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Implicitization of rational surfaces using toric varieties ☆

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Abstract

A parameterized surface can be represented as a projection from a certain toric surface. This generalizes the classical homogeneous and bihomogeneous parameterizations. We extend to the toric case two methods for computing the implicit equation of such a rational parameterized surface. The first approach uses resultant matrices and gives an exact determinantal formula for the implicit equation if the parameterization has no base points. In the case the base points are isolated local complete intersections, we show that the implicit equation can still be recovered by computing any non-zero maximal minor of this matrix.

The second method is the toric extension of the method of moving surfaces, and involves finding linear and quadratic relations (syzygies) among the input polynomials. When there are no base points, we show that these can be put together into a square matrix whose determinant is the implicit equation. Its extension to the case where there are base points is also explored.

Keywords: Implicitization; Toric varieties; Resultants; Syzygies

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1. Introduction

A rationally parameterized surface $\Phi(s,t)$ in affine three space is defined by a map $\phi:\mathbb{C}^2\to\mathbb{C}^3$ given by three rational components:

$$\begin{cases} X_1=\frac{x_1(s,t)}{x_4(s,t)},\\ X_2=\frac{x_2(s,t)}{x_4(s,t)},\\ X_3=\frac{x_3(s,t)}{x_4(s,t)}.\end{cases} \tag{1}$$

Here x_1,x_2,x_3,x_4 are (Laurent) polynomials in two variables s and t with coefficients in $\mathbb{C}$ (or $\mathbb{R}$ or $\mathbb{Q}$ or any arbitrary subfield $\mathbb{K}$ of $\mathbb{C}$). Let $\Phi\subset\mathbb{C}^3$ be the smallest algebraic surface containing (1). The *implicitization problem* [10,12] is to compute the polynomial equation $P(X_1,X_2,X_3)$ defining Φ. At times we will also consider the same surface in projective space, where there are four coordinates with equations given by X_1, X_2, X_3, and X_4.

The last few decades have witnessed a rise of interest in the implicitization problem for geometric objects motivated by applications in computer aided geometric design and geometric modelling [1,3,5,7,8,10,11,16,19,25,27]. A very common approach is to write the implicit equation as the determinant of a matrix whose entries are easy to compute.

Our approach is also to look for matrix formulas, but we recast the parameterization in terms of a projection from a certain toric surface built out of the specific monomials which appear in x_1,x_2,x_3,x_4. This generalizes the standard approaches of projections from tensor product surfaces (Segre embeddings of $\mathbb{P}^1\times\mathbb{P}^1$) or from total degree surfaces (Veronese embeddings of $\mathbb{P}^2$). So while previously x_1,x_2,x_3,x_4 have been considered only as "generic" homogeneous or bihomogeneous polynomials, we can exploit sparsity present in the parameterization.

Standard homogenization of sparse polynomials can result in numerous spurious base points of the projection at infinity. By using the more general toric surface, customized for the equations on hand, many of these extraneous base points at infinity can be avoided. This results in smaller matrices and fewer extraneous factors in the computation of the implicit equation. Toric projections can also be exploited in the construction of the parameterization. The work of Krasauskas [22] shows how "toric surface patches" can be used to parametrize regions on a surface shaped like arbitrary sided polygons.

In this article we extend to the toric case two methods for computing the implicit equation: computing a Chow form and computing syzygies on the input polynomials x_1,x_2,x_3,x_4.

A classical method for finding the implicit equation is to compute the bivariate resultant or Chow form of the three polynomials:

$$\begin{aligned} f_1&=x_1(s,t)-X_1x_4(s,t),\\ f_2&=x_2(s,t)-X_2x_4(s,t),\\ f_3&=x_3(s,t)-X_3x_4(s,t). \end{aligned}\tag{2}$$

Our first approach essentially follows the classical method using the sparse resultant in place of the classical bivariate resultant. Formally, we will reduce the computation of F to the computation of the Chow form of a toric surface which projects onto Φ. Exact matrix formulas for computing this Chow form were found by the first author in [20].

We show that if the projection has no *base points*, points on the toric variety such that f_1, f_2, f_3 are simultaneously zero, the matrix constructed gives an exact determinantal formula for the implicit equation. New to our approach is an analysis when base points are present. We show that if the base points are isolated local complete intersections, the implicit equation can still be recovered by computing a non-zero maximal minor of this matrix.

The second method involves finding linear and quadratic relations (syzygies) among the polynomials x_1, x_2, x_3, x_4 of a certain fixed type. When there are no base points, we will see how these can be put together into a square matrix whose determinant is exactly the implicit equation. This is precisely the technique used in the method of moving surfaces for tensor product or total degree surfaces [5,10,11,27,28]. Our contribution is to exploit the structure of the sparsity of the polynomials to avoid extra base points. Moreover, we present a novel proof of the validity of the method of moving surfaces which ties together the complexes of moving planes and quadrics with the resultant complex in a natural way.

The method of moving surfaces can also be applied in the presence of basepoints and often still produces the correct implicit equation. Validity in the presence of basepoints was proved under certain conditions in the total degree [5] and tensor product [2] situations. We do not have a proof in the general toric setting but we illustrate the situation with a few examples.

The paper is organized as follows: in Section 2, we recall some properties of toric surfaces and introduce some notation. In Section 3, we present the first of our methods and show that it works if the base points are a local complete intersection. In Section 4, we present the method of moving quadrics. In Section 5, we prove its validity in the absence of base points. In this section we also provide some examples and moreover explore what happens in the presence of base points.

2. Toric varieties, parameterizations, and base points

Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{Z}^2$, a finite subset of points, and Q the convex hull of the points in $\mathcal{A}$. The *toric variety* $X_{\mathcal{A}}$ associated with $\mathcal{A}$ is defined as the Zariski closure of the set of points $(x^{\alpha_1} : \cdots : x^{\alpha_N})$ in $\mathbb{P}^{N-1}$ where x ranges over $(\mathbb{C}^*)^2$ (the "algebraic" torus). See [13,18] for details.

If each of the polynomials x_i has its support contained in $\mathcal{A}$, then it is a linear combination of monomials in $\mathcal{A}$, hence can be thought of as a linear functional on $\mathbb{P}^{N-1}$ defining a hyperplane section of $X_{\mathcal{A}}$. Let $\mathcal{A}_i$ be the support of x_i. Define $\mathcal{A}$ as the union of the supports; that is

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4.$$

Therefore, we can consider the set of zeros Z of x_1, x_2, x_3, x_4 in $X_{\mathcal{A}}$. Note that Z will contain those common zeros of the x_i in $(\mathbb{C}^*)^2$. Now the map ϕ can be realized as (the affine part of) a projection from $X_{\mathcal{A}}$ to $\mathbb{P}^3$ via the hyperplane sections x_1, x_2, x_3, x_4.

The points in Z correspond to basepoints of this projection. We will assume that $\gcd(x_1, x_2, x_3, x_4) = 1$, so that Z is finite. For each $p \in Z$, we get a certain multiplicity $e(\mathcal{I}_{Z,p}, \mathcal{O}_{X_{\mathcal{A}},p})$. The degree of the parameterization ϕ is the generic number of points in $X_{\mathcal{A}}$ which map to a point in Φ. The degree of Φ is the total degree of its implicit equation.

Now, as in [10, Appendix], we have the following *degree formula*:

Proposition 1.

$$\deg(\phi)\deg(\Phi) = \mathrm{Area}(Q) - \sum_{p \in Z} e(\mathcal{I}_{Z,p}, \mathcal{O}_{X_{\mathcal{A}},p})$$

where $\mathrm{Area}(Q)$ *is the normalized area of the polygon* Q *equal to twice its usual Euclidean area* (*in particular* $\mathrm{Area(Q)}$ *is always an integer*).

In the next section we will consider the case when there are no basepoints, that is $Z = \emptyset$. If x_1, x_2, x_3, x_4 are each generic with respect to their supports $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ then this will be the case provided a certain geometric condition on the supports $\mathcal{A}_i$ holds. This is expressed in the next result.

Proposition 2. *Fix subsets* $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k \subset \mathbb{Z}^2$ *with* $k \geqslant 3$ *and define* $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$. *Let* $x_1, \ldots, x_k$ *be generic polynomials with* x_i *supported on* $\mathcal{A}_i$. *Let* $Q_i = \mathrm{Conv}(\mathcal{A}_i)$ *and* $Q = \mathrm{Conv}(\mathcal{A})$ *be the associated polytopes. Assume that* $\dim(Q) = 2$. *The polynomials* x_i *viewed as sections of* $X_{\mathcal{A}}$ *have no common zeros if and only if every edge of* Q *intersects at least two of the polytopes* Q_i.

Proof. The torus orbits of $X_{\mathcal{A}}$ correspond to the faces of Q. The restriction of x_i to a particular orbit corresponds to intersecting Q_i with the corresponding face of Q and setting all terms of x_i not in the intersection to 0. A zero-dimensional face of Q corresponds to a vertex, which by construction must be a vertex of some Q_i. The corresponding x_i restricts to a single non-zero monomial with a generic (non-zero) coefficient, hence does not vanish at this point. On the orbit corresponding to the dense 2-dimensional torus, any 3 of the polynomials do not generically have a common zero. Finally, a one-dimensional orbit corresponds to an edge of Q. By hypothesis, intersecting with the Q_i yields at least two non-zero polynomials with generic coefficients which do not have a common zero on the one-dimensional space. Conversely, if an edge of Q intersects only one of the Q_i then it must be an edge of that Q_i, so that x_i restricts to a polynomial in one variable, while all other x_j restrict to zero. Thus every root of this restriction of x_i is a common zero of all of the x_j. $\square$

Of course if all of the x_i have the same support, the case most often of interest for implicitization, then the condition above is automatically satisfied. In general it corresponds to a mild geometric compatibility of the supports.

3. Implicitization from the Chow form

The *Chow form* of $X_{\mathcal{A}}$ is a polynomial $\mathrm{Ch}_{\mathcal{A}}$ in the coefficients of three linear sections f_1, f_2, f_3 which is zero whenever f_1, f_2, f_3 have a common root on $X_{\mathcal{A}}$. In the case where $Z = \emptyset$, we get the following result.

Theorem 3. *Let x_1, x_2, x_3, x_4 be Laurent polynomials with complex coefficients. Let Z be the set of common zeros on the toric variety $X_{\mathcal{A}}$ corresponding to the union of their supports. Let* $\mathrm{Ch}_{\mathcal{A}}$ *be the Chow form of the toric variety $X_{\mathcal{A}}$. Let f_1, f_2, f_3 be as in* (2) *and $P(X_1, X_2, X_3)$ the implicit equation of Φ. If $Z = \emptyset$, then there exists a non-zero constant $c \in \mathbb{C}$ such that*

$$\mathrm{Ch}_{\mathcal{A}}(f_1, f_2, f_3) = c P^{\deg(\phi)}. \tag{3}$$

Proof. Let $G(X_1, X_2, X_3)$ be the left-hand side of (3). For a generic point on the surface there is an associated common zero of f_1, f_2, f_3. Conversely if X_1, X_2, X_3 are such that f_1, f_2, f_3 have a common zero (s, t) then as $Z = \emptyset$, $x_4(s, t) \neq 0$ and thus (X_1, X_2, X_3) is a point on the surface. As P is irreducible, it follows that $G = cP^d$, with $c \neq 0$ and $d \in \mathbb{N}$. In order to verify that $d = \deg(\phi)$, by Proposition 1 it is enough to see that the degree of G is $\mathrm{Area}(Q)$. This follows easily by applying the Chow form to the dual Plücker coordinates of the polynomials (2) (see [18]) and by noting that the dual Plücker coordinates have degree one in X_1, X_2, X_3. The degree of $\mathrm{Ch}_{\mathcal{A}}$ in the Plücker coordinates is $\mathrm{Area}(Q)$. □

In [21] there is a construction for computing the Chow form of any toric surface. Given a toric surface $X_{\mathcal{A}}$ with $Q = \mathrm{conv}(\mathcal{A})$, and three sections f_1, f_2, f_3 with $f_i = \sum_{a \in \mathcal{A}} C_{ia} x^{\alpha}$. Then $\mathrm{Ch}_{\mathcal{A}}(f_1, f_2, f_3)$ is the determinant of a matrix of the following block form:

$$\begin{pmatrix} B & L \\ \tilde{L} & 0 \end{pmatrix}.$$

Here the entries of L and $\tilde{L}$ are linear forms, and the entries of B are cubic forms in the coefficients C_{ia}, as described below. The rows of $\tilde{L}$ are called *Sylvester* rows. The columns of L are called *Sylvester* columns. The rows and columns of B will be called *Bézout* rows and columns.

The columns of B and $\tilde{L}$ are indexed by the lattice points in Q, the rows of B and L are indexed by the interior lattice points in $2 \cdot Q$, the Minkowski sum of Q with itself. The matrix $\tilde{L}$ has three rows indexed by $\{f_1, f_2, f_3\}$, and the columns of the matrix L are indexed by pairs (f_i, a) where $i \in \{1, 2, 3\}$ and a runs over the interior lattice points of Q. Each entry of L and $\tilde{L}$ is either zero or is a coefficient of some f_i and is determined in the following straightforward manner. The entry of $\tilde{L}$ in row f_i and column a is the coefficient of x^a in f_i. The entry of L in row b and column (f_i, a) is the coefficient of x^{b-a} in f_i. The entries of the matrix B are linear forms in *bracket variables*. A bracket variable is defined as

$$[abc] = \det \begin{bmatrix} C_{1a} & C_{1b} & C_{1c} \\ C_{2a} & C_{2b} & C_{2c} \\ C_{3a} & C_{3b} & C_{3c} \end{bmatrix}.$$

There is an explicit, combinatorial construction of the matrix B given in [20]. By virtue of Theorem 3 above we get the immediate corollary.

Corollary 4. *If $Z = \emptyset$, then there is a determinantal formula $M_{\mathcal{A}}$ for computing $P^{\deg(\phi)}$.*

Example 5. Consider the surface parameterized by

$$x_1 = s^3 + t^2,$$
$$x_2 = s^2 + t^3,$$
$$x_3 = s^2 t + st^2,$$
$$x_4 = st.$$

The associated polygon Q is a quadrilateral in the first quadrant with vertices $(2,0)$, $(3,0)$, $(0,2)$, $(0,3)$. So we compute the Chow form, with respect to this polygon, of $s^3 + t^2 - X_1 st$, $s^2 + t^3 - X_2 st$, and $s^2 t + st^2 - X_3 st$ which results in the following 7×7 matrix:

$$\begin{bmatrix} 0 & 1 & -X_1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -X_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & -X_3 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -X_3 & -X_2 & X_1 & X_1 - 1 \\ 0 & 0 & 0 & 0 & X_3 & -1 & -1 \\ 0 & 0 & -X_1 & X_2 - 1 & 1 & X_2 & -X_3 \\ 0 & 0 & 1 - X_1 & X_2 & 1 - X_2 & -X_3 & X_1 \end{bmatrix}.$$

The determinant of this matrix is

$$\begin{aligned} &2 + X_1 - 5X_3^3 - X_1^2 X_2 - X_2^2 X_1 + X_3 X_1^3 + X_3 X_2^3 + X_3^5 + X_2 + 5X_3 \\ &\quad + 4X_2^2 X_3^2 - X_3 X_1 - 2X_2 X_1 - X_2 X_3 - 3X_2 X_3^2 + X_2^2 X_3 - 3X_2 X_1 X_3^3 \\ &\quad - X_2^2 X_3^2 X_1 + 4X_1^2 X_3^2 - 3X_1 X_3^2 + X_2^2 X_1 X_3 + X_3 X_1^2 X_2 - X_1^2 X_3^2 X_2 \\ &\quad + 2X_2 X_3^3 + X_1^2 X_3 - 5X_1 X_2 X_3 + 2X_1 X_3^2. \end{aligned}$$

This is the degree 5 (equal to Area(Q)) affine implicit equation.

It is an immediate consequence from the proof of Theorem 3 that when $Z \neq \emptyset$ the Chow form $\mathrm{Ch}_{\mathcal{A}}$ is identically zero. However, we shall see in the next section that the implicit equation can still be recovered from maximal minors of the resultant matrix. This shows how a matrix resultant formula encodes much more information than just the Chow form.

3.1. Base points

In this section we take a closer look at the Chow form matrix described above in order to determine what happens in the presence of base points. Throughout this section we will assume x_1, x_2, x_3, x_4 are specific choices of polynomials supported on $\mathcal{A}$ which may in particular have base points.

We will see that we can always get a matrix whose determinant is a non-trivial multiple of the implicit equation. In order to still get an exact formula we will need a hypothesis on the structure of the basepoints. By the construction of $\mathcal{A}$, we will always be able to assume that the points in Z are smooth points of $X_{\mathcal{A}}$ (see the explanation in the proof of Theorem 8). In that case the local ring $\mathcal{O}_{X,p}$ is just the localized polynomial ring in two variables x, y.

Definition 6. Let X be a variety of dimension n. A *zero-dimensional local complete intersection* (*LCI*) is a subscheme Z in the smooth locus of X, such that for each point p in Z, the ideal $I_{Z,p}$ of the local ring $\mathcal{O}_{X,p}$ is defined by n equations.

The main property of local complete intersections that we will use is contained in the next proposition.

Proposition 7. *If Z is a local complete intersection then the multiplicity $e(\mathcal{I}_{Z,p}, \mathcal{O}_{X,p})$ is equal to the vector space dimension of the finite local algebra $\mathcal{O}_{Z,p} = \mathcal{O}_{X,p}/I_{Z,p}$. In particular $\sum_{p\in Z} e(\mathcal{I}_{Z,p}, \mathcal{O}_{X,p})$ is equal to the vector space dimension of the affine coordinate ring of Z.*

This proposition is a consequence of [6, Theorem 4.7.4] as $I_{Z,p}$ is generated by a regular sequence. Hence, the Euler characteristic is just the length of $\mathcal{O}_{X,p}/I_{Z,p}$, which is the vector space dimension in the zero-dimensional case.

We can now state the main result of this section.

Theorem 8. *Let $\pi : X_{\mathcal{A}} \to \mathbb{P}^3$ be a projection onto a surface Φ parameterized by x_1, x_2, x_3, x_4 with no common factor such that $\mathcal{A}$ is the union of the supports of the x_i. Let $Z \subset X_{\mathcal{A}}$ be the finite set of basepoints of π. Now, let $M_{\mathcal{A}}$ be the determinantal formula for* $\mathrm{Ch}_{\mathcal{A}}$ *from* [20], *where f_1, f_2, f_3 are the polynomials $x_1(s,t) - X_1x_4(s,t)$, $x_2(s,t) - X_2x_4(s,t)$, and $x_3(s,t) - X_3x_4(s,t)$, respectively. Then the implicit equation $P^{\deg(\phi)}$ divides any maximal minor of $M_{\mathcal{A}}$.*

Moreover, a maximal minor of $\mathrm{Ch}_{\mathcal{A}}(f_1, f_2, f_3)$ *using all of the Sylvester rows and columns exists and has determinant equal to exactly $P^{\deg(\phi)}$ if*:

(1) *Z is a local complete intersection on $X_{\mathcal{A}}$.*
(2) *The Sylvester columns in L, indexed by* int(Q)*, are linearly independent for generic choices of X_1, X_2, X_3. Equivalently, f_1, f_2, f_3 have no syzygies supported on* $\mathrm{int}(Q)$ *with coefficients in* $\mathbb{C}[X_1, X_2, X_3]$.

The LCI hypothesis seems to be ubiquitous in implicitization [4,5]. Note that in particular isolated basepoints are always LCI so that "generically" even if x_1, x_2, x_3, x_4 have basepoints, e.g., if the geometric condition of Theorem 2 is not satisfied, the basepoints they do have will be LCI.

The second condition is somewhat more subtle and is not really well understood except that it was quite difficult to construct examples for which it fails (see Example 12). It can be compared with Assumption 20 in the method of moving surfaces, i.e., that the "moving plane" matrix MP is of maximal rank. Even if the second condition fails we can still recover the implicit equation as the GCD of the maximal minors. This is not true if the first condition fails as illustrated by Example 11.

Example 9. Consider the surface parameterized by

$$\begin{aligned} x_1 &= 1+s-t+st-s^2t-st^2, \\ x_2 &= 1+s-t-st+s^2t-st^2, \\ x_3 &= 1-s+t-st-s^2t+st^2, \\ x_4 &= 1-s-t+st-s^2t+st^2. \end{aligned}$$

There is one basepoint at $(s,t)=(1,1)$. The corresponding polygon Q is a pentagon with vertices $(0,0)$, $(1,0)$, $(0,1)$, $(2,1)$, $(1,2)$. Computing the Chow form matrix gives a singular 9×9 matrix. However, we can remove one row and column to get the 8×8 matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 1-X_1 & 1+X_1 & -1+X_1 & 1-X_1 & -1+X_1 \\ 0 & 0 & 0 & 1-X_2 & 1+X_2 & -1+X_2 & -1+X_2 & -1-X_2 \\ 0 & 0 & 0 & 1-X_3 & -1+X_3 & 1+X_3 & -1-X_3 & -1+X_3 \\ 1-X_1 & 1-X_2 & 1-X_3 & 0 & 4X_3-4X_2+4X_1-4 & 0 & -4X_3-4X_1 & 4X_2+4X_3 \\ 1-X_1 & -1-X_2 & -1-X_3 & 0 & -4+4X_1 & 0 & 8-4X_3-4X_1 & 0 \\ -1+X_1 & -1+X_2 & 1+X_3 & 0 & 4-4X_1 & 0 & 4X_3-4X_2+4X_1-4 & -4X_3+4X_1 \\ 1+X_1 & 1+X_2 & -1+X_3 & 0 & 0 & 0 & 0 & 0 \\ -1+X_1 & 1+X_2 & -1+X_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The determinant is 256 times the irreducible implicit equation which is

$$\begin{aligned} &2X_1-X_2+X_3-X_3^3X_1-X_2^2X_1^2+X_3X_1^2X_2-5X_3X_1+3X_2X_1-2X_3X_1X_2-2X_1^2 \\ &\quad -3X_1^2X_2+2X_3X_2-2X_2^2+4X_2^2X_1+X_3X_1^2-X_2^3+3X_3X_2^2-2X_3^2-X_3^2X_2 \\ &\quad +2X_2^3X_1-X_2^4-2X_3^2X_2^2-X_3X_2^3+X_3^2X_1X_2+X_3^3+4X_3^2X_1. \end{aligned}$$

This has degree 4 since $\mathrm{Area}(Q)=5$ and there is one basepoint of multiplicity 1.

Example 10. Let us now consider an example where the basepoint has multiplicity:

$$\begin{aligned} x_1 &= \bigl(t+t^2\bigr)(s-1)^2+\bigl(1+st-s^2t\bigr)(t-1)^2, \\ x_2 &= \bigl(-t-t^2\bigr)(s-1)^2+\bigl(-1+st+s^2t\bigr)(t-1)^2, \\ x_3 &= \bigl(t-t^2\bigr)(s-1)^2+\bigl(-1-st+s^2t\bigr)(t-1)^2, \\ x_4 &= \bigl(t+t^2\bigr)(s-1)^2+\bigl(-1-st-s^2t\bigr)(t-1)^2. \end{aligned}$$

Once again there is a single basepoint at $(s,t) = (1,1)$. But since locally the ideal at this basepoint is generated by $((s-1)^2, (t-1)^2)$, the basepoint is an LCI. So applying the method above we get a 15×15 matrix and an 11×11 maximal minor.

The determinant, after removing the integer constant, is

$$\begin{aligned}
&-12 - 4X_1 - 9X_2 + 5X_3 - X_2^5 - 4X_3^2 - 20X_3^2X_2^3\\
&\quad - 16X_3^2X_2 - 32X_3^2X_2^2 - 12X_3^2X_1 - 12X_3^2X_1^2\\
&\quad + 8X_3X_2^4 - 12X_3^2X_1^2X_2 - 36X_3^2X_2^2X_1 - 48X_3^2X_2X_1\\
&\quad + 2X_1^3 - 6X_3X_1^2 + 11X_2X_1^2 + X_1^2 - 13X_2X_1 - 3X_1^4\\
&\quad - 3X_1^3X_2 + 14X_2^2X_1^2 - 9X_2^2X_1 - 16X_2^2 - 7X_1^4X_2\\
&\quad - X_3X_1^4 - X_1^5 - 19X_3X_1^3X_2 + 9X_3X_1 - 15X_3X_1^3\\
&\quad + 19X_3X_2 - 11X_2^2X_1^3 - 43X_3X_1^2X_2 + 27X_3X_2X_1\\
&\quad - 6X_2^4 + 3X_2^4X_1 + 4X_2^3X_1 - 14X_2^3 + 33X_3X_2^2\\
&\quad + 10X_3X_1X_2^2 - 43X_3X_1^2X_2^2 + 28X_3X_2^3 - 12X_3X_2^3X_1.
\end{aligned}$$

The degree of this equation is 5 and the area of the support polygon Q is 9.

Example 11. Let us now modify the above example so that the basepoints no longer form an LCI. We will see that we can no longer recover the implicit equation exactly from our Chow form matrix.

$$\begin{aligned}
x_1 &= \left(t+t^2\right)(s-1)^2 + \left(1+st-s^2t\right)(t-1)^2 + \left(t+st+st^2\right)(s-1)(t-1),\\
x_2 &= \left(-t-t^2\right)(s-1)^2 + \left(-1+st+s^2t\right)(t-1)^2 + \left(t+st+st^2\right)(s-1)(t-1),\\
x_3 &= \left(t-t^2\right)(s-1)^2 + \left(-1-st+s^2t\right)(t-1)^2 + \left(t+st+st^2\right)(s-1)(t-1),\\
x_4 &= \left(t+t^2\right)(s-1)^2 + \left(-1-st-s^2t\right)(t-1)^2 + \left(t+st+st^2\right)(s-1)(t-1).
\end{aligned}$$

Because of the additional $(s-1)(t-1)$ term, the degree of the basepoint at $(1,1)$ drops to 3, however, the multiplicity remains 4. Indeed, a maximal minor of the 15×15 Chow form matrix now has rank 12. And the determinant of *any* maximal minor is (up to a constant):

$$\begin{aligned}
&(-X_2 + 2X_3 - 1)\big(101 - 224X_3^5 + 8X_1^5 - 525X_1 + 75X_2 + 2689X_1X_3 - 573X_3\\
&\quad + 5519X_3^2X_1^2 + 3830X_3^3X_1 + 2948X_1^3X_3 + 1310X_3^2 + 155X_1X_3^2X_2^2\\
&\quad - 169X_1X_3X_2^3 - 1970X_3^2X_1^3 - 2308X_1^2X_3X_2 - 487X_3X_2^2X_1 - 1182X_3^4X_1\\
&\quad - 2296X_3^3X_1^2 + 1707X_1X_3X_2 + 1006X_1X_3^3X_2 + 1487X_1^2X_3^2X_2\\
&\quad + 956X_1^3X_3X_2 - 1512X_3^3 - 4795X_1X_3^2 - 2118X_1X_3^2X_2 - 624X_1^4X_3 + X_2^5
\end{aligned}$$

$$- 13X_2^4 - 88X_2^2 - 76X_2^3 - 948X_1^3X_2 + 244X_3^4X_2 - 646X_3^3X_2 - 513X_1X_2$$
$$- 211X_3^2X_2^2 + 191X_2^3X_1 - 105X_1^2X_2^3 + 1140X_1^2X_2 + 185X_1^2X_3X_2^2$$
$$+ 143X_3X_2^3 + 255X_1^4X_2 + 3X_3X_2^4 - 42X_3^2X_2^3 + 19X_1X_2^4 + 264X_1X_2^2$$
$$- 214X_1^2X_2^2 + 48X_1^3X_2^2 - 385X_3X_2 + 248X_3X_2^2 + 729X_3^2X_2 + 18X_3^3X_2^2$$
$$+ 337X_1^4 - 1050X_1^3 + 898X_3^4 - 4445X_1^2X_3 + 1133X_1^2\big).$$

The second factor, of degree 5, is the desired implicit equation.

In the last example, there is a linear extraneous factor of $-X_2 + 2X_3 - 1$. One can show that this extraneous factor divides every maximal minor of $M_{\mathcal{A}}$. Hence, the extraneous factor is somehow intrinsic to the resultant matrix and cannot be removed. It would be interesting to have some theoretical explanation for this factor.

We conclude with an example where the Sylvester rows are not linearly independent.

Example 12.

$$x_1 = s + s^2 + s^3t + s^2t^2 + st^3,$$
$$x_2 = t^2(s + 1),$$
$$x_3 = st(s + 1),$$
$$x_4 = t(s + 1).$$

The Newton polygon has three interior points st, s^2t, st^2. This system turns out to have a degree 7 LCI basepoint locus on $X_{\mathcal{A}}$. However, one can easily check that $(s^2t - stX_3)(x_2 - X_2x_4) = (st^2 - stX_2)(x_3 - X_3x_4)$ so that there is indeed a syzygy of f_1, f_2, f_3 supported in $\mathrm{int}(Q)$. So there is no maximal minor using all of the Sylvester columns. We can still construct maximal minors using as many Sylvester columns as possible, in this case 8 of the 9. The determinant of such a minor depends on which choice of Sylvester columns we remove. If we remove the column in the Sylvester block corresponding to $st \cdot f_3$ we get a matrix whose determinant is

$$X_2\big(X_2^3X_1 + X_1^2X_2^2 + X_3^2X_2^2 - X_1X_2X_2 + X_2X_3^3 - X_1X_3^2\big).$$

If, on the other hand, we remove a column corresponding to $s^2t \cdot f_3$ the determinant is exactly the implicit equation without the extraneous factor of X_2.

3.2. *Proof of Theorem 8*

In this section we prove Theorem 8. The Chow form matrix described above, and indeed most of the formulas for Chow forms in the literature, are applications of a general setup due to Weyman [30]. A constructive approach using exterior algebras was described by Eisenbud, Schreyer and Weyman [17]. They start with an arbitrary projective variety

$X \subset \mathbb{P}^N$ of dimension n and try to compute its Chow form. Hence they consider the incidence correspondence:

$$\begin{tikzcd} & V \subset X \times G_{n+1} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \subset \mathbb{P}^N & & G_{n+1}. \end{tikzcd}$$

Here G_{n+1} is the Grassmanian of codimension $n+1$ planes in $\mathbb{P}^N$ and $V = \{(x, F)\colon F(x) = 0\}$ the incidence subvariety of $X \times G_{n+1}$. Now given any sheaf $\mathcal{F}$ supported on X which is generically a vector bundle, there is a complex, denoted $U_{n+1}(\mathcal{F})$ in [17], of vector bundles on G_{n+1} equivalent in the derived category to $R(\pi_2)_*\pi_1^*\mathcal{F}$. This leads to the following completely general result.

Theorem 13. *Let $X \subset \mathbb{P}^N$ be any variety of dimension n. Let $\mathcal{F}$ be any sheaf supported on X that is generically of rank* 1. *Let $F_0, \ldots, F_n$ be any linearly independent sections of $\mathbb{P}^N$ which simultaneously meet X only at finitely many points at all of which $\mathcal{F}$ is of rank* 1. *The last map in the complex $U_{n+1}(\mathcal{F})$ has cokernel of rank equal to the degree of the zero-dimensional subscheme of X cut out by F.*

Proof. Consider the incidence correspondence as above. As $U_{n+1}(\mathcal{F})$ is isomorphic in the derived category to $R(\pi_2)_*\pi_1^*\mathcal{F}$, the cokernel of the last map in particular is just $(\pi_2)_*\pi_1^*\mathcal{F}$ itself. So all we need to show is that the dimension of the fiber of this sheaf at a point $F \in G_{n+1}$ satisfying the above properties is the degree of the subscheme X_F of X defined by F.

First consider the fiber of the morphism π_2 over F. Let R be the coordinate ring of X and S the Stiefel coordinate ring of G_{n+1} with variables $\underline{a}$. The ideal of V in $R \otimes S$ is denoted $I(\underline{a})$. Now, by definition the fiber over the point F defined by a choice $\underline{a} = a$ with corresponding maximal ideal m_a in S is $(R \otimes S)/I(\underline{a}) \otimes_S S/m_a$. But this is just $R/I(a)$ which is the coordinate ring of X_F. Hence the fiber of π_2 over F is $X_F \times F$. (Note that different choices of a realizing the same point F give the same ideal $I(a)$.)

Next, since X_F is a zero-dimensional subscheme of the generic locus of $\mathcal{F}$ it is actually affine and $\mathcal{F}$ is trivial on X_F. Let $R/I(a)$, as above, be the (dehomogenized) coordinate ring of X_F and hence also of $X_F \times F$. As our sheaf was trivial, the pushforward onto the closed point F is just $R/I(a)$ itself viewed as a vector space over the residue field of F. The dimension of this vector space is by definition the degree of X_F as desired. $\square$

We can now prove Theorem 8 as a corollary.

Proof. We consider, in this case, $\mathcal{F} = \mathcal{O}(\mathrm{int}(2Q))$ the divisor corresponding to the interior of the polytope $2Q$. In [20], it was shown that $U_3(\mathcal{F})$ reduced to a two term complex with matrix exactly as described above. The sheaf $\mathcal{F}$ is locally free of rank 1, except possibly on the singular points of $X_{\mathcal{A}}$. For a toric surface the only possible singularities can occur on the torus fixed points which correspond to the vertices of Q. By the construction of $\mathcal{A}$, at least one of $x_1, \dots, x_4$ does not vanish on each vertex, hence the base point locus always misses the singular locus.

Now, we can apply Theorem 13. Pick a maximal minor of our matrix. For a generic X_1, X_2, X_3 not on the surface S, this remains a maximal minor of the specialization. Moreover, the corank of this minor is the degree of $I(f_1(X_1), f_2(X_2), f_3(X_3))$. However, for a point X_1, X_2, X_3 on the surface, the number of basepoints increases, therefore the rank of our matrix M decreases, hence the determinant of our chosen minor must be zero. Moreover, the rank drop of the minor for a generic point on the surface is exactly $\deg\phi$ (the number of "new basepoints" mapping on to our point). Since any order k derivative of the determinant of a matrix of linear forms is in the ideal of corank k minors (easy to see from the expansion of determinant), the first $\deg\phi - 1$ derivatives of the determinant are also zero for a generic point on the surface. Since P was irreducible, $P^{\deg\phi}$ must divide our chosen maximal minor.

For the second part, in the case of an LCI, the corank of our maximal minor, i.e., the degree of the base point locus, is the same as the sum of the multiplicities of our base points. If moreover, the maximal minor is chosen to contain all Sylvester rows and columns, only Bézout rows and columns are removed, each of which drops the degree by 1. Thus the degree of our determinant is equal to the degree of $P^{\deg\phi}$ and so they must be equal up to a constant. □

4. The method of moving surfaces

We now switch gears and present an entirely different method for constructing matrix formulas in implicitization. For the rest of this paper we will work with the projective surface $\Phi \subset \mathbb{P}^3$ defined by the four coordinates X_1, X_2, X_3, X_4.

The idea will be to construct linear and quadratic syzygies on the polynomials x_1, x_2, x_3, x_4 and put them together into a matrix of linear and quadratic forms in the X_i. For the case of homogeneous and bihomogeneous polynomials, this is the method of moving planes and surfaces introduced by Sederberg and Chen [28]. However, the proof we present in Section 5 is quite different, and in our opinion more insightful, than the ones in the literature. Our goal will to be to extend the method to general toric surfaces which will require looking at certain "degrees" of the homogeneous coordinate ring of the toric variety.

We shall see that the syzygy method has certain advantages and disadvantages to the Chow form/resultant method described above. It will always give smaller matrices due to the fact that some of the entries are quadratic in the X_i. Second, the algorithm will be relatively easy to describe and efficient in practice; all of the computations are just numerical linear algebra. Finally, the method appears to be surprisingly flexible in the presence of base points. We shall see empirical evidence supporting this at the end of the section.

On the other hand, rigorous proofs of the method in any of the more complicated situations have been hard to come by. Also, as pointed out above, all of the computations are linear algebra in the coefficients of the x_i. In particular, the method becomes much more inefficient with a generic parameterization or whenever the coefficients of the x_i are not numerical. The Chow form matrix constructed above, on the other hand, works the same for arbitrary coefficients and is therefore preferred when implicitizing a family of surfaces.

4.1. Moving planes and quadrics

Given a rational surface Φ parameterized by

$$\begin{cases} X_1 = x_1(s,t), \\ X_2 = x_2(s,t), \\ X_3 = x_3(s,t), \\ X_4 = x_4(s,t), \end{cases} \tag{4}$$

a *moving plane* is a syzygy on $I = \langle x_1, x_2, x_3, x_4 \rangle$, i.e., an equation of the form

$$A_1(s,t)X_1 + A_2(s,t)X_2 + A_3(s,t)X_3 + A_4(s,t)X_4$$

which is identically zero as a polynomial in s and t after the specialization $X_i \mapsto x_i$. Notice that each particular choice of (s,t) gives the equation of a plane which intersects the surface Φ at the point $(x_1(s,t), \ldots, x_4(s,t))$. Hence, this is said to be a plane that follows the surface Φ and justifies the terminology moving plane.

Similarly, a *moving quadric* is a syzygy on I^2:

$$A(s,t)X_1^2 + B(s,t)X_1X_2 + \cdots + J(s,t)X_4^2.$$

Once again a choice of (s,t) gives the equation of a quadric meeting the surface Φ. Hence, the moving quadric is said to follow the surface.

If we rewrite the moving planes and quadrics in terms of the monomial bases in s and t we get vectors of linear or quadratic forms in the X_i. Clearly multiplying each moving plane by X_1, X_2, X_3, X_4 gives a moving quadric. Therefore, we will only look for "new" moving quadrics. If we can now get enough of these vectors, we may be able to build a square matrix out of them. The determinant of this square matrix will hopefully be equal to the implicit equation of S. The following well-known result is our starting point.

Proposition 14. *Let $M(X_1, X_2, X_3, X_4)$ be any square matrix constructed from moving planes and quadrics as above. Then* $\det(M(x_1, x_2, x_3, x_4)) = 0$. *In particular the implicit equation always divides the determinant of M* (*which may, quite possibly, be identically* 0).

Proof. This has been proved in even more generality in [28]. □

The big question is now, of course, how should the moving planes and quadrics be chosen? In the case of homogeneous polynomials they were chosen to also be homogeneous of an appropriate degree. In the case of bihomogeneous polynomials, the moving planes and quadrics can be chosen to be bihomogeneous. In the more general toric setting we will need to work in appropriate homogeneous coordinates for the set $\mathcal{A}$.

4.2. Homogeneous coordinate ring of $X_{\mathcal{A}}$

Let $\mathcal{A}$ be the union of monomials in the x_i as before and $Q = \mathrm{conv}(\mathcal{A})$ the associated polygon. Let $E_1, \dots, E_s$ be the edges of Q and $\eta_1, \dots, \eta_s$ the primitive lattice vectors for the corresponding inner normal rays.

We can therefore define Q by its facet inequalities:

$$Q = \{m \in \mathbb{R}^2\colon \langle m, \eta_i\rangle \geqslant -a_i \text{ for } i = 1, \dots, s\}$$

for some $(a_1, \dots, a_s) \in \mathbb{Z}^s$.

$X_{\mathcal{A}}$ is a toric variety with a given very ample line bundle determined by the polytope Q. We will need to consider other divisors on $X_{\mathcal{A}}$. David Cox [9] defined a single ring that encapsulates all torus invariant divisors on $X_{\mathcal{A}}$.

Definition 15. The *homogeneous coordinate ring* for $X = X_{\mathcal{A}}$ is the polynomial ring $S_X = \mathbb{K}[y_1, \dots, y_s]$ where the monomials are graded as described below.

Consider the exact sequence of maps:

$$0 \to \mathbb{Z}^2 \xrightarrow{\phi} \mathbb{Z}^s \xrightarrow{\pi} G \to 0.$$

Here ϕ is the map $m \to (\langle m, \eta_1\rangle, \dots, \langle m, \eta_s\rangle)$. The ring S_X is graded by elements of G where $\deg y^\alpha = \pi(\alpha)$.

The graded pieces of this ring have bases corresponding to lattice points in polygons. More precisely the monomials in $S_{\pi(b)}$ are in one to one correspondence with the lattice points in $Q_b = \{m \in \mathbb{R}^2\colon \langle m, \eta_i\rangle \geqslant -b_i\}$. And moreover, $\pi(b) = \pi(b')$ iff Q_b is a translate of $Q_{b'}$.

So it will make sense to talk about S_{Q_b}, the graded piece of S defined by Q_b.

Remark 16. In truth the divisors and homogeneous coordinate ring are really defined for the *normal* toric variety X_Q obtained from the normal fan of Q. This variety is the normalization of our $X_{\mathcal{A}}$. The projection and all prior and subsequent results can be lifted up to X_Q without affecting any of the calculations.

4.3. Picking moving planes and quadrics

Also associated to the polygon Q is a certain polynomial $E(k)$, the Ehrhart polynomial defined in [29], which counts the number of lattice points in $k \cdot Q$. In the case Q is two-dimensional, it turns out that

$$E(x) = Ax^2 + \frac{B}{2}x + 1$$

where $A = \frac{\mathrm{Area}(Q)}{2}$ and B equals the number of boundary points.

Let I be a non-empty proper subset of $\{1, \ldots, s\}$ such that the corresponding edges form a connected set. Let E_I be this connected set of edges of Q, let B_I be the sum of the lattice edge lengths of E_I. It is easy to see that the number of lattice points in the set of edges E_I in $k \cdot Q$ is $B_I k + 1$.

Assumption 17. We choose E_I in such a way that $B \geqslant 2B_I$.

Remark 18. Observe that this can always be done, for instance, by taking as E_I the shortest edge of Q. In practice, we will want to pick E_I in such a way that B_I is as big as possible consistent with Assumption 17.

Now we can define a degree of S denoted $S_{Q \setminus E_I}$ obtained by "pushing in" all of the edges of Q in E_I by one, whose monomial basis consists of all lattice points in Q not on any of the edges E_I. In the case of homogeneous polynomials of degree n, the only E_I satisfying Assumption 17 consist of a single edge and the degree in question in just $n-1$. In the case of bihomogeneous polynomials of bidegree (m, n), we can take E_I to be two consecutive edges and the degree is $(m-1, n-1)$. Note that in the latter case $B - 2B_I = 0$ which, as we shall see, means that we will not need to take any moving planes and can build a matrix entirely out of moving quadrics. We now formally define what we mean by moving planes and quadrics of this degree.

Consider the following $\mathbb{K}$-linear map:

$$\psi_1 : \begin{array}{rcl} S^4_{Q \setminus E_I} & \to & S_{2Q \setminus E_I} \\ (p_1, p_2, p_3, p_4) & \mapsto & \sum_{i=1}^4 p_i x_i, \end{array} \tag{5}$$

and let MP be the matrix of this map in the monomial bases.

Definition 19. As in [11], any element of the form $(A_1, A_2, A_3, A_4) \in \ker(\psi)$ will be called a *moving plane* of "degree" $Q \setminus E_I$ that follows the surface (1). Sometimes, we will write moving planes as $A_1 X_1 + A_2 X_2 + A_3 X_3 + A_4 X_4$.

Now for moving quadrics we consider the following map:

$$\psi_2 : \begin{array}{rcl} S^{10}_{Q \setminus E_I} & \to & S_{3Q \setminus E_I} \\ (A_{i,j,k,l})_{i+j+k+l=2} & \mapsto & \sum_{i+j+k+l=2} A_{i,j,k,l} x_1^i x_2^j x_3^k x_4^l, \end{array} \tag{6}$$

and let MQ be the matrix of ψ_2 in the monomial bases. Then

$$\begin{aligned} \#\,\text{rows of } MQ &= \#\,(3Q \setminus E_I) \cap \mathbb{Z}^2 \\ &= \left(9A + \frac{3}{2}B + 1\right) - (3B_I + 1) = 9A + \frac{3}{2}B - 3B_I \end{aligned}$$

and

$$\begin{aligned}\#\text{ columns of } MQ &= 10\ \#\ (Q \setminus E_I) \cap \mathbb{Z}^2 \\ &= 10\left(A + \frac{B - 2B_I}{2}\right) = 10A + 5B - 10B_I.\end{aligned}$$

Now a *moving quadric* of degree $Q \setminus E_I$ which follows our surface S is just an element of the kernel of MQ.

We now describe the method of moving quadrics. It differs from the presentations in the literature not only in its application to general toric surfaces but also in that we allow the bases of moving planes and quadrics to be chosen freely. Earlier papers specify that moving planes and quadrics be chosen of a specific form to ensure that the resulting matrix has determinant non-zero. Our more intrinsic proof of Section 5 makes this unnecessary.

- Compute a basis P of the kernel of MP. The entries are $P_i = A_1^i X_1 + A_2^i X_2 + A_3^i X_3 + A_4^i X_4$. Where the A_j^i are polynomials in $S_{Q \setminus E_I}$.
- Each $P_i \cdot X_j$ for $j = 1, \ldots, 4$ is in the kernel of MQ. We will see that these are linearly independent. Extend this set to an entire basis for the kernel of MQ. Let $Q_1, \ldots, Q_d$ be the new moving quadrics in this basis.
- Construct a matrix $\mathbb{M}$ out of the P_i and Q_j such that the columns correspond to the monomial basis of $S_{Q \setminus E_I}$ and the entries are the linear (or quadratic) polynomial in $X_1, \ldots, X_4$ corresponding to the coefficient of that monomial in P_i (or Q_j).

Our hope is that the resulting matrix will be square and that the determinant is the implicit equation. To start with, by Theorem 14, if the matrix $\mathbb{M}$ has more rows than columns, then the determinant of any maximal minor (possibly 0) is divisible by the implicit equation.

5. Validity of the method of moving quadrics without basepoints

In this section we verify, in the absence of basepoints, that the method of moving quadrics gives a square, nonsingular matrix whose determinant is exactly the implicit equation raised to the power the degree of the parameterization. We will need to make one assumption:

Assumption 20. The moving plane matrix MP, or the map ψ_1, has maximal rank.

This assumption also appears in the papers by Cox, Goldman, and Zhang [11] and D'Andrea [14]. Empirical evidence suggests that it is almost always satisfied. It appears that for a fixed Q and E_I, and any generic set of x_1, x_2, x_3, x_4 without basepoints, MP has maximal rank.

We now build a complex containing both the moving plane map ψ_1 and the moving quadric map ψ_2 in Fig. 1.

The terms K_1 and K_2 are the kernels of the moving plane and moving quadric maps ψ_1 and ψ_2, respectively. The term $\tilde{K}_2$ is the cokernel of the map X of K_1^4 into K_2, generated precisely by a basis of K_2 extending the image of moving planes multiplied by linear

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d, dashed] & (S_{Q\setminus E_I})^6 \arrow[r, equal] \arrow[d, "i"'] & (S_{Q\setminus E_I})^6 \arrow[r] \arrow[d, "x'"'] & 0 \\
0 \arrow[r] & K_1^4 \arrow[r] \arrow[d, "X"'] & (S_{Q\setminus E_I})^{16} \arrow[r, "\psi_1^4"] \arrow[d, "X"'] & (S_{2Q\setminus E_I})^4 \arrow[r] \arrow[d, "x"'] & 0 \\
0 \arrow[r] & K_2 \arrow[r] \arrow[d] & (S_{Q\setminus E_I})^{10} \arrow[r, "\psi_2"] \arrow[d] & S_{3Q\setminus E_I} \arrow[r, dashed] \arrow[d] & 0 \\
 & \tilde{K}_2 & 0 & 0 &
\end{tikzcd}
$$

Fig. 1. Complex of moving planes and quadrics.

forms. In this new language a matrix $\mathbb{M}$ of moving planes and quadrics is a basis for $K_1^4 \oplus \tilde{K}_2$ taken as vectors in $(S_{Q\setminus E_I})$ with coefficients that are linear or quadratic forms in (X_1, X_2, X_3, X_4).

We now prove out two main theorems that together prove the validity of the method of moving quadrics.

Theorem 21. *If MP has maximal rank, then* $\dim(K_1) + \dim(\tilde{K}_2) = \dim(S_{Q\setminus E_I})$ *and* $\dim(K_1) + 2\dim(\tilde{K}_2) = \mathrm{Area}(Q)$. *Therefore, the method of moving quadrics yields a square matrix with determinant of degree equal to the implicit equation.*

Theorem 22. *Let $p = (p_1, p_2, p_3, p_4)$ be a point not on the surface X. The moving plane matrix $\mathbb{M}$ is non-singular at p. Consequently, if ψ_1 has maximal rank* $\det(\mathbb{M}) = P^{\deg(\phi)}$ *where P is the implicit equation as desired.*

Before proceeding we further describe the maps in the complex. The second row consists of four copies of the moving plane complex. An element of $(S_{Q\setminus E_I})^{16}$ is represented as a four tuple of linear forms in X_1, X_2, X_3, X_4 with coefficients in $S_{Q\setminus E_I}$. Similarly the bottom row is the moving quadric complex. An element of $(S_{Q\setminus E_I})^{10}$ is a quadratic form in X_1, X_2, X_3, X_4 with coefficients in $S_{Q\setminus E_I}$ generated by the 10 monomials X_iX_j with $i \leqslant j$. The map X, multiplication by (X_1, X_2, X_3, X_4), sends the four tuple (u_1, u_2, u_3, u_4) of linear forms to the quadratic form $\sum u_iX_i$. This has the effect of sending X_i in position j and X_j in position i both to X_iX_j.

The kernel of X is isomorphic to $(S_{Q\setminus E_I})^6$ indexed by pairs (i, j) with $i < j$. The injection i sends the term p_{ij} to $(0, \ldots, p_{ij}X_j, \ldots, -p_{ij}X_i, \ldots, 0)$ with X_j in position i and $-X_i$ in position j. The rightmost column is a graded piece of the Koszul complex on (x_1, x_2, x_3, x_4), with x mapping a four tuple (s_1, s_2, s_3, s_4) to $\sum s_ix_i$ and x' sending p_{ij} with $i < j$ to $(0, \ldots, p_{ij}x_j, \ldots, -p_{ij}x_i, \ldots, 0)$.

Commutativity of the diagram is immediate. The rows are all exact by construction. The second column is also clearly exact. The rightmost column is more interesting. When (x_1, x_2, x_3, x_4) have no basepoints, the map x' is injective and x is surjective. This can be seen by investigating the complex $U_4(\mathcal{O}(3Q \setminus E_I))$ arising from the Tate resolution in the theory of [17]. However, the spot in the middle is not exact. We shall see later that obstruction to exactness comes from a certain 'Bezoutian' map determined exactly by elements of $\tilde{K}_2$.

Now, to prove Theorem 21 we will need three lemmas.

Lemma 23. *If MP has maximal rank, then the number of linearly independent moving planes of degree $Q \setminus E_I$ which follow the surface is $B - 2B_I$.*

Proof. There are

$$(4A + B + 1) - (2B_I + 1) = 4A + B - 2B_I$$

integer points in $2Q \setminus E_I$, and

$$\left(A + \frac{B}{2} + 1\right) - (B_I + 1) = A + \frac{B - 2B_I}{2}$$

integer points in $Q \setminus E_I$. If *MP* has maximal rank, then the number we want to compute is the dimension of the kernel of ψ_1 which equals

$$4\left(A + \frac{B - 2B_I}{2}\right) - (4A + B - 2B_I) = B - 2B_I$$

as claimed. □

Lemma 24. *If ψ_1 is surjective then so is ψ_2.*

Proof. This is an easy diagram chase. Given $s \in S_{3Q\setminus E_I}$ pull it back to $S^4_{2Q\setminus E_I}$ and then to $(S_{Q\setminus E_I})^{16}$ via the surjectivity of the corresponding maps. Finally, map this down to $t \in (S_{Q\setminus E_I})^{10}$. Commutativity of the diagram yields $\psi_2(t) = s$. □

Lemma 25. *The map X from K_1^4 to K_2 is injective.*

Proof. Given k in the kernel, it is a non-zero element of $(S_{Q\setminus E_I})^{16}$ mapping to zero in $(S_{Q\setminus E_I})^{10}$. By exactness it comes from a non-zero element in $S^6_{Q\setminus E_I}$ mapping to a non-zero element k' in $(S_{2Q\setminus E_I})^4$. But commutativity implies $k' = \psi_1(k) = 0$, a contradiction. □

We are now ready for the proof of Theorem 21.

Proof. By Lemma 24, MQ has maximal rank. From the computations of the last section, the dimension of $(S_{Q\setminus E_I})^{10}$ is $10A + 5B - 10B_I$, while the dimension of $S_{3Q\setminus E_I}$ is $9A +$

$3B/2 - 3B_I$. Thus the rank of K_2 is $A + 7B/2 - 7B_I$. By Lemma 23 the rank of K_1^4 is $4(B - 2B_I)$, so by Lemma 25 the rank of $\tilde{K}_2$ is $A - B/2 + B_I$.

So, the sum of the ranks of K_1 and $\tilde{K}_2$ is

$$B - 2B_I + \left(A - \frac{B}{2} + B_I\right) = A + \frac{B}{2} - B_I = \dim S_{Q \setminus E_I}.$$

Moreover, the total degree of the determinant is

$$B - 2B_I + 2\left(A - \frac{B}{2} + B_I\right) = 2A.$$

This is twice the Euclidean area, hence equal to the normalized area of Q as desired. □

The theorem just proved shows that $\mathbb{M}$ is square of the right rank. Theorem 22 will show that its determinant does not vanish outside of the surface.

Let $p = (p_1, p_2, p_3, p_4) \in \mathbb{P}^3$ be a point not on the surface parametrized by X. WLOG assume that $p_4 = 1$. Make a change of coordinates $X_1' = X_1 - p_1 X_4$, $X_2' = X_2 - p_2 X_4$, $X_3' = X_3 - p_3 X_4$ and $X_4' = X_4$. The point (p_1, p_2, p_3, p_4) is transformed to $(0, 0, 0, 1)$. Since the parameterization has no base points $\mathrm{Ch}_{\mathcal{A}}(x_1', x_2', x_3') \neq 0$ by Theorem 3.

We now use two facts arising from resultant complexes.

Lemma 26. *The restricted map $\tilde{\psi}_1 : S^3_{Q \setminus E_I} \to S_{2Q \setminus E_I}$ given by $(s_1, s_2, s_3) \to \sum s_i x_i'$ is injective. In particular no moving plane $A_1 X_1' + A_2 X_2' + A_3 X_3' + A_4 X_4'$ vanishes at $(X_1', X_2', X_3', X_4') = (0, 0, 0, 1)$.*

Proof. In [20, Theorem 3.4.1], a matrix whose determinant gives $\mathrm{Ch}_{\mathcal{A}}(x_1, x_2, x_3)$ is constructed, and this matrix has a Sylvester part coming from $\tilde{\psi}_1$. As we have $\mathrm{Ch}_{\mathcal{A}}(x_1', x_2', x_3') \neq 0$, it turns out that $\tilde{\psi}_1$ must be injective. Any vanishing moving plane as above has $A_4 = 0$ so must in fact be in the kernel of $\tilde{\psi}_1$. □

Lemma 27. *The restriction of the last column*:

$$0 \to S^3_{Q \setminus E_I} \to S^3_{2Q \setminus E_I} \to S_{3Q \setminus E_I} \to 0,$$

is just the Koszul complex on x_1', x_2', x_3', is exact.

Proof. We may consider x_1', x_2', x_3' as sections of sheaves on the toric variety $X_{\mathcal{A}}$. As in [15, Section 4], we start with the Koszul complex of these sheaves in degree $3\beta - \beta_I$, where β is the degree associated to $Q \cap \mathbb{Z}^2$ and β_I the divisor associated to all the edges whose union equals E_I. As in the proof of Theorem 3.1 in [15], one can see that we can apply the Weyman's complex (see [18, Section 3.4.E]) to this complex. By the toric version of Kodaira vanishing (see [26]), all higher cohomology terms vanish and we get that the complex above is generically exact. Indeed, the determinant of the complex equals the Chow form of x_1', x_2', x_3'. $\square$

Corollary 28. *Any moving quadric* $\sum_{1\leqslant i\leqslant j\leqslant 4} A_{ij}X_i'X_j'$ *vanishing at* $(0,0,0,1)$ *is in the image of* K_1^4 *under the multiplication map* X.

Proof. Start with such a vanishing moving quadric q. Plugging in we see that $A_{44}=0$. Hence $q = q_1X_1' + q_2X_2' + q_3X_3'$ where $q_1 = A_{11}X_1' + A_{12}X_2' + A_{13}X_3' + A_{14}X_4'$, $q_2 = A_{22}X_2' + A_{23}X_3' + A_{24}X_4$ and $q_3 = A_{33}X_3' + A_{34}X_4'$.

Pulling back to $q' = (q_1, q_2, q_3, 0) \in (S_{Q\setminus E_I})^{16}$ and mapping to $(S_{2Q\setminus E_I})^4$ by substituting x_1', x_2', x_3' into q_1, q_2, q_3 we get an element of the subspace $(S_{2Q\setminus E_I})^3$ as in the restricted complex above which is still in the kernel of X. Thus, by Lemma 24 we can pull back via X' to $S^3_{Q\setminus E_I} \subset S^6_{Q\setminus E_I}$. Let q'' be the image of this element in $S^{16}_{Q\setminus E_I}$. By construction $\psi_1^4(q'-q'')=0$ and $X(q'-q'')=q$. But now we can pull back $q'-q''$ to $k=(k_1,k_2,k_3,k_4)$ with $X(k)=q$ as desired.

It is now straightforward to finish the proof of Theorem 22.

Proof. Suppose $(u(p), v(p))$ is in the kernel. Write $u = \sum u_iX_i'$ and $v = \sum v_{ij}X_i'X_j'$. Substituting in for p we have $u_4 + v_{44} = 0$. Therefore the moving quadric $X_4'u + v$ has no $(X_4')^2$ term and thus vanishes at p. By Corollary 28 this must be in the image of K_1^4 so we must have $v=0$. But now $u(p)=0$ violating Lemma 26. Hence $\mathbb{M}$ is singular only on points of X. If ψ_1 is maximal rank then $\mathbb{M}$ is square, hence its determinant is a power of the implicit equation. Since the degree of $\det(\mathbb{M}) = \mathrm{Area}(Q)$, the exponent must be $\deg(\phi)$. $\square$

Example 29. Consider the system from Example 5:

$$x_1 = s^3 + t^2,$$
$$x_2 = s^2 + t^3,$$
$$x_3 = s^2t + st^2,$$
$$x_4 = st.$$

The total boundary length of the quadrilateral Q is 7. We can pick E_I to be the long edge of length 3. Hence $B - 2B_I = 1$. Applying the method of moving quadrics then gives a matrix with one moving plane and two moving quadrics:

$$\begin{bmatrix} -X_1 - X_2 - X_3 & X_3 + X_4 & X_3 + X_4 \\ X_1X_3 - X_2X_4 + X_4^2 & X_1X_3 - X_2X_4 - X_3X_4 & -X_3^2 + X_4^2 \\ \begin{matrix} -X_1^2 - X_1X_2 - 3X_1X_3 \\ + 2X_2X_4 - X_3^2 + X_3X_4 \end{matrix} & X_1X_4 + X_2X_4 + X_3X_4 & \begin{matrix} X_1X_2 + X_2X_3 + 2X_3^2 \\ - X_3X_4 - 2X_4^2 \end{matrix} \end{bmatrix}.$$

The determinant is exactly the degree 5 implicit equation.

5.1. Moving quadrics in the presence of base points

In the case of homogeneous parameterizations ($X_{\mathcal{A}} = \mathbb{P}^2$), [5] gives a series of conditions for when the method of moving quadrics works even with basepoints. The conditions are labelled (BP1)–(BP5) but essentially they boil down to assuming the basepoints form an LCI, there are no syzygies on linear combinations of x_1, x_2, x_3, x_4 of the desired degree, and that there are the "right number" of moving planes of the degree in question.

The last assumption can be rephrased into a regularity assumption on the ideal of basepoints I. Using commutative algebra on graded rings they deduce a corresponding regularity bound on I^2 which implies that there are also the "right number" of linearly independent moving quadrics.

To extend these conditions to the toric setting would seem to require a notion of "toric regularity" using the homogeneous coordinate ring S_X in place of the usual graded polynomial ring. Perhaps the definition proposed by Maclagan and Smith [23,24] can be applied here. Instead of delving into the theory of toric commutative algebra and what does and does not extend, we simply present some examples to illustrate how the toric method of moving quadrics can often work in the presence of basepoints.

Example 30. We repeat Example 9 using moving quadrics.

$$\begin{aligned}
x_1 &= 1 + s - t + st - s^2t - st^2,\\
x_2 &= 1 + s - t - st + s^2t - st^2,\\
x_3 &= 1 - s + t - st - s^2t + st^2,\\
x_4 &= 1 - s - t + st - s^2t + st^2.
\end{aligned}$$

Recall that we have one basepoint at $(1,1)$ with multiplicity 1. If there were no basepoints then we can choose $B - 2B_I = 1$ and we would expect one moving plane and two moving quadrics. Applying the algorithm gives two planes and one quadric but still a 3×3 square matrix:

$$\begin{bmatrix}
-X_3 + X_4 & 0 & X_1 - X_2\\
X_2 - X_3 + 2X_4 & X_2 + X_3 & -X_2 - X_3 + 2X_4\\
X_2X_1 + X_3X_1 & X_3X_1 - X_1X_4 + X_2^2 + X_2X_4 & -2X_1^2 + X_2^2 + X_2X_4 - X_3X_4 + X_4^2
\end{bmatrix}.$$

The method of moving quadrics works perfectly here and gives the implicit equation of degree 4.

Example 10 which had an LCI basepoint of multiplicity 4 also works with the method of moving quadrics. In this case we get 5 moving planes and no moving quadrics. The implicit equation is recovered as the determinant of the corresponding 5×5 matrix of linear forms.

Example 11 has a basepoint which is not an LCI. In this case, the moving quadric matrix was not square. Indeed there were four moving planes and two moving quadrics on a space of five monomials.

However, taking the maximal minor consisting of the four planes and either one of the two quadrics gives the implicit equation with a linear extraneous factor. Unlike the Chow form matrix of Example 11, this extraneous factor is not intrinsic to the construction. The two different maximal minors give different extraneous factors, hence the implicit equation is the gcd of the maximal minors.

6. Conclusion

In this paper we extend two of the most important implicitization techniques, resultants and syzygies, to general toric surfaces. There is a couple of interesting open questions remaining.

For the resultant method, when the basepoints are not an LCI, every maximal minor of the resultant matrix will have an extraneous factor. Is there a way to compute this extraneous factor apriori?

For the syzygy method, the biggest open question is how to extend the method when basepoints are present. Our examples show that the method may often still work. The second open problem is an understanding of exactly when the moving plane matrix has maximal rank.

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