Non-separating Induced Cycles in Graphs

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In this paper we consider non-separating induced cycles in graphs. A basic result is that any 2-connected graph with at least six vertices and without such a cycle has at least four vertices of degree 2, and this is best possible. For any 3-connected graph $G$ we prove that there exists a non-separating induced cycle $C$, such that all cycles in $G - V(C)$ are contained in the same block of $G - V(C)$. We apply our results in various directions. In particular, we obtain an extension of a conjecture of Hobbs (first proved by Jackson), and a new proof of Tutte's theorem on 3-connected graphs. Moreover, we show that any graph with minimum degree at least 3 contains a subdivision of $K_4$ in which the three edges of a Hamiltonian path of the $K_4$ are left undivided. This is an extension of a conjecture by Toft and implies an extension of a conjecture of Bollobás and Erdős (first proved by Larson) on the existence of an odd cycle with at least one diagonal. Finally, we obtain a result on the existence of a vertex joined by edges to three vertices of a cycle in a graph. This implies an extremal result conjectured by Bollobás and Erdős (first proved by Thomassen), as well as the conjecture of Toft that every 4-chromatic graph contains such a configuration.

1. INTRODUCTION

In [18] Thomassen gave, in the form of a catalogue, a complete description of all finite graphs (without loops or multiple edges) with no separating cycles. From this it follows that a graph has a separating cycle unless it has a very special structure. In the present paper we consider the analogous problem of finding a non-separating cycle in a graph, and we demonstrate how the existence of such a cycle is useful in several contexts.

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Non-separating cycles were previously considered in [12] by Krusenstjerna–Hafström and Toft, who proved that it never happens that all odd cycles in a critical 4-chromatic graph are separating. In a 3-connected graph the non-separating induced cycles play a special role since, by the results of Tutte [23], these cycles generate the cycle space of the graph and, if the graph is planar, then the non-separating induced cycles are precisely the facial cycles in any planar representation of the graph.

We shall use the terminology of [2, 3, 18]. In particular, $G[S]$ denotes the subgraph of $G$ spanned (or induced) by the subset $S$ of the vertex set $V(G)$ of $G$. By a $k$-rail $P_k$ in a graph $G$ between two vertices $x$ and $y$ we shall understand a system of $k$ internally disjoint paths joining $x$ and $y$, and all interior vertices of the paths having degree 2 not only in $P_k$, but in the whole of $G$. One (and only one) of these $k$ paths may have length 1.

We first consider the class $\mathcal{F}$ (resp. $\mathcal{F}'$) of connected graphs in which every cycle (resp. every induced cycle) is separating. The class $\mathcal{F}$, and hence $\mathcal{F}'$, is very large. Any $k$-rail ($k \geq 4$) is in $\mathcal{F}$, and any subdivision of a graph in $\mathcal{F}$ is in $\mathcal{F}$. Also, if we take any connected graph $G$ of order $\geq 3$ and replace each edge $(x, y)$ by a $k$-rail ($k \geq 3$, and if $k = 3$ the edge $(x, y)$ is not one of the $k$ paths of the $k$-rail), then the resulting graph is in $\mathcal{F}$ (in fact only one edge of each non-separating cycle of $G$ needs to be replaced by a $k$-rail). These examples seem to indicate that no complete catalogue of the graphs in $\mathcal{F}$ or $\mathcal{F}'$ is possible. In the examples $k$-rails play an important role. We show that this is generally so. Specifically, we prove that a 2-connected graph $G$ in $\mathcal{F}'$, which is not a $k$-rail itself, contains at least two 3-rails between different pairs of vertices (the type of graph indicated in Fig. 1 demonstrates that this is best possible). In particular, a 2-connected graph in $\mathcal{F}'$ is either the 4-rail of order 5 or it contains at least four vertices of degree 2.

This result will be shown to imply that a connected graph $G$ with minimum degree 3 or more has an induced cycle $C$ such that $G - V(C)$ is connected. The complete bipartite graph $K_{3,n-3}$, which is 3-connected, shows that $G - V(C)$ need not be 2-connected for any such cycle $C$. However, we show that if $G$ is 3-connected, then there exists an induced cycle $C$ such that $G - V(C)$ is connected and has at most one block containing cycles. On the basis of this result new simple proofs of Tutte's theorem on 3-connected graphs [22] and of Kuratowski's theorem on planar graphs may be given.
The proof of Kuratowski's theorem is more combinatorial in nature than the well-known short proof by Dirac and Schuster [4]: however, we shall only give a brief indication of this proof, because an alternative very short and simple proof was recently found by Thomassen [19].

As a further application we present a proof of a conjecture of Hobbs that every 2-connected graph $G$ of minimum degree 4 or more contains a cycle $C$ such that $G - E(C)$ is 2-connected (this conjecture was first proved by Jackson [10]). We show that in fact $C$ can be chosen such that, in addition, $G - V(C)$ is connected.

We also show that any graph with minimum degree 3 or more contains a cycle with two crossing diagonals from neighbouring vertices on the cycle. This extends the result of Dirac [5] that any such graph contains a subdivision of $K_4$, and it also implies that if the graph, in addition, is non-bipartite and of order at least 5, then it contains an odd cycle with a diagonal. This implies the conjecture of Bollobás and Erdős [7], first proved by Larson [8, 13], that any 4-chromatic graph not containing a $K_4$, contains an odd cycle with a diagonal.

Finally, we give conditions for a graph to contain a vertex joined by edges to three vertices of a cycle (not containing the given vertex) in the graph. This implies the conjecture of Bollobás and Erdős [6] (first proved by Thomassen [17]) that every graph with $n$ vertices ($n \geq 4$) and at least $2n - 2$ edges contains such a configuration. Also, together with the previous result, it implies the conjecture of Toft [21], that any 4-chromatic graph contains a subdivision of the complete graph $K_4$ in which the three edges of a given spanning tree of the $K_4$ are left undivided.

The above results for subdivisions of $K_4$ with some edges left undivided are related to a result on special subdivisions of $K_n$ by Bollobás [1].

2. GRAPHS IN WHICH EVERY INDUCED CYCLE IS SEPARATING

The proof technique of the basic Lemmas 1 and 2 below was used earlier by Krusenstjerna-Hafstrøm and Toft [12].

**Lemma 1.** Let $G$ be a 2-connected graph and let $G'$ be a connected subgraph of $G$ such that $G - V(G')$ contains at least one cycle. Then either $G - V(G')$ contains an induced cycle $C$ such that $G - V(C)$ is connected or there is a connected subgraph $G^*$ of $G$ with $G' \subseteq G^*$ such that $G - V(G^*)$ is a $k$-rail in $G$ with $k \geq 3$.

**Proof.** Let $C$ denote an induced cycle in $G - V(G')$ that maximizes the order of the connected component $G^*$ of $G - V(C)$ that contains $G'$. If $C$ is not separating, the first alternative holds. Hence we shall consider the case where $G - V(C)$ has a connected component $H$ different from $G^*$. 
Since \( G \) is 2-connected there are at least two vertices \( x \) and \( y \) on \( C \) joined by edges to vertices of \( H \). Then \( x \) and \( y \) are joined in \( G - V(G*) \) by a path \( P \) whose interior vertices are all in \( H \). The graph \( C \cup P \) contains three cycles one of which is \( C \). If we compare \( C \) with two induced cycles of the graphs induced by the two other cycles of \( C \cup P \), it follows, by the maximality of \( G* \), that the only vertices of \( C \) that can possibly be joined by edges to \( G* \) are \( x \) and \( y \). On the other hand, both \( x \) and \( y \) are then joined to \( G* \), since \( G \) is 2-connected. Since \( x \) and \( y \) were two arbitrarily chosen vertices of \( C \) joined to vertices of \( H \), it follows that they are the only such vertices. Moreover, if \( G - V(C \cup G*) \) has connected components other than \( H \), then the vertices \( x \) and \( y \) will be the same for any other such connected component. Hence \( \{x, y\} \) is a separating set of \( G \), and the connected components of \( G - V(C) \) are also connected components of \( G - \{x, y\} \) (but \( G - \{x, y\} \) has more connected components than \( G - V(C) \) has).

Again by the maximality of \( G* \), the graphs \( G[V(H) \cup \{x\}] \) and \( G[V(H) \cup \{y\}] \) do not contain any cycles. It follows that \( x \) and \( y \) both have degree 1 in \( G[V(H) \cup \{x, y\}] \) and that this graph is a tree. Since \( G \) is 2-connected this tree cannot have vertices of degree 1 other than \( x \) and \( y \); hence the tree is a path, i.e., \( G[V(H) \cup \{x, y\}] = P \). But then \( C \cup P \) is a 3-rail \( P_3 \) of \( G \).

Since \( H \) was an arbitrary connected component of \( G - V(C) \) different from \( G* \), and since the vertices \( x \) and \( y \) are independent of the chosen \( H \), Lemma 1 follows.

The above proof of Lemma 1 also directly implies the following lemma:

**Lemma 2.** Let \( G \) be a 2-connected graph in which all vertices have degree at least 3 (except perhaps one vertex of degree 2). Let \( G' \) be a connected subgraph of \( G \) such that \( G - V(G') \) contains at least one cycle. If \( C \) is an induced cycle in \( G - V(G') \) that maximizes the order of the connected component \( G* \) of \( G - V(C) \) that contains \( G' \), then \( G - V(C) \) is connected (i.e., \( G - V(C) = G* \)).

**Theorem 1.** Let \( G \) be a 2-connected graph in which every induced cycle is separating. Then there exists a pair of vertices \( \{x, y\} \) such that there is a 3-rail between \( x \) and \( y \) in \( G \). Moreover, either \( G \) is equal to a \( k \)-rail with \( k \geq 4 \) or there exists another pair of vertices \( \{x', y'\} \) (distinct but not necessarily disjoint from \( \{x, y\} \)) such that there is also a 3-rail between \( x' \) and \( y' \) in \( G \).

**Proof.** That there is a 3-rail \( P_3 \) in \( G \) between two vertices \( x \) and \( y \) follows immediately from Lemma 1 with \( G' = \emptyset \). If \( G \) is not equal to a \( k \)-rail between \( x \) and \( y \), then either \( G - (V(P_3) \setminus \{x\}) \) or \( G - (V(P_3) \setminus \{y\}) \), say \( G - (V(P_3) \setminus \{x\}) \), contains a cycle. Then we may use Lemma 1 again, this
time with $G' = P_3 - x$. Hence there is another 3-rail $P'_3$ in $G$ between two vertices $x'$ and $y'$. Since $P'_3 \subseteq G - V(G') = G - (V(P_3) \setminus \{x\})$ it is clear that $\{x, y\} \neq \{x', y'\}$. This proves Theorem 1.

If the first alternative of Lemma 1 is made weaker by removing the word "induced," then the second alternative may be made stronger by adding the requirement that, if $k = 3$, then each of the paths of the 3-rail has length at least 2. It follows that if $G$ is a 2-connected graph in which every cycle is separating, then either $G$ is a $k$-rail or there exist two different pairs of vertices $\{x, y\}$ and $\{x', y'\}$ such that there is a 3-rail between $x$ and $y$ and another 3-rail between $x'$ and $y'$, and each of the six paths of the two 3-rails has length at least 2.

**Corollary 1.** Any 2-connected graph $G$ such that all vertices (except perhaps two, or, if $|V(G)| \leq 6$, perhaps three) have degree at least 3 contains an induced cycle $C$ such that $G - V(C)$ is connected.

Corollary 1 follows immediately from Theorem 1.

**Corollary 2.** Any connected graph $G$ (with $|V(G)| \geq 2$) which has

(i) at most one vertex of degree 1, and

(ii) at most two vertices of degree 2, and

(iii) at most one vertex of degree 2 if there is a vertex of degree 1,

contains an induced cycle $C$ such that $G - V(C)$ is connected.

**Proof.** If $G$ is 2-connected, then Corollary 2 follows immediately from Corollary 1. If $G$ is not 2-connected, then $G$ has at least two endblocks $B_1$ and $B_2$. By (i), we may assume that $B_1$ is 2-connected. Let the cutvertex of $G$ belonging to $B_1$ be denoted by $x_1$. If $B_1 - x_1$ contains a cycle, then we may use Lemma 1 on $B_1$ with $G' = x_1$. If the first alternative of Lemma 1 holds, then $B_1 - x_1$ contains an induced cycle $C_1$ such that $B_1 - V(C_1)$ is connected. Then also $G - V(C_1)$ is connected. Hence we may assume that the second alternative of Lemma 1 holds. This implies that there are two vertices of degree 2 in $G$ contained in $B_1 - x_1$. The same conclusion holds if $B_1 - x_1$ does not contain any cycle, because, by (ii), $B_1 - x_1$ is in this case a path. By (iii), it then follows that $B_2$ is 2-connected. Since there are no vertices of degree 2 in $G$ left to $B_2$ we conclude as above that $B_2 - x_2$ (where $x_2$ is the cutvertex of $G$ belonging to $B_2$) contains an induced cycle $C_2$ such that $B_2 - V(C_2)$, and hence also $G - V(C_2)$, is connected. This proves Corollary 2.

If we just ask for a cycle $C$ (not necessarily induced) such that $G - V(C)$ is connected, then the conditions of Corollaries 1 and 2 may be weakened. For example, the number of exceptional vertices in Corollary 1 may be
raised to 3, or to 5 if $|V(G)| \geq 8$. In Corollary 2 the upper limits 2 and 1 on
the number of vertices of degree 2 in (ii) and (iii) may be raised to 3 and 2,
respectively.

Theorem 1 is best possible as shown by the type of graph in Fig. 1. This
type of graph is 2-connected, any induced cycle is separating, it has only two
pairs of vertices joined by 3-rails, and there need only be four vertices of
degree 2. Also note that condition (ii) of Corollary 2 cannot be relaxed
because of the 4-rail of order 5.

In fact, using Lemma 1, it is possible to prove that any 2-connected graph
in which any induced cycle is separating and which has only two pairs of
vertices joined by 3-rails can be obtained from the type of graph in Fig. 1 by
omitting some edges, subdividing others, and replacing some of the non-
crossing edges joining the upper path (from $x_1$ to $x_2$) to the lower path (from
$y_1$ to $y_2$) by 2-rails. Note that such graphs have many separating sets of two
vertices. For any 2-connected graph $G$ let $s_2(G)$ denote the number of
separating sets of two vertices and let $v_2(G)$ denote the number of vertices of
degree 2. Then maybe the following is true:

**Conjecture.** There is a positive constant $c$ such that for any 2-connected
graph $G$ in $\mathcal{G}$

$$v_2(G) + s_2(G) > c \cdot |E(G)|.$$  

However, instead of going deeper into the structure of the graphs in $\mathcal{G}$ and
$\mathcal{G}'$ we shall use the above results to study certain properties of connected
graphs with minimum degree at least 3. By Lemma 2, every such graph has
an induced non-separating cycle.

A result similar to that of Theorem 1 has recently been announced by
Kelmans [11]. Kelmans' result also follows easily from Lemma 1.

### 3. Non-separating Induced Cycles in 3-Connected Graphs

In this section we consider the block-structure of $G - V(C)$, where $C$ is a
non-separating induced cycle of a 3-connected graph $G$.

**Lemma 3.** Let $G$ be a 3-connected graph. Suppose that $C$ is an induced
cycle of $G$ (which may or may not be separating) that maximizes the order of
the largest block $B^*$ of $G - V(C)$. Then $B^*$ is the only block of $G - V(C)$
that may contain cycles, i.e., any other block of $G - V(C)$ is a $K_2$.

**Proof.** Let $G$, $C$ and $B^*$ be as described, and let $B$ be a block of
$G - V(C)$ different from $B^*$. Denote the connected component of $G - V(C)$
containing $B^*$ by $G^*$. The graph $G[V(G^*) \cup V(C)]$ is 2-connected; hence if
B \notin G^*, then B contains no cycles by the maximality of $B^*$. If $B \subseteq G^*$, then there exists a cutvertex $z$ in $G^*$, such that $G^* = G^*_1 \cup G^*_2$, where $V(G^*_1) \cap V(G^*_2) = \{z\}$ and $B^* \subseteq G^*_1$ and $B \subseteq G^*_2$. The graph $G[V(G^*_1 - z) \cup V(C)]$ is 2-connected and larger than $B^*$; hence again we conclude that $B$ contains no cycles. This proves Lemma 3.

A similar argument gives the following:

**Lemma 4.** Let $G$ be a 3-connected graph and let $x \in V(G)$. Suppose that $C$ is an induced cycle of $G - x$ that maximizes the order of the largest block $B^*$ of $G - V(C)$ containing $x$. Then for any cycle $C'$ of $G - V(C)$, either $C'$ is contained in $B^*$ or $C'$ and $B^*$ have precisely the vertex $x$ in common, and $x$ is a cutvertex of $G - V(C)$.

**Proof.** Let $G$, $C$ and $B^*$ be as described, and let $B$ and $G^*$ be as in the proof of Lemma 3. If $B \notin G^*$, then we conclude as in that proof that $B$ contains no cycles. So assume that $B \subseteq G^*$, and let $G^*_1$, $G^*_2$ and $z$ be as in the proof of Lemma 3, where we may suppose that $z \notin V(B)$. As in that proof the graph $G[V(G^*_1 - z) \cup V(C)]$ is 2-connected and larger than $B^*$; hence either $B$ contains no cycles, by the maximality of $B^*$, or else $z = x \in V(B^*)$.

If $z = x \in V(B^*)$ and $B^* = K_2$, then, by the maximum property of $B^*$, also $B = K_2$ since $x = z \in V(B)$; hence $B$ contains no cycles.

If finally $z = x \in V(B^*)$ and $B^* \neq K_2$, then $z$ has degree at least 2 in $B^*$, and hence $G[V(G^*_1) \cup V(C)]$ is 2-connected since $G[V(G^*_1 - z) \cup V(C)]$ is. In this case $B$ may contain a cycle $C'$, but if it does, then by the maximality of $B^*$, we conclude that $x \in V(C')$, and hence $V(C') \cap V(B^*) = \{x\}$ for any cycle $C'$ of $B$.

This proves Lemma 4.

Again, by similar arguments, we get the following:

**Lemma 5.** Let $G$ be a 3-connected graph and let $(x, y) \in E(G)$. Suppose that $C$ is an induced cycle of $G - \{x, y\}$ that maximizes the order of the largest block $B^*$ of $G - V(C)$ containing $(x, y)$, and suppose furthermore that $B^*$ has at least three vertices. Then, for any cycle $C'$ of $G - V(C)$, either $C'$ is contained in $B^*$, or $C'$ and $B^*$ have precisely one of $x$ or $y$ in common, and the common vertex is a cutvertex of $G - V(C)$.

A graph consisting of a graph of the type in Fig. 1 with precisely one vertex of degree 2 at each end and an edge $(x, y)$ joining these two vertices, shows that the assumption that $B^*$ has at least three vertices is necessary in Lemma 5.

In this and the next section we shall only use Lemma 3. The main result of this section is the following:
THEOREM 2. Let $G$ be a 3-connected graph. Then $G$ contains a non-separating induced cycle $C$ such that $G - V(C)$ has at most one block $B^*$ containing cycles, and such that

either $G - V(C) = B^*$,

or $G$ contains an induced $K_{2,3}$ containing $C$, such that $G - V(K_{2,3})$ is connected and contains $B^*$, and such that the three vertices of degree 2 in $K_{2,3}$ all have degree 3 in $G$,

or $G$ contains an induced $K_4^-$ (i.e., a $K_4$ with one edge missing) containing $C$, such that $G - V(K_4^-)$ is connected and contains $B^*$, and such that the two vertices of degree 2 in the $K_4^-$ both have degree 3 in $G$,

or $G$ contains a $K_4$ containing $C$, such that $G - V(K_4) = B^*$, and such that each vertex of the $K_4$ has degree 4 in $G$ except perhaps one such vertex of degree 3 in $G$.

Before we prove Theorem 2, let us remark that none of the last three alternatives can be left out. This is shown by $K_{3,n-3}$ ($n \geq 7$), the graph of Fig. 2 without the dotted edges, and the graph of Fig. 2 with the dotted edges, respectively. Note that the graphs of Fig. 2 are planar.

Proof of Theorem 2. Let $G$ be 3-connected, and let $C_1$ be an induced cycle of $G$ that maximizes the order of the largest block $B^*$ of $G - V(C_1)$. Let $C$ be an induced cycle of $G - V(B^*)$ that maximizes the order of the connected component $G^*$ of $G - V(C)$ containing $B^*$. Then by Lemma 2, $G - V(C)$ is connected, and by Lemma 3, the block $B^*$ of $G - V(C)$ is the

\[ \text{Figure 2} \]
only block of $G - V(C)$ containing cycles. Hence $G - V(C)$ consists of $B^*$ together with some trees attached to $B^*$. This proves the first part of Theorem 2.

If $G - V(C) = B^*$ we have finished, so assume this is not the case. Then there exists at least one tree $T$ attached to $B^*$ in $G - V(C)$. Let $x$ be an endvertex of $T$ of degree 1 in $G - V(C)$. Hence $x$ is joined to vertices $x_1, x_2, ..., x_k$ with $k \geq 2$ on $C$.

If $k = 2$, then $G[V(C) \cup \{x\}]$ contains two cycles $C_2$ and $C_3$ containing $x$. If $C_2$ has length at least 5, then, since all vertices of $C$ are joined to at least one vertex of $G - V(C)$ by an edge, the graph $G - V(C_1)$ is connected and has order greater than the order of $G - V(C)$, contradicting the maximality of $G* = G - V(C)$. Hence $C_2$ has length at most 4, and similarly, $C_3$ has length at most 4. This implies that $C$ has length either 3 or 4, and if $C$ has length 4, then the vertices $x_1$ and $x_2$ are not neighbours on $C$. It follows that each vertex of $C$ different from $x_1$ and $x_2$ has degree 3 in $G$ (otherwise, two edges from such a vertex to $G - V(C)$ would be contained in a cycle of $G - V(C_2)$ or $G - V(C_3)$, contradicting either the maximality of $B^*$ or Lemma 3). Thus if $C$ has length 4, we have the second alternative of Theorem 2 with $K_{2,3} = G[V(C) \cup \{x\}]$, and if $C$ has length 3, we have the third alternative of Theorem 2 with $K_4 = G[V(C) \cup \{x\}]$.

If $k \geq 3$, then, by the maximality of $G^*$, the cycle $C$ has length $k$, i.e., $x$ is completely joined to $C$. Moreover, since at least one vertex of $C$ is joined by an edge to $G - (V(C) \cup \{x\})$ it follows that $k = 3$ and that $G[V(C) \cup \{x\}] = K_4$. By the maximality of $B^*$ and Lemma 3, it follows that each vertex of $C$ has degree 3 or 4 in $G$. Since $k = 3$ the vertex $x$ has degree 4 in $G$. In order to prove that the fourth alternative of Theorem 2 holds, we shall assume that none of the three first alternatives holds. Since $G$ is 3-connected, it only remains to be shown that $G - (V(C) \cup \{x\}) = B^*$.

We claim that $G - V(C)$ contains no other vertex $y$ of degree 1 in $G - V(C)$ (for if this were the case, then also $y$ has degree 4 and is joined completely to $C$ since otherwise we would have an earlier alternative. Then, since $G$ is 3-connected, $G - \{x, y\}$ has an edge from $C$ to $G - V(C)$ contradicting that each vertex of $C$ has degree 3 or 4 in $G$). Hence $G - V(C)$ consists of $B^*$ with a path $P$ attached to it. If $P$ has length at least 2, then let $y$ be the neighbour of $x$ on $P$. Since $G$ is 3-connected, $y$ is joined to a vertex of $C$, say $x_1$. Since $\{y, x_2\}$ is not a cutset of $G$ the vertex $x_3$ is joined to a vertex of $G - (V(C) \cup \{x, y\})$, and similarly for $x_2$. But then the cycle spanned by $x, y, x_2$ and $x_3$ contradicts the maximality of $B^*$ or Lemma 3.

This proves Theorem 2.

**Corollary 3.** If $G$ is 3-connected and either has girth at least 5 or minimum degree at least 4 (or both), then $G$ has an induced non-separating cycle $C$ such that $G - V(C)$ is a block.
Proof. If \( G \) has girth at least 5, this follows immediately from Theorem 2. In the case where \( G \) has minimum degree at least 4, we proceed by induction. The statement holds for the smallest possible such graph, which is a \( K_5 \). So let \( G \) be a 3-connected graph of minimum degree at least 4, such that the statement holds for any such graph smaller than \( G \). If the first alternative of Theorem 2 holds for \( G \), we are finished; hence we may assume that the fourth alternative holds with all four vertices \( x_1, x_2, x_3 \), and \( x_4 \) of the \( K_4 \) having degree 4 in \( G \).

If the neighbours of \( x_1, x_2, x_3 \) and \( x_4 \) in \( G - \{x_1, x_2, x_3, x_4\} \) are \( y_1, y_2, y_3 \) and \( y_4 \), respectively, and these are all distinct, then let \( G' \) denote a new graph obtained from \( G - \{x_1, x_2, x_3, x_4\} \) by joining a new vertex \( x' \) to each of \( y_1, y_2, y_3 \) and \( y_4 \) by an edge. This graph \( G' \) is 3-connected and has minimum degree 4; hence by the induction hypothesis, \( G' \) contains an induced cycle \( C' \) such that \( G' - V(C') \) is a block. If \( x' \notin V(C') \), then also \( G - V(C') \) is a block. If \( x' \in V(C') \), then \( C' \) contains precisely two of \( y_1, y_2, y_3 \) and \( y_4 \), say \( y_1 \) and \( y_2 \), since \( C' \) is induced. But then the induced cycle \( C = G[V(C' - x') \cup \{x_1, x_2\}] \) of \( G \) has the property that \( G - V(C) \) is a block.

If the neighbours of \( x_1, x_2, x_3 \) and \( x_4 \) in \( G - \{x_1, x_2, x_3, x_4\} \) are \( y_1, y_2, y_3, y_4 \) and \( y_3 = y_4 \), then the cycle of \( G \) spanned by \( x_3, x_4 \) and \( y_3 \) has the desired property. For \( G - y_3 \) is 2-connected, since \( G \) is 3-connected, and in \( G - y_3 \) the vertices \( x_3 \) and \( x_4 \) are only joined by edges to each other and to the neighbouring vertices \( x_1 \) and \( x_2 \); hence also \( G - \{y_3, x_2, x_3\} \) is 2-connected.

This proves Corollary 3.

In [14] Lovász proposed the problem of finding a function \( f \) (and the best such function, if possible) with the following property: If \( G \) is \( f(k) \)-connected, then \( G \) contains a cycle \( C \) such that \( G - V(C) \) is still \( k \)-connected. Lovász remarked that by Tutte's theorem [22] \( f(1) = 3 \), and our results in Section 2 may be regarded as extensions of this fact. Moreover the results presented above may be regarded as extensions of the problem of determining \( f(2) \). Corollary 3 implies that any 4-connected graph \( G \) has an induced cycle \( C \) such that \( G - V(C) \) is a block, and it is easy to see that \( C \) can be chosen such that the block \( G - V(C) \) has at least three vertices unless \( G = K_5 \).

**Corollary 4.** Let \( G \) be a 3-connected graph of girth at least 5. Then \( G \) contains an induced cycle \( C \) such that for any subset \( E' \) of \( E(C) \), the contraction of all edges of \( E' \) results in a 3-connected graph, unless \( E' \) consists of all edges of \( C \) except two, which are both incident with the same vertex \( x \) of \( C \), and \( x \) has degree 3 in \( G \).

**Proof.** By Corollary 3, \( G \) contains an induced cycle \( C \) such that
$G - V(C)$ is a block. Let $G_1$ and $G_2$ be two connected subgraphs of $G$, where for $i = 1, 2$ $G_i$ is either a single vertex of $G - V(C)$ or $G_i \subseteq C$. By considering the three cases (i) $G_1 \subseteq C$ and $G_2 \subseteq C$, (ii) $G_1$ and $G_2$ are both single vertices of $G - V(C)$, and (iii) $G_1 \subseteq C$ and $G_2$ is a single vertex of $G - V(C)$, it is easy to see that $G - (V(G_1) \cup V(G_2))$ is connected, unless $G_1 = C - x$, where $x$ is a vertex of $C$ of degree 3 in $G$, and $G_2$ is the single neighbour of $x$ in $G - V(C)$. From this it follows that the graph $G'$ obtained from $G$ by contracting a set $E'$ of edges of $C$ cannot have a cutset consisting of two vertices, unless we have the exceptional situation described in Corollary 4. Hence Corollary 4 follows.

**COROLLARY 5.** Let $G$ be a 3-connected graph of girth at least 4. Then $G$ contains a cycle $C$ such that for any edge $e$ of $C$, the contraction of $e$ results in a 3-connected graph.

**Proof.** Let $C$ and $B^*$ be as in Theorem 2. If $G - V(C) = B^*$, then let $e = (x_1, x_2)$ be an edge of $C$, and let $G'$ denote the graph obtained from $G$ by contracting $e$ into a single vertex $z$. If $G'$ is not 3-connected, then $G'$ has a cutset of two vertices, one of which must be $z$. Denote the other by $y$. Then $\{x_1, x_2, y\}$ is a cutset of $G$. But then $y \in V(B^*)$ and, since $G$ contains no triangles, there exists a vertex of $C - \{x_1, x_2\}$ joined by an edge to a vertex of $B^* - y$. Hence $G - \{x_1, x_2, y\}$ is connected. This contradiction shows that $G'$ is 3-connected.

If $G - V(C) \neq B^*$, then the situation is as described in the second alternative of Theorem 2, since $G$ contains no triangles. In this case $G - V(K_{2,3})$ consists of $B^*$ with possibly one or more trees attached. Let in this case $e = (x_1, x_2)$ be an edge of the $K_{2,3}$ with $x_1$ of degree 2 in the $K_{2,3}$. Let $G'$ denote the graph obtained by contracting $e$ into a single vertex $z$. If $G'$ is not 3-connected, then $G'$ has a cutset of two vertices, one of which must be $z$. Denote the other by $y$. Then $\{x_1, x_2, y\}$ is a cutset of $G$, and $y$ is a cutvertex of $G - \{x_1, x_2\}$, hence $y \in V(G) \setminus V(K_{2,3})$.

Consider any connected component $H$ of $G - \{x_1, x_2, y\}$ with $H \subseteq G - V(K_{2,3})$. Then $H$ is joined in $G$ by edges to precisely the vertices $x_1$ and $x_2$ of the $K_{2,3}$. Moreover, $H$ does not contain any vertex which has degree 1 in $G - V(K_{2,3})$, since such a vertex would be joined to $x_1$ and $x_2$ and thus be contained in a triangle, contradicting that the girth of $G$ is at least 4. But then it follows from the structure of $G - V(K_{2,3})$ that there is at most one cutvertex $y'$ of $G - V(K_{2,3})$ with $y' \in V(B^*)$, i.e., $B^*$ is an endblock of $G - V(K_{2,3})$ (otherwise there would be two trees attached to two different vertices of $B^*$ in $G - V(K_{2,3})$, and $H$ would contain a vertex which has degree 1 in $G - V(K_{2,3})$).

Consider first the case $G - V(K_{2,3}) \neq B^*$. In this case $G - V(K_{2,3})$ consists of $B^*$ with one or more trees attached to $y'$. Then $B^* - y' \subseteq H$
(otherwise \( H \) would contain an endvertex of such a tree and thus contain a vertex which has degree 1 in \( G - V(K_{2,3}) \)). Hence \( B^* - y' \) is joined by edges to at most the vertices \( x_1 \) and \( x_2 \) of the \( K_{2,3} \). However, \( G \) is 3-connected, so \( B^* - y' \) is joined to precisely the vertices \( x_1 \) and \( x_2 \) of the \( K_{2,3} \). But then we cannot have this situation if we consider any edge of the \( K_{2,3} \) other than \((x_1, x_2)\). Thus the contraction of any edge of the \( K_{2,3} \) other than \((x_1, x_2)\) results in a 3-connected graph and so the 4-cycle \( K_{2,3} - x_1 \) may be used as the desired cycle.

In the case \( G - V(K_{2,3}) = B^* \) we have \( H = B^* - y \). The two vertices \( x_3 \) and \( x_4 \) other than \( x_1 \) of degree 2 in \( K_{2,3} \) (and of degree 3 in \( G \)) are both joined to exactly the same vertex (which must be \( y \)) outside \( K_{2,3} \), since \( B^* - y \) is a connected component of \( G - \{x_1, x_2, y\} \). The vertex \( x_1 \) is joined to exactly one vertex \( x_1^* \) outside \( K_{2,3} \), and \( x_1^* \neq y \) since, otherwise, \( \{y, x_2\} \) would be a cutset of \( G \). Then again we cannot have a similar situation for any edge of the 4-cycle \( K_{2,3} - x_1 \), and we finish as before.

This proves Corollary 5.

Corollary 5 raises the following question: If \( G \) is a \( k \)-connected graph of girth at least 4 and \( G' \) is the spanning subgraph of \( G \) consisting of those edges \( e \) of \( G \) for which the contraction of \( e \) results in a \( k \)-connected graph, then what can be said about the structure of \( G' \)? In [20] it was shown that \( G' \) has at least one edge, and Corollary 5 shows that \( G' \) cannot be a forest when \( k = 3 \).

As a further corollary of Theorem 2 it is possible to give a short proof of the non-trivial part of Kuratowski's theorem on planar graphs, which says that if \( G \) is a graph not containing any subdivision of \( K_{3,3} \) or \( K_5 \), then \( G \) can be embedded in the plane. The proof is by induction on the number of edges of \( G \). If \( G \) is not 3-connected, we finish in the usual way (see, e.g., [3] or [4]). For a 3-connected graph \( G \) satisfying the hypothesis of Kuratowski's theorem, either the first or the third or the fourth alternative (with a vertex \( x \) of the \( K_4 \) having degree 3) of Theorem 2 holds. If the first alternative holds and \( |V(B^*)| \leq 2 \), then it is easy to finish. If the first alternative holds and \( |V(B^*)| \geq 3 \) then contract an edge of \( C \). If the third alternative holds, then remove the edge which is a diagonal in the 4-cycle of the \( K_4^- \). If the fourth alternative holds, then remove \( x \). Clearly, in each case the resulting graph \( G' \) contains no subdivision of \( K_{3,3} \) or \( K_5 \), and from a planar embedding of \( G' \), which exists by the induction hypothesis, it is now easy to find an embedding of \( G \).

For the reason explained in the Introduction we shall not go deeper into the proof.
4. TUTTE'S THEOREM ON 3-CONNECTED GRAPHS

COROLLARY 6 (Tutte [22]). Let $G$ be a 3-connected graph which is not a wheel. Then $G$ contains an edge $e$ such that either the graph obtained from $G$ by deleting $e$ or the graph obtained from $G$ by contracting $e$ (where $e$ is not contained in a $K_3$ in this case) is 3-connected.

Proof. By Corollary 5, we may assume that $G$ contains a $K_3$. Let its vertices be $x$, $y$ and $z$. We may assume that $G - \{x, y, z\}$ is connected, because otherwise there are three internally disjoint paths each of length at least 2 from $x$ to $y$, and $G - (x, y)$ is 3-connected.

If $G - \{x, y, z\}$ is not a block, then let $B_1$ and $B_2$ be two endblocks of $G - \{x, y, z\}$ containing the cutvertices $x_1$ and $x_2$ of $G - V(K_3)$, respectively. Since $G$ is 3-connected we may assume that $x$ has a neighbour in $B_1 - x_1$, that $y$ has a neighbour in $B_2 - x_2$, and that $z$ has neighbours in both $B_1 - x_1$ and $B_2 - x_2$, because otherwise two of $x$, $y$ and $z$, say $x$ and $y$, would both be joined to $B_1 - x_1$ and to $B_2 - x_2$, and $G - (x, y)$ would be 3-connected. If $x$ has degree at least 4, then $G - (x, z)$ contains three internally disjoint paths from $x$ to $z$ and is thus 3-connected. Hence we may assume that $x$, and similarly $y$, has degree 3 in $G$.

The same conclusion holds if $G - V(K_3)$ is a block. That is, if two of $x$, $y$ and $z$, say $x$ and $z$, in this case both have degree at least 4, then $G - (x, z)$ is 3-connected.

If the neighbour of $x$ different from $y$ and $z$, say $u$, is not joined to $y$ or $z$, then by contracting $(x, u)$ the resulting graph is 3-connected. This follows since $G - u$ is 2-connected and $x$ is only joined to two neighbouring vertices in $G - u$, hence also $G - \{u, x\}$ is 2-connected.

Hence $G$ contains triangles, and in any triangle at least two vertices have degree 3, and any vertex of degree 3 in a triangle is contained in at least two triangles. This implies easily that $G$ is a wheel, and hence Corollary 6 has been proved.

Only a much weaker statement than that of Corollary 5 is needed to prove Corollary 6. An alternative formulation of the proof of Corollary 6 using only the first part of Theorem 2 may be given. This may be regarded as an easy proof of Tutte's theorem. Halin [9] also gave a simple proof. This proof was based on the existence of a vertex of degree 3 in a minimally 3-connected graph. We note that this result also follows immediately from Theorem 2. For if $G$ has minimum degree at least 4, then in the first alternative of Theorem 2, $G - e$ is 3-connected for any edge $e$ of $C$, and in the fourth alternative $G - e$ is 3-connected for any edge $e$ of the $K_4$. 

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5. A Conjecture of Hobbs

Mader [15] proved that any \( n \)-connected graph \( G \) \((n \geq 2)\) contains a cycle \( C \) such that \( G - E(C) \) is \((n-2)\)-connected; in particular, any 4-connected graph contains a cycle \( C \) such that \( G - E(C) \) is 2-connected. Hobbs (see [10]) conjectured that any 2-connected graph with minimum degree at least 4 contains a cycle \( C \) such that \( G - E(C) \) is 2-connected. Jackson [10] proved a stronger version of this conjecture. Here we extend conjecture in a different direction:

**Theorem 3.** Let \( G \) be a 2-connected graph of minimum degree at least 4. Then \( G \) contains an induced cycle \( C \) such that \( G - V(C) \) is connected and \( G - E(C) \) is 2-connected.

In the proof of Theorem 3 we shall use the following lemma:

**Lemma 6.** Let \( G \) be a 3-connected graph and let \( C \) be a non-separating cycle of length 3 in \( G \), where each vertex of \( C \) has degree at least 4 in \( G \). Then \( G - E(C) \) is 2-connected.

**Proof.** \( G - V(C) \) is connected, hence also \( G - E(C) \) is connected. Since \( G \) is 3-connected each endblock of \( G - V(C) \) is joined by edges to at least two of the three vertices of \( C \). Since each vertex of \( C \) has degree at least 4 in \( G \) this implies that \( G - E(C) \) has no cutvertex. Hence \( G - E(C) \) is 2-connected, and Lemma 6 has been proved.

**Proof of Theorem 3**

If \( G \) is 3-connected, then Theorem 3 follows by Corollary 3, because if \( G - V(C) \) is a block, then \( G - E(C) \) is 2-connected. Hence we may assume that \( G \) contains a cutset \( \{x, y\} \), and we select this cutset in such a way that the smallest connected component \( H \) of \( G - \{x, y\} \) is as small as possible. Let \( H^* = G[V(H) \cup \{x, y\}] \cup \{x, y\} \) (where we only add the edge \((x, y)\) if it is not already present in \( G[V(H) \cup \{x, y\}] \)). Then \( H^* \) is 3-connected. Assume that in \( H^* \) the degree of \( y \) is greater than or equal to the degree of \( x \).

**Case 1.** \( y \) has degree at least 4 in \( H^* \). In this case \( H^* - x \) contains a cycle since it has minimum degree at least 3, and we select an induced cycle \( C_1 \) of \( H^* - x \) that maximizes the order of the block \( B^* \) of \( H^* - V(C_1) \) containing \( x \). Clearly, \( |V(B^*)| \geq 2 \).

Let \( C \) be an induced cycle of \( H^* - V(B^*) \) that maximizes the order of the connected component \( G^* \) of \( H^* - V(C) \) containing \( B^* \). Then by Lemma 2, \( H^* - V(C) \) is connected, and since \( x \) belongs to \( H^* - V(C) \), also \( G - V(C) \) is connected. By Lemma 4, any cycle \( C' \) of \( H^* - V(C) \) is either contained in \( B^* \), or \( C' \) and \( B^* \) have precisely \( x \) in common.
If $H^* - V(C) = B^*$, then $H^* - E(C)$ is 2-connected, since each vertex of $C$ has degree at least 4 in $H^*$ (here we use that the degree of $y$ is at least 4 in case $y \in V(C)$). Then also $G - E(C)$ is 2-connected, and we have found the desired cycle. Hence we may assume that $H^* - V(C)$ contains more than one block, and hence it has an endblock $B \neq B^*$.

If $B = K_2$, then $B$ contains a vertex $z$ of degree 1 in $H^* - V(C)$ and, by the maximality of $G^*$, the cycle $C$ is a $K_3$. But then $C$ may be used as the desired cycle by Lemma 6.

If $B \neq K_2$ then $B$ contains a cycle. By Lemma 4, the cutvertex of $B$ in $G^*$ is $x$, and $B - x$ is a tree $T$ with at least two vertices. An endvertex $z$ of $T$ has degree 2 in $H^* - V(C)$; hence $z$ is joined to at least two vertices of $C$. If $z$ is joined to two neighbours on $C$, then again, by the maximality of $G^*$, the cycle $C$ has length 4, say with vertices $t_1, t_2, t_3$ and $t_4$ in this order, and we may assume that $z$ is joined to $t_1$ and $t_3$, and not to $t_2$ and $t_4$. The graph $H^* - t_2$ is 2-connected, and in $(H^* - t_2) - \{(z, t_1), (z, t_3)\}$ it is still possible to find two internally disjoint paths from $z$ to $t_1$ and two other such paths from $z$ to $t_3$ (this may be seen by considering two cases: (i) $t_1$ is joined to a vertex of $T - z$ and (ii) $t_1$ is joined to a vertex of $H^* - V(T)$. In case (i) the first path from $z$ to $t_1$ has all its interior vertices in $T$, and the second path starts with the edge $(z, x)$ and has only the vertex $z$ in $T$. In case (ii) the first path from $z$ to $t_1$ goes via another endvertex of $T$ and $t_4$, and the second path starts with the edge $(z, x)$ and has all its interior vertices in $H^* - (V(C) \cup V(T))$. Similarly for $t_3$.

Hence $(H^* - t_2) - \{(z, t_1), (z, t_3)\}$ is 2-connected and, since $t_2$ has degree at least 4 in $H^*$, also $H^* - \{(z, t_1), (t_1, t_2), (t_2, t_3), (t_3, z)\}$ is 2-connected. Thus the 4-cycle $H^*[\{z, t_1, t_2, t_3\}]$ may be used as the desired cycle.

Case 2. $H^*$ contains two disjoint cycles, one of which contains the edge $(x, y)$. In this case we let $C_1$ be an induced cycle of $H$ that maximizes the order of the largest block $B^*$ of $H^* - V(C_1)$ containing the edge $(x, y)$. Clearly $|V(B^*)| \geq 3$. Again let $C$ denote an induced cycle of $H^* - V(B^*)$ that maximizes the order of the connected component $G^*$ of $H^* - V(C)$ containing $B^*$.

Again by using Lemma 2, and Lemma 5 instead of Lemma 4, we conclude as in Case 1 that the desired cycle exists.

Case 3. We are left with the case where both $x$ and $y$ have degree 3 in $H^*$, and where $H^*$ does not contain two disjoint cycles one of which contains the edge $(x, y)$.

Since $H^*$ is 3-connected, the graph $H^* - (x, y)$ contains two internally disjoint paths $P_1$ and $P_2$ from $x$ to $y$. Let the vertices of $P_1$ and $P_2$ be $x, z_1, z_2, \ldots, z_k, y$ and $x, t_1, t_2, \ldots, t_p, y$ in this order. Then necessarily $V(H^*) = V(P_1 \cup P_2)$, because otherwise $H^* - V(P_1 \cup P_2)$ contains a connected
component joined to at least two of the internal vertices on either $P_1$ or $P_2$ (since $H^*$ is 3-connected and both $x$ and $y$ have degree 3 in $H^*$), and then $H^*$ contains two disjoint cycles one of which contains $(x, y)$, which is a contradiction.

The paths $P_1$ and $P_2$ are both spanned subgraphs of $H^* - (x, y)$, since otherwise it would be possible to replace one of $P_1$ or $P_2$, say $P_1$, by a shorter path $P'_1$, contradicting that we must also have $V(H^*) = V(P'_1 \cup P_2)$. Since $z_1$ has degree $\geq 4$ in $H^*$ it follows that $z_1$ is joined by edges to at least two of $t_1, t_2, \ldots, t_p$. In fact, $z_1$ must be joined to $t_1$ and $t_2$, since otherwise it would be possible to replace $P_1$ and $P_2$ by two paths $P'_1$ and $P'_2$, where $t_2 \notin V(P'_1 \cup P'_2)$. But again we must have $V(H^*) = V(P'_1 \cup P'_2)$; hence this is a contradiction. Then by Lemma 6, the triangle $H^* \{z_1, t_1, t_2\}$ may be used as the desired cycle $C$.

This completes the proof of Theorem 3.

It is not true that any 2-connected graph with minimum degree at least 4 necessarily contains an induced cycle $C$ such that $G - V(C)$ is a block. A simple counterexample is shown in Fig. 3. However:

**Theorem 4.** Let $G$ be a 2-connected graph with minimum degree at least 5. Then $G$ contains an induced cycle $C$ such that $G - V(C)$ is 2-connected.

**Outline of a proof of Theorem 4.** If $G$ is 3-connected, we finish by Corollary 3. If $G$ is only 2-connected, we define $x, y, H$ and $H^*$ as in the proof of Theorem 3. We first assume that we have Case 2 in the proof of Theorem 3 and we define $B^*, G^*$ and $C$ as in that case.

If $H^* - V(C) = B^*$, the cycle $C$ may be used as the desired cycle. Hence let $B$ be an endblock of $H^* - V(C)$ with $B \neq B^*$. If $B = K_2$, then a vertex $z$ of $B$ has degree 1 in $H^* - V(C)$, and hence $z$ is joined to at least four vertices of $C$. On the other hand, $C$ is a $K_3$ by the maximality of $G^*$. This is a contradiction. Hence $B \neq K_2$. By Lemma 5, we may assume that the cutvertex of $B$ in $G^*$ is $x$ and that $B - x$ is a tree $T$ with at least two vertices. Let $z_1$ be any vertex of degree 1 in $T$. Then $z_1$ is joined to at least three vertices of $C$; hence by the maximality of $G^*$, the cycle $C$ is a $K_3$. Since $H^*$
is 2-connected at least one vertex $t_1$ of $C$ is joined to a vertex of the connected component of $H^* - (V(C) \cup \{x\})$ containing $y$. Let the two other vertices of $C$ be $t_2$ and $t_3$. The cycle $C' = H^* \setminus \{z_1, t_2, t_3\}$ is non-separating, and as before, by Lemma 5, the only possible cutvertices of $H^* - V(C')$ are $x$ and $y$. However, it is easy to exclude both possibilities (remember that $t_1$ is joined to an endvertex of $T$ other than $z_1$). Hence $C'$ may be used as the desired cycle.

We are left with the case where $H^*$ does not contain two disjoint cycles one of which contains the edge $(x, y)$. Define $P_1$ and $P_2$ as in Case 3 of the proof of Theorem 3. If $V(H^*) \neq V(P_1 \cup P_2)$ we let $T'$ be a connected component of $H^* - V(P_1 \cup P_2)$. Then $T'$ is a tree and any endvertex of $T'$ is joined to at least four vertices of $P_1 \cup P_2$ (or at least five if $T'$ is a single vertex). But then it is easy to find two disjoint cycles one of which contains $(x, y)$. This is a contradiction; hence $V(H^*) = V(P_1 \cup P_2)$. Let the vertices of $P_1$ and $P_2$ be named as in the proof of Theorem 3. Then $z_1$ is joined to at least three of $t_1, t_2, \ldots, t_p, y$. But then it is easy to replace $P_1$ and $P_2$ by two paths $P'_1$ and $P'_2$, where $V(H^*) \neq V(P'_1 \cup P'_2)$. Again this gives a contradiction.

This proves Theorem 4.

6. ON THE EXISTENCE OF A CYCLE WITH TWO CROSSING DIAGONALS FROM NEIGHBOURING VERTICES OF THE CYCLE

A cycle with two crossing diagonals from neighbouring vertices of the cycle is a subdivision of $K_4$ in which the three edges of a Hamiltonian path of the $K_4$ are left undivided. We shall follow the notation of [12] and call such a special subdivision a $K_4H$. Toft [21] conjectured that any 4-chromatic graph contains a $K_4H$, and Krusenstjerna-Hafström and Toft [12] proved a best possible extremal result for the existence of a $K_4H$. We shall prove a result which implies both the conjecture and the extremal result and extends a result of Dirac [5]:

**Theorem 5.** Let $G$ be a graph with at least two vertices in which all vertices have degree at least 3, except perhaps one vertex $x_0$. Then $G$ contains a $K_4H$.

**Proof.** The proof is by induction on the number $n$ of vertices of $G$. If $n \leq 4$, then the theorem is true. Hence assume that $G$ has $n$ vertices, with $n \geq 5$, and that the theorem is true for any graph smaller than $G$.

If $G$ is not connected, then we apply the induction hypothesis on a connected component of $G$, and if $G$ is connected, but not 2-connected, then
we apply the induction hypothesis on an endblock of $G$ not containing the exceptional vertex $x_0$ (except perhaps as the cutvertex).

If $G$ is 2-connected, but not 3-connected, then let $\{x, y\}$ be a cutset of $G$ selected in such a way that the smallest connected component $H$ of $G - \{x, y\}$ not containing $x_0$ is as small as possible. Then $H^* = G[V(H) \cup \{x, y\}] \cup \{(x, y)\}$ (where we only add the edge $(x, y)$ if it is not already present in $G[V(H) \cup \{x, y\}]$) is 3-connected. If $G$ is 3-connected, we define $H^* = G$ and $(x, y)$ as any edge of $G$.

We shall prove that $H^*$ contains a $K_4H$, where $(x, y)$ is not one of the undivided edges. From this Theorem 5 follows.

If $H^*$ contains two disjoint cycles one of which contains the edge $(x, y)$, then let $C_1$ be an induced cycle of $H^* - \{x, y\}$ that maximizes the order of the block $B^*$ of $H^* - V(C_1)$ containing the edge $(x, y)$. If $H^*$ does not contain two disjoint cycles one of which contains $(x, y)$, then let $C_1$ be an induced cycle of $H^* - x$ that maximizes the order of the block $B^*$ of $H^* - V(C_1)$ containing the vertex $x$.

Let $C$ be an induced cycle of $H^* - V(B^*)$ that maximizes the order of the connected component $G^*$ of $H^* - V(C)$ containing $B^*$. Then by Lemma 2, $H^* - V(C)$ is connected, and by Lemmas 4 and 5, any cycle $C'$ of $H^* - V(C)$ is either contained in $B^*$ or has precisely $x$ or $y$ in common with $B^*$.

If $H^* - V(C) = B^*$, we take a vertex $z$ of $C$ and its two neighbours $p$ and $q$ on $C$, where we may assume that $p \neq y \neq q$. Let $p^*$, $z^*$ and $q^*$ denote vertices of $B^*$ joined to $p$, $z$ and $q$, respectively. If $p^* \neq z^*$, there exist two internally disjoint paths in $B^*$ from $q^*$ to $p^*$ and $z^*$. These two paths together with $(p, p^*)$, $(z, z^*)$, $(q, q^*)$ and $C$ are a $K_4H$ with $(q^*, q)$, $(q, z)$ and $(z, p)$ left undivided. If $p^* = z^*$, then a path from $p^*$ to $q^*$ in $B^*$ together with $(p, p^*)$, $(z, z^*)$, $(q, q^*)$ and $C$ is a $K_4H$ with $(p^*, p)$, $(p, z)$ and $(z, q)$ left undivided. Hence we may assume that $H^* - V(C)$ contains more than one block, and hence it has an endblock $B \neq B^*$.

If $B = K_2$, then it contains a vertex $z$ of degree 1 in $H^* - V(C)$. Now $z$ is joined to at least two vertices of $C$, and by the maximality of $G^*$, the cycle $C$ has length either 3 or 4. If $C$ has length 3, then $H^* [ V(C) \cup \{z\} ]$ is either a $K_4$ or a $K_4$, and it is easy to find the desired subdivision. If $C$ has length 4, say with vertices $t_1$, $t_2$, $t_3$ and $t_4$ in this order, where $z$ is joined to $t_1$ and $t_3$, then $t_2$ and $t_4$ are both joined to the connected graph $H^* - V(C) - z$, and hence it is easy to find the desired subdivision of $K_4$ with all the edges of $C$ left undivided.

We are left with the situation where any endblock of $H^* - V(C)$ different from $B^*$ contains cycles, in particular $B$ does. From the structure of $H^* - V(C)$ we may assume that the cutvertex of $B$ is $x$ and that $B - x$ is a tree $T$ with at least two vertices, among them two endvertices $z_1$ and $z_2$. The unique path in $T$ from $z_1$ to $z_2$ together with the vertex $x$ and the edges
(x, z₁) and (x, z₂) is a cycle, which is disjoint from C. Hence z₁ ≠ y ≠ z₂, because otherwise one of the edges (x, z₁) or (x, z₂) would be equal to (x, y) and belong to B*, by the definition of B*.

Since z₁ and z₂ both have degree at least 3 in H* they are joined to vertices t₁ and t₂ on C, respectively. There is a vertex t₁ on C different from t₁ joined to another endblock of H* - V(C) than B. But then it is easy to find the desired subdivision of K₄ with either (t₁, z₁), (z₁, x), (x, z₂) (in case t₂ ≠ t₁) or (t₂, z₂), (z₂, x), (x, z₁) (in case t₂ = t₁) left undivided.

This proves Theorem 5.

A K₃-cockade is a graph defined recursively as follows:

(i) a K₃ is a K₃-cockade,

(ii) if G₁ and G₂ are two disjoint K₃-cockades and e₁ ∈ E(G₁) for i = 1, 2, then the graph obtained from G₁ and G₂ by identifying e₁ and e₂ (and their respective endvertices) is a K₃-cockade.

A K₃-cockade with n vertices has 2n - 3 edges, and it does not contain any K₄H. We now obtain

**Corollary 7** [12]. If G has n vertices (n ≥ 4) and at least 2n - 3 edges, then G contains a K₄H unless G is a K₃-cockade.

**Proof.** Suppose G is a counterexample of least possible order n. Then n ≥ 5. By Theorem 5, G contains a vertex x of degree at most 2; hence G - x has n - 1 vertices and at least 2(n - 1) - 3 edges. Since G - x is not a counterexample, a contradiction easily follows.

**Corollary 8.** Let G be a 2-connected non-bipartite graph with at least five vertices and with minimum degree at least 3. Then G contains a subdivision of K₄ in which a 4-cycle becomes an odd cycle and one of the two remaining edges is left undivided.

**Proof.** By Theorem 5, there is a K₄H in G. Let the four branch-vertices of the K₄H be x₁, x₂, x₃ and x₄ and let (x₁, x₂), (x₂, x₃) and (x₃, x₄) be the undivided edges. Then it is easy to see that, if we do not already have the desired situation, then the lengths of the paths P₁₃ from x₁ to x₃ in K₄H - {x₂, x₄} and P₂₄ from x₂ to x₄ in K₄H - {x₁, x₃} have the same parity, and moreover the length of the path P₁₄ from x₁ to x₄ in K₄H - {x₂, x₃} is odd.

Since G is 2-connected and non-bipartite any edge of G is contained in an odd cycle; hence since |V(G)| ≥ 5 and each vertex has degree at least 3, there exists an odd cycle C such that C is not contained in the K₄H. But C may be chosen such that it has at least two vertices in common with the K₄H. The cycle C' = K₄H - {x₁, x₂}, (x₃, x₄}) is even and has at least two vertices in common with the odd cycle C. Then there exists a segment P of C joining
two vertices $x$ and $y$ on $C'$ such that $P$ has only $x$ and $y$ in common with $C'$, and such that the parity of the length of $P$ is different from the parity of the two segments into which $x$ and $y$ divide $C'$ (otherwise $C$ has a 2-colouring, which is a contradiction). But then the desired subdivision exists in $K_4H \cup P$ with either $(x_1, x_2)$, $(x_2, x_3)$ or $(x_3, x_4)$ as the undivided edge.

This proves Corollary 8.

**Corollary 9.** Let $G$ be a 4-chromatic graph. Then

(a) $G$ contains a $K_4H$, and

(b) if $G$ contains no $K_4$, then $G$ contains a subdivision of $K_4$ in which a 4-cycle becomes an odd cycle and one of the two remaining edges is undivided.

**Proof.** It is sufficient to prove Corollary 9 for 4-critical graphs $G$. But such a graph is 2-connected with minimum degree at least 3, hence Corollary 9 follows immediately from Theorem 5 and Corollary 8.

Corollaries 8 and 9(b) are extensions of conjectures by Bollobás and Erdős [7], first proved by Larson [13], implying that any 4-chromatic graph not containing a $K_4$ contains an odd cycle with a diagonal. Extensions of this result in other directions have been obtained by Voss [25].

**Corollary 10.** If $G$ is a 2-connected non-bipartite graph with $n$ vertices ($n \geq 5$) and at least $2n - 3$ edges, then $G$ contains an odd cycle with a diagonal unless $G$ consists of a $K_2$ completely joined to $n - 2$ independent vertices.

**Outline of proof.** By induction on $n$. For $n = 5$ the result is true. If $G$ has minimum degree at least 3, the result follows by Corollary 8. Hence let $x$ be a vertex of degree 2 in $G$, and let $x_1$ and $x_2$ be the neighbours of $x$ in $G$.

If $(x_1, x_2) \in E(G)$, then either there is an even cycle in $G - x$ containing $(x_1, x_2)$ or there is an odd such cycle. In the first case we finish directly and in the second case we apply the induction hypothesis on $G - x$, and either we obtain an odd cycle with a diagonal or $G$ consists of a $K_2$ completely joined to $n - 2$ independent vertices.

If $(x_1, x_2) \notin E(G)$, then there is a connected component $H_i$ of $G - \{x, x_1, x_2\}$ such that $G_i = G[V(H_i) \cup \{x_1, x_2\}]$ has $n_i$ vertices and at least $2n_i - 3$ edges. Since $(x_1, x_2) \notin E(G)$ we have $n_i \geq 4$. If $n_i = 4$, then $G_i$ is a $K_4^-$ and $G[V(G_i) \cup \{x\}]$ is a cycle of length 5 with two diagonals. If $n_i = 5$, then the number of edges in $G_i$ is at least 7, and again it is easy to find an odd cycle with a diagonal in $G[V(G_i) \cup \{x\}]$. Hence we may assume that $n_i \geq 6$.

If $G_i$ is bipartite with $x_1$ and $x_2$ in the same colour class, then we add the edge $(x_1, x_2)$ to $G_i$ and use induction to obtain an odd cycle $C$ in
If \( G_i \) is bipartite with \( x_1 \) and \( x_2 \) in different colour classes, or if \( G_i \) is non-bipartite and not 2-connected, then we identify \( x_1 \) and \( x_2 \) in \( G_i \) and apply the induction hypothesis on the resulting graph which is 2-connected and non-bipartite. Then \( G[V(G_i) \cup \{x\}] \) contains the desired cycle.

If finally \( G_i \) is both non-bipartite and 2-connected, we apply the induction hypothesis on \( G_i \).

This proves Corollary 10.

7. **ON THE EXISTENCE OF A VERTEX JOINED BY EDGES TO THREE VERTICES OF A CYCLE**

A vertex joined by edges to three vertices of a cycle (not containing the vertex) gives rise to a subdivision of \( K_4 \) in which the three edges of a \( K_{1,3} \) of the \( K_4 \) are left undivided. We shall call such a special subdivision a \( K_4 T \). Toft [21] conjectured that any 4-chromatic graph contains a \( K_4 T \), and Thomassen [17] proved a best possible extremal result for the existence of a \( K_4 T \). Again, we shall prove a result which implies both the conjecture and the extremal result. However, as pointed out by Thomassen [17], there exist infinitely many 3-connected graphs which do not contain a \( K_4 T \); hence the direct counterpart of Theorem 5 is not true. A substantial class of examples can be obtained as follows: take any 3-connected cubic graph \( G \) and replace each vertex \( x \) by a \( K_{2,3} \), in such a way that each of the three vertices of the \( K_{2,3} \) of degree 2 becomes incident with precisely one of the three edges incident with \( x \) in \( G \). The resulting graph is also a 3-connected cubic graph, and it contains no \( K_4 T \). In these examples \( K_{2,3} \) plays an important role. This is generally so in graphs of minimum degree at least 3 containing no \( K_4 T \) as demonstrated by the following result:

**THEOREM 6.** Let \( G \) be a graph with at least two vertices one of which is denoted \( x_0 \). Suppose all vertices have degree at least 3, except perhaps \( x_0 \). Then either \( G \) contains a \( K_4 T \), or else \( G \) contains an induced \( K_{2,3} \) (not containing \( x_0 \)) of which four vertices of a \( K_{2,2} \) all have degree 3 in \( G \).

If \( G \) is connected, then in the second alternative also \( G - V(K_{2,3}) \) is connected.

If \( G \) is 3-connected, then in the second alternative all five vertices of the \( K_{2,3} \) have degree 3 in \( G \).

**Proof.** The proof is by induction on the number \( n \) of vertices of \( G \), and it starts exactly as the proof of Theorem 5. In particular, if \( G \) is not 2-
connected, we proceed as we did there. If \( G \) is 2-connected, but not 3-connected, then the 3-connected graph \( H^* \) and the edge \((x, y)\) are defined as there. If \( G \) is 3-connected, then we define \( H^* = G \) and \((x, y)\) as any edge of \( G \) with \( x = x_0 \).

We shall then prove that either \( H^* \) contains a \( K_4 \) with \((x, y)\) not one of the undivided edges, or else \( H^* \) contains an induced \( K_{2,3} \) not containing \( x \), where all five vertices of the \( K_{2,3} \) have degree 3 in \( H^* \), and where \( H^* - V(K_{2,3}) \) is connected. From this Theorem 6 follows.

The graphs \( B^* \), \( C \) and \( G^* \) are defined as in the proof of Theorem 5 with the addition that \( G^* \) is not only maximum with respect to the number of vertices, but it also has a maximum number of edges among all possible graphs with a maximum number of vertices. By Lemmas 4 and 5, any cycle \( C' \) of \( H^* - V(C) \) is either contained in \( B^* \) or has precisely \( x \) or \( y \) in common with \( B^* \).

If \( H^* - V(C) = B^* \), then by arguments partly similar to those in the proof of Theorem 5, there exists a \( K_4 \) in \( H^* \) (where \((x, y)\) is not one of the undivided edges), unless \( B^* = K_2 \) and \( C \) has length 4 and \( H^* = K_{1,3} \). If \( C \) has length 4, and \( H^* \) has a \( K_4 \) with \((p, z), (z, q)\) and \((z, z^*)\) undivided. If we cannot choose \( p, z, q, p^*, z^* \) and \( q^* \) in this way, then \( C \) has length at least 4. If furthermore, \( C \) has length at least 5, then there exist five consecutive vertices \( p_1, p_2, p_3, p_4, p_5 \) on \( C \) with \( y \notin \{p_2, p_3, p_4, p_5\} \) and \( p_1^* = p_4^* = p_2^* = p_2^* = p_3^* \). In this case \( H^* \) has a \( K_4 \) with \((p_2, p_3), (p_3, p_4) \) and \((p_3, p_5)\) left undivided. If \( C \) has length 4, the exceptional situation arises.) In the exceptional case \( H^* \) contains a \( K_{2,3} \) as desired. Hence we may assume that \( H^* - V(C) \) contains more than one block, and hence it has an endblock \( B \neq B^* \).

We consider again first the case where \( B = K_2 \), i.e., there is a vertex \( z \) in \( B \) of degree 1 in \( H^* - V(C) \). Then \( z \) is joined to at least two vertices of \( C \). If \( z \) is joined to at least three vertices of \( C \), we have a \( K_4 \). Hence we may assume that \( z \) is joined to precisely two vertices of \( C \), and by the maximality of \( G^* \), the cycle \( C \) has length either 3 or 4. If \( C \) has length 3, then \( H^* [V(C) \cup \{z\}] \) is a \( K_3 \), and in this case it is easy to find a \( K_4 \). If \( C \) has length 4, say with vertices \( t_1, t_2, t_3 \) and \( t_4 \), in this order, where \( z \) is joined to \( t_1 \) and \( t_3 \), then \( t_2 \) and \( t_4 \) are both joined to \( H^* - (V(C) \cup \{z\}) \); in fact, by the edge-maximality of \( G^* \), they are both joined to precisely one vertex in \( H^* - (V(C) \cup \{z\}) \). If \( y = t_2 \) or \( y = t_4 \), say \( y = t_2 \), then we may use \( H^*[\{z, t_1, t_2, t_4\}] \) instead of \( C \). Hence we may assume that \( t_2 \neq y \) and \( t_4 \neq y \).

The graph \( H^* [\{t_1, t_2, t_3, t_4, z\}] \) is a \( K_{2,3} \) and \( t_2, t_4 \) and \( z \) all have degree 3 in \( H^* \). If also \( t_1 \) and \( t_3 \) have degree 3 in \( H^* \) we have a \( K_{2,3} \) as desired. Hence assume that \( t_1 \) is joined to a vertex \( t_1^* \) in \( H^* - (V(C) \cup \{z\}) \). Let \( t_2 \) be joined to the vertex \( t_2^* \) in \( H^* - (V(C) \cup \{z\}) \).
In $H^* - V(C)$ there is a path from $z$ containing $t^*_x$ and ending in an endblock $B'$ of $H^* - V(C)$ different from $B$. Let the cutvertex of $B'$ in $H^* - V(C)$ be $x'$. Since $H^*$ is 3-connected, there are at least two edges from $B' - x'$ to different vertices of $C$. If there is an edge from $B' - x'$ to either $t_1$ or $t_3$, say $t_1$, then there is a path $P$ in $H^* - V(C)$ from $z$ to a vertex $t^*_1 \in B'$ such that $P$ contains $t^*_1$. The cycle, whose edges consists of the edges of $P$, $(t^*_1, t_1), (t_1, t_4), (t_4, t_3)$ and $(t_3, z)$, has three vertices $t^*_1, t_1$ and $t_3$ joined to $t_2$. Hence in this case there is a $K_4T$. We may now assume that all edges from $B' - x'$ to $C$ are incident with $t_2$ or $t_4$; hence we may assume that $t^*_2 \in V(B' - x')$, and that $B' - x'$ also contains a vertex $t^*_3$ joined to $t_4$ in $H^*$. In this case there is a path $P$ in $H^* - (V(C) \cup \{z\})$ from $t^*_2$ to the vertex $t^*_1$ joined to $t_1$ in $H^*$, such that $P$ contains $t^*_3$. The cycle, whose edges consist of the edges of $P$, $(t^*_1, t_1), (t_1, z), (z, t_3), (t_3, t_2)$ and $(t_2, t^*_2)$, has three vertices $t_1, t_3$ and $t^*_3$ joined to $t_4$. Hence we have again a $K_4T$. This finishes the case where $B = K_2$.

We are then left with the situation where any endblock of $H^* - V(C)$ different from $B^*$ contains cycles, in particular $B$ does. From the structure of $H^* - V(C)$ we may assume that the cutvertex of $B$ is $x$ and that $B - x$ is a tree $T$ with at least two vertices, among them two endvertices $z_1$ and $z_2$. The vertices $z_1$ and $z_2$ are both different from $y$ by the definition of $B^*$. If $x$ is joined to an interior vertex of the unique path $P$ joining $z_1$ and $z_2$ in the tree $T$, then it is easy to find a $K_4T$. If $x$ is not joined to any such vertex, then let $z'_2$ be the neighbour of $z_1$ on $P$ (the case $z'_2 = z_2$ is possible). Since $z'_2$ has degree at least 3 in $H^*$ there is a path $P'$ from $z'_2$ to a vertex $t'_2$ on $C$, where $P'$ and $P$ have only $z'_2$ in common, and $P'$ and $C$ have only $t'_2$ in common, and all interior vertices of $P'$ are in $T$. There is a vertex $t'_3$ on $C$ different from $t'_2$ joined to a vertex $t^*_3$, in another endblock of $H^* - V(C)$ than $B$. Let $t'_1$ be a vertex on $C$ joined to $z_1$ by an edge in $H^*$. The cycle consisting of the segment of $C$ from $t'_1$ to $t'_3$ containing $t'_1$, the edge $(t'_3, t^*_3)$, a path from $t^*_3$ to $x$ in $H^* - (V(C) \cup V(T))$, the edge $(x, z_2)$, the path $P - z_2$ and the path $P'$ is a cycle not containing $z_1$, but containing the three neighbours $t'_1, x$ and $z'_2$ of $z_1$. Hence $H^*$ contains a $K_4T$.

The graphs of type $K_4T$ found in the various cases never have $(x, y)$ as an undivided edge. Thus Theorem 6 has been proved.

A $(K_3, K_3, 3)$-cockade is defined as a $K_3$-cockade with the addition:

(iii) a $K_{3,3}$ is a $(K_3, K_{3,3})$-cockade.

Figure 4 shows a $(K_3, K_3, 3)$-cockade built from the copies of $K_{3,3}$ and a $K_3$. It contains no $K_4T$ and only four vertices of each $K_{2,3}$ in the graph has degree 3. This shows that Theorem 6 in the 2-connected case is best possible.

**Corollary 11** [17]. If $G$ has $n$ vertices ($n \geq 3$) and at least $2n - 3$ edges, then $G$ contains a $K_4T$ unless $G$ is a $(K_3, K_{3,3})$-cockade.
Outline of proof. By induction on $n$. If $x$ is a vertex of $G$ of degree at most 2, then $G - x$ has $n - 1$ vertices and at least $2(n - 1) - 3$ edges, and hence we may use the induction hypothesis. If all vertices of $G$ have degree at least 3, then $G - K_{2,2}$, where the $K_{2,2}$ is as described in Theorem 6, has $n - 4$ vertices and at least $2(n - 4) - 3$ edges, and hence also in this case we may use the induction hypothesis. The proof of Corollary 11 easily follows.

Since a $K_4T$ contains an even cycle with a diagonal, Corollary 11 implies the following counterpart to Corollary 10:

**Corollary 12.** If $G$ has $n$ vertices ($n \geq 4$) and at least $2n - 3$ edges then $G$ contains an even cycle with a diagonal.

Note that $K_{2,n-2}$ has $2n - 4$ edges and contains no cycles with diagonals at all. The fact that a graph with $n$ vertices and $2n - 3$ edges always has a cycle with a diagonal was first proved by Pósa [16].

Corollary 12 also follows from the following result:

**Corollary 13.** Let $G$ be a graph with $n$ vertices ($n \geq 4$) in which all vertices have degree at least 3, except perhaps one vertex $x_0$. Then $G$ contains an even cycle with a diagonal.

*Proof.* We may assume that $G$ is connected. If $G$ contains a $K_4T$ we finish as above. If $G$ does not contain a $K_4T$, then by Theorem 6, $G$ contains a $K_{2,3}$ not containing $x_0$. Since $G - V(K_{2,3})$ is connected (by Theorem 6) there exists a vertex in $G - V(K_{2,3})$ joined by three internally disjoint paths to the three vertices of the $K_{2,3}$ of degree 2 in the $K_{2,3}$. The length of two of these paths, say $P$ and $P'$, have the same parity. But then it is easy to find an even cycle with a diagonal in the graph consisting of the $K_{2,3}$, $P$ and $P'$. This proves Corollary 13.

**Corollary 14.** If $G$ is 4-chromatic, then $G$ contains a $K_4T$. 
Proof. It is sufficient to prove Corollary 14 for 4-critical graphs. But such a graph has minimum degree at least 3 and no two non-adjacent vertices have the same neighbours. Hence Corollary 14 follows immediately from Theorem 6.

8. Concluding Remarks

In this paper we have investigated the structure of graphs using non-separating induced cycles as a basic tool. The obtained method, based on the simple Lemmas 1–5, seems to be rather powerful. We have thus in this paper demonstrated how one can apply it in various directions to obtain new results and new proofs of old results. Unfortunately, some of the proofs are involved in special cases where small cycles occur. This may seem surprising since the presence of a small cycle intuitively makes a graph more likely to have the properties described in our results.

Possibly other graph theoretic results can be obtained by these ideas, and maybe some of our results can be extended to matroids or at least to regular or binary matroids. A cycle $C$ in a 2-connected graph $G$ with minimum degree at least 3 is an induced non-separating cycle if and only if the contraction of all edges of $C$ results in a non-separable graph, i.e., a graph with only one block. Thus one might suggest that perhaps every non-separable matroid in which every cycle and cocycle has at least three elements contains a cycle whose contraction results in a non-separable matroid. Separation properties of cycles play a role in the proof of Tutte’s theorem characterizing graphic matroids [24].

Another approach to non-separating cycles, based on connectivity-preserving edge-contractions, is indicated in [19]. By that method Thomassen [20] recently solved the problem of Lovász [14] mentioned earlier by showing that a $(k+3)$-connected graph always contains a cycle whose deletion results in a $k$-connected graph.

It seems difficult to extend the theory of this paper to infinite graphs since one can construct infinite graphs of arbitrarily high (finite) connectivity with no non-separating cycles. Also, there exist, for each $k \geq 3$, infinite $k$-connected graphs of arbitrarily large girth such that the contraction of any edge decreases the connectivity. This shows that Corollaries 4–6 cannot be extended to infinite graphs. However, non-separating induced cycles play a role in extensions of the planarity criteria of MacLane and Whitney, respectively, to infinite graphs [19].

Note added in proof: While this paper was in print there appeared a more detailed version of Kelman's work on non-separating cycles with applications (in other directions than those of the present paper) in A. K. Kelman's, The concept of a vertex in a matroid, the non-separating

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