtain constants $\alpha_{ij\sigma} \in C$, not all zero, such that

$$\sigma(\zeta_i) - \sum_{j=1}^{t} \alpha_{ij\sigma}\zeta_i \in L. \tag{4.2}$$

We view the quotient space E/L as a C-vector space (infinite dimensional) and denote its element by $\bar{y}$, where $y \in E$. There is a natural action of H on E/L, namely, $\sigma \cdot \bar{y} = \overline{\sigma(y)}$. This action is well defined since H keeps L and E invariant. From Eq. (4.2) we see that

$$\sigma \cdot \bar{\zeta}_i = \sum_{j=1}^{t} \alpha_{ij\sigma} \bar{\zeta}_i \tag{4.3}$$

for every $\sigma \in H$. Thus, the finite dimensional subspace $W := \mathrm{span}_C\{\bar{\zeta}_1, \dots, \bar{\zeta}_t\}$ of E/L is kept invariant under the action of H. The above equation induces a group homomorphism $\Phi : H \to End(W)$ and since H is commutative, $\Phi(H)$ is commutative as well. It is a well-known fact that any commuting set of endomorphisms of a vector space over an algebraically closed field[1] can be triangularized (see [2, p. 100]). That is, there is a basis $\{\bar{\eta}_1, \bar{\eta}_2, \dots, \bar{\eta}_t\}$ of W and there are constants $\gamma_{ij\sigma} \in C$ such that

$$\sigma \cdot \bar{\eta}_i = \overline{\sigma(\eta_i)} = \sum_{j=1}^{i} \gamma_{ij\sigma} \bar{\eta}_j. \tag{4.4}$$

For each i, we have $\bar{\eta}_i = \sum_{j=1}^{m} \beta_{ij} \bar{\zeta}_j$ and therefore there are elements $r_i \in L$ such that $\eta_i = \sum_{j=1}^{m} \beta_{ij} \zeta_j + r_i$. Thus, from Proposition 3.2, each η_i is an antiderivative of L. The linear independence of $\{\bar{\eta}_i \mid 1 \leqslant i \leqslant t\}$ over C and Theorem 3.1 together will guarantee the algebraic independence of $\{\eta_i \mid 1 \leqslant i \leqslant t\}$ over L. Since $L(\eta_1, \dots, \eta_t) \subseteq E$ and $\mathrm{tr.d.}(E|L) = \mathrm{tr.d.}(L(\eta_1, \dots, \eta_t)|L)$, we may apply Theorem 2.1 and obtain $E = K$. For each i, we set $L_i := L(\eta_1, \dots, \eta_i)$ and observe from Eq. (4.4) that $HL_i \subseteq L_i$. From Eq. (4.4), we see that $\sigma\eta_i - \gamma_{ii\sigma}\eta_i - \sum_{j=1}^{i-1} \gamma_{ij\sigma}\eta_j = a_{i\sigma}$ for some $a_{i\sigma} \in L$. Thus $\sigma\eta_i = \delta_{i\sigma}\eta_i + \sum_{j=1}^{i-1} \gamma_{ij\sigma}\eta_j + a_{i\sigma}$, where $\delta_{i\sigma} := \gamma_{ii\sigma}$. Clearly, $\sigma\eta_i - \delta_{i\sigma}\eta_i \in L_{i-1}$. □

Corollary 4.1.1. *Let F be a differential field with an algebraically closed field of constants C. Let E be an iterated antiderivative extension of F and let H be a commutative subset of $G(E|F)$. Then there are H-invariant iterated antiderivatives $\eta_1, \eta_2, \dots, \eta_t$ of F such that $E = F(\eta_1, \eta_2, \dots, \eta_t)$. Moreover, for each i and each $\sigma \in G$,*

$$\sigma(\eta_i) = \delta_{i\sigma}\eta_i + r_{i\sigma},$$

for some $\delta_{i\sigma} \in C$ and $r_{i\sigma} \in L_{i-1}$.

Proof. Let $E = E_m \supset E_{m-1} \supset \cdots \supset E_1 \supset E_0 = F$ be the normal tower of F. Note that E_j is an antiderivative extension of E_{j-1} and from Section 4.1 we know that $HE_j \subseteq E_j$ for each j. Thus applying Lemma 4.1 with $M := E_j$ and $L := E_{j-1}$, we obtain elements η_{ji} and H-invariant differential fields L_{ji} for $i = 1, 2, \dots, t_j$. Now we rename $\eta_{11}, \dots, \eta_{1t_1}, \dots, \eta_{m1}, \dots, \eta_{mt_m}$ as $\eta_1, \dots, \eta_t$ and $L_{11}, \dots, L_{1t_1}, \dots, L_{m1}, \dots, L_{mt_m}$ as $L_1, \dots, L_t$, where $t := \sum_{i=1}^{m} t_i$. One can easily check that L_i and η_i satisfy the desired properties. □

We need the following technical (rather computational) lemma to prove Theorem 5.3.

Lemma 4.2. *Let F be a differential field with an algebraically closed field of constants C and let E be an iterated antiderivative extension of F. Suppose that $K \supseteq F$ is a differential subfield of E such that E is an antiderivative extension of K and let $G := G(E|K)$. Then, there are G-invariant iterated antiderivatives $\eta_1, \eta_2, \dots, \eta_t$ of F such that $E = F(\eta_1, \dots, \eta_t)$. Let $L^* := F(\eta_1, \dots, \eta_{t-1})$. Then, either $K \subseteq L^*$ or there is an element $a \in L^*$ such that $\eta_t + a \in K$. Moreover, $\eta_t + a \notin F\langle \eta_t' + a' \rangle$ and thus $F\langle \eta_t' + a' \rangle$ is a proper differential subfield of K.*

Proof. Since G is a commutative group, from Corollary 4.1.1, it follows that there are G-invariant iterated antiderivatives $\eta_1, \eta_2, \dots, \eta_t$ of F such that $E = F(\eta_1, \dots, \eta_t)$. Assume that $K \nsubseteq L^* := F(\eta_1, \dots, \eta_{t-1})$ and let $u \in K \cap (E - L^*)$. Since $E = L^*(\eta_t)$, we may write $u = P/Q$, where $P, Q \in L^*[\eta_t]$, P, Q relatively prime, and Q is monic. From Corollary 4.1.1, we have

[1] Here we use the assumption that the field of constants C of F is algebraically closed.

$$\sigma(\eta_t) = \delta_\sigma \eta_t + r_\sigma \tag{4.5}$$

for every $\sigma \in G$, where $\delta_\sigma \in C$ and $r_\sigma \in L^*$. Thus G consists of differential automorphisms of the ring $L^*[\eta_t]$. Since $u \in K$, we have $\sigma(u) = u$ for all $\sigma \in G$. Thus $\sigma(P)Q = \sigma(Q)P$. Since P and Q are relatively prime, P divides $\sigma(P)$ and Q divides $\sigma(Q)$. But from Eq. (4.5), we see that $\deg \sigma(P) = \deg P$ and $\deg \sigma(Q) = \deg Q$ and thus $\sigma(P) = f_\sigma P$ and $\sigma(Q) = g_\sigma Q$ for some $f_\sigma, g_\sigma \in L^*$. Since $\sigma(P/Q) = P/Q$, we must have $f_\sigma = g_\sigma$. Now writing $Q = \sum_{i=0}^{l} b_i \eta_t^i$ with $b_i \in L^*$ (note that $b_l = 1$), we observe that

$$\sum_{i=0}^{l} \sigma(b_i)(\delta_\sigma \eta_t + r_\sigma)^i = f_\sigma \left(\sum_{i=0}^{l} b_i \eta_t^i \right).$$

Thus comparing the coefficients of η_t^l, we obtain $\delta_\sigma^l = f_\sigma$. Hence, for all $\sigma \in G$, $\sigma(P) = \delta_\sigma^l P$ and $\sigma(Q) = \delta_\sigma^l Q$, where $\delta_\sigma^l \in C$. Then $P'/P, Q'/Q \in E^G$—the fixed field of the group G. From Proposition 3.4, we know that $E^G = K$ and thus $P'/P, Q'/Q \in K$, where $P, Q \in E$. Now from Theorem 2.2 we obtain that $P, Q \in K$. Hence G fixes both P and Q.

Since $u \notin L^*$, we have P or Q does not belong to L^*. Without loss of generality, assume $P \notin L^*$. Then there are an $n \geqslant 1$ and $a_i \in L^*$ such that $P = \sum_{i=0}^{n} a_i \eta_t^i$. Now, for any $\sigma \in G$, we have $\sigma(P) = P$ and therefore

$$\sigma(a_n)(\delta_\sigma \eta_t + r_\sigma)^n + \sigma(a_{n-1})(\delta_\sigma \eta_t + r_\sigma)^{n-1} + \cdots + \sigma(a_0) = a_n \eta_t^n + a_{n-1}\eta_t^{n-1} + \cdots + a_0.$$

Comparing the coefficients of η_t^n, and respectively of η_t^{n-1}, we obtain

$$\sigma(a_n) = \delta_\sigma^{-n} a_n \quad \text{and} \tag{4.6}$$

$$n\delta_\sigma^{n-1}\sigma(a_n) r_\sigma + \delta_\sigma^{n-1}\sigma(a_{n-1}) = a_{n-1}, \tag{4.7}$$

for every $\sigma \in G$. Since $\delta_\sigma \in C$, from Eq. (4.6), we have $a_n'/a_n \in E^G = K$ and therefore applying Theorem 2.2, we obtain $a_n \in K$. In particular $\delta_\sigma^n = 1$. Now from Eq. (4.7), we obtain

$$\sigma(a_{n-1}) = \delta_\sigma(a_{n-1}) - n a_n r_\sigma \quad \text{and thus}$$

$$\sigma(a_{n-1}/n a_n) = \delta_\sigma(a_{n-1}/n a_n) - r_\sigma. \tag{4.8}$$

We add Eqs. (4.8) and (4.5) to get

$$\sigma\left(\eta_t + \frac{a_{n-1}}{n a_n}\right) = \delta_\sigma\left(\eta_t + \frac{a_{n-1}}{n a_n}\right) \quad \text{for all } \sigma \in G. \tag{4.9}$$

Let $a := a_{n-1}/n a_n$ and observe that $(\eta_t + a)'/(\eta_t + a) \in E^G = K$. Again by Theorem 2.2 we should then have $\eta_t + a \in K$. Note that $\eta_t' + a' \in L^*$ and thus $F\langle \eta_t' + a' \rangle \subseteq L^*$. And since $\eta_t \notin L^*$ and $a \in L^*$ we know that $\eta_t + a \notin F\langle \eta_t' + a' \rangle$. Thus $\eta_t + a \in K - F\langle \eta_t' + a' \rangle$ is an antiderivative of $F\langle \eta_t' + a' \rangle$. Thus $\eta_t + a$ is transcendental over $F\langle \eta_t' + a' \rangle$ and therefore $\text{tr.d.}(K|F\langle \eta_t' + a' \rangle) \geqslant 1$. Hence $F\langle \eta_t' + a' \rangle$ is a proper differential subfield of K. □

5. Structure theorem

We recall that M is a *minimal* differential field extension of F if $M \supseteq F$ is a differential field extension such that if K is a differential subfield of M and $K \supseteq F$ then $M = K$ or $M = F$.

Proposition 5.1. *Let E be an iterated antiderivative extension of F. Suppose that for any containments of differential fields $F \subseteq F^* \subset M^* \subseteq E$ such that M^* is a minimal differential field extension of F^*, there is an antiderivative $\eta \in E$ of F^* such that $M^* = F^*(\eta)$. Then, if K is a differential subfield of E such that $K \supseteq F$ then K is an iterated antiderivative extension of F.*

Proof. Let K be a differential subfield of E such that $E \supseteq K \supset F$. Let F^*, $K \supseteq F^* \supseteq F$ be a maximal iterated antiderivative extension of F contained in K. If $F^* \neq K$, then by Corollary 2.1.2, there is a minimal differential field extension M^* of F^* in K. By the hypothesis of the proposition, we have $M^* = F^*(\eta)$, where $\eta' \in F^*$. This contradicts the maximality of F^*. □

We note that to prove Theorem 5.3, it is necessary and sufficient to prove that the supposition statement of Proposition 5.1 is always true for any iterated antiderivative extension of F.

Theorem 5.2. *Let F be a differential field with an algebraically closed field of constants C and let E be an iterated antiderivative extension of F. Let K be a minimal differential field extension of F such that $E \supseteq K \supset F$. Then $K = F(\zeta)$ for some antiderivative $\zeta \in E$ of F.*

Proof. We will use an induction on $n := \text{tr.d.}\, E|F$ to prove this theorem. From Theorem 2.1, we know that $\text{tr.d.}(K|F) \geqslant 1$. In particular, $n \geqslant 1$.

Case $n = 1$: we have $\text{tr.d.}(E|F) = \text{tr.d.}(K|F) = 1$ and $E \supseteq K$. Applying Corollary 2.1.1, we obtain that $E = K$.

Let $n \geqslant 2$ and assume that the theorem holds for iterated antiderivative extensions of transcendence degree $\leqslant n-1$. Let $E = E_m \supset E_{m-1} \supset \cdots \supset E_1 \supset E_0 = F$ be the normal tower of E. Since $E \neq F$, from Corollary 2.1.1, we have $\text{tr.d.}(E_1|F) > 0$ and thus E is an iterated antiderivative extension of E_1 with $\text{tr.d.}(E|E_1) \leqslant n-1$. Note that if $E \supseteq F^* \supseteq E_1$ then $\text{tr.d.}(E|F^*) \leqslant \text{tr.d.}(E|E_1) = n-1$. Then by induction, if M^* and F^* are differential fields such that $E \supseteq M^* \supset F^* \supseteq E_1$ and that M^* is a minimal differential field extension of F^* then $M^* = F^*(\eta)$ for some antiderivative $\eta \in E$ of F^*. Therefore, by Proposition 5.1, we obtain that every differential subfield of E that contains E_1 is an iterated antiderivative extension of E_1. Since $E \supseteq KE_1 \supseteq E_1$, we obtain KE_1 is an iterated antiderivative extension of E_1. And since E_1 is an antiderivative extension of F, we obtain that KE_1 is an iterated antiderivative extension of F as well. If $\text{tr.d.}(KE_1|F) < \text{tr.d.}(E|F) = n$ then by induction, we have proved that K is of the required form. Therefore we may assume $\text{tr.d.}(KE_1|F) = \text{tr.d.}(E|F)$, that is, $KE_1 = E$. Then since E_1 is an antiderivative extension of F and $K \supset F$, we obtain that E is an antiderivative extension of K as well and thus $G(E|K)$ is a commutative group.

Now we apply Lemma 4.2 and obtain $G(E|K)$-invariant iterated antiderivatives $\eta_1, \eta_2, \ldots, \eta_t$ of F such that $E = F(\eta_1, \ldots, \eta_t)$. If $K \subseteq L^* := F(\eta_1, \ldots, \eta_{t-1})$ then since $\text{tr.d.}(L^*|F) = \text{tr.d.}(E|F) - 1$ and L^* is an iterated antiderivative extension of F, by induction, we are done. Otherwise, by Lemma 4.2, there is an element $a \in L^*$ such that $\eta_t + a \in K$, $\eta_t + a \notin F\langle \eta_t' + a' \rangle$ and that $F\langle \eta_t' + a' \rangle$ is a proper differential subfield of K. Then, since K is minimal extension of F, $F\langle \eta_t' + a' \rangle = F$. Thus we have $(\eta_t + a)' = \eta_t' + a' \in F$ and $\eta_t + a \notin F$. Then $F(\eta_t + a)$ is a differential field and $K \supseteq F(\eta_t + a) \supset F$. Again, since K is a minimal extension of F, we should have $K = F(\eta_t + a)$ and by setting $\zeta := \eta_t + a$, we complete the proof. □

Theorem 5.3. *Let F be a differential field with an algebraically closed field of constants C and let E be an iterated antiderivative extension of F. Let $K \supseteq F$ be a differential subfield of E. Then K is an iterated antiderivative extension of F.*

Proof. Follows from Theorem 5.2 and Proposition 5.1. □

6. Concluding remarks

In this section we will see an application of Theorem 5.3. Throughout this section let C be an algebraically closed field of characteristic zero and we view C as a differential field with the trivial derivation. Consider the field of rational functions $C(z)$ and set $z' := 1$. Then it is easy to check that $C(z)$ is a no new constants extension of C. Let $C(z)(z_1, z_2, \dots, z_t)$ be any iterated antiderivative extension of $C(z)$. We may also assume that $z_1, z_2, \dots, z_t$, are algebraically independent over $C(z)$. For any $u \in C(z, z_1, z_2, \dots, z_t) - C$, Theorem 5.3 tells us the differential field $C\langle u\rangle = C(u, u', u'', \dots)$ contains an antiderivative $\eta \in C\langle u\rangle - C$ of C. Then, $\eta' = \alpha$ for some $\alpha \in C - \{0\}$ and we see that $\eta' = (\alpha z)'$. Therefore, there is a $\beta \in C$ such that $\eta = \alpha z + \beta$, where $\alpha \in C - \{0\}$. Thus $z \in C\langle u\rangle$. Therefore, for each $u \in C(z, z_1, z_2, \dots, z_t) - C$, there are an integer $n \geqslant 0$ and relatively prime polynomials $P, Q \in C[x_1, \dots, x_{n+1}]$ such that

$$z = \frac{P(u, u^{(1)}, \dots, u^{(n)})}{Q(u, u^{(1)}, \dots, u^{(n)})}, \tag{6.1}$$

where $u^{(i)}$ denotes the i-th derivative of u.

Example 6.1. Consider the differential field $\mathbb{C}(z, \log z)$ with the usual derivation d/dz. Then, for even a simple expression like $u := \frac{\log z}{z}$, it can be tedious to write z in terms of u and its derivatives as in Eq. (6.1). In fact $z = \frac{u'' + uu'}{uu'' - 3(u')^2}$. Since $z_1 = uz$, we see that $z_1 = \frac{uu'' + u^2u'}{uu'' - 3(u')^2}$ and thus $\mathbb{C}\langle u\rangle = \mathbb{C}(z, \log z)$.

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