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Multiple Categories: The Equivalence of a Globular and a Cubical Approach

Fahd Ali Al-Agl[1]

Um-Alqura University, Makkah, Saudi Arabia

Ronald Brown[2]

School of Informatics, Mathematics Division, University of Wales, Bangor, Gwynedd LL57 1UT, United Kingdom
E-mail: r.brown@bangor.ac.uk

and

Richard Steiner

Department of Mathematics, University of Glasgow, University Gardens, Glasgow G12 8QW, United Kingdom
E-mail: r.steiner@maths.gla.ac.uk

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We show the equivalence of two kinds of strict multiple category, namely the well-known globular ω-categories, and the cubical ω-categories with connections.

Key Words: multiple category; globular ω-category; cubical ω-category; folding operation.

Contents.

[1] We have been unable to contact Dr. Al-Agl and so the responsibility for the paper rests with the last two authors.
[2] To whom correspondence should be addressed.

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0. INTRODUCTION

An essential feature for the possibility of 'higher-dimensional group theory' (see the expository article [5]) is the extension of the domain of discourse from groups to groupoids, that is from a set with a binary operation defined on all elements, to a set with an operation defined only on pairs satisfying a geometric condition. This fact itself leads to various equivalent candidates for 'higher-dimensional groups,' namely those based on different geometric structures, for example balls, globes, simplices, cubes and even polyhedra. The proofs of these equivalences are non-trivial—the basic intuitions derive from the foundations of relative homotopy theory. Some of these equivalences have proved crucial for the applications: theorems may be easily proved in one context and then transferred into another, more computational context. Notable examples are the advantages of cubical methods for providing both a convenient 'algebraic inverse to subdivision,' for use in local-to-global problems [10], and also a simple monoidal closed structure, which may then be translated into other situations [13].

It has proved important to extend these ideas from groupoids to categories. The standard notion of (strict) higher-dimensional category is that of globular ω-category. Our main result is that there is an adjoint equivalence of categories

$$\lambda : \text{globular } \omega\text{-categories} \rightleftarrows \text{cubical } \omega\text{-categories with connections} : \gamma.$$

Precise definitions are given below. The proof has interest because it is certainly much harder than the groupoid case, and because at one stage it uses braid relations among some key basic folding operations (Proposition 5.1, Theorem 5.2). The equivalence between the two forms should prove useful. In Section 9, we use this equivalence to define the notion of 'commutative n-cube.' In Section 10, we follow methods of Brown and Higgins in [13] to show that cubical ω-categories with connection form a monoidal closed category. The equivalence of categories transfers this structure to the globular case—the resulting internal hom in the globular case gives various higher-dimensional forms of 'lax natural transformation.'

Cubical ω-categories with connection have been applied to concurrency theory by Goubault [21] and by Gaucher [20], and again relations with the globular case are important for these studies.

The origin of this equivalence is as follows.

In developing the algebra of double groupoids as a framework for potential 2-dimensional Van Kampen Theorems, Brown and Spencer in [15] were led to the notion of double groupoid with an extra structure of 'connection'—this was essential to obtain an equivalence of such a double groupoid with the classical notion of crossed module. This structure was also essential for the proof of the 2-dimensional Van Kampen Theorem given by Brown and Higgins [8].

The double groupoid case was generalised by Brown and Higgins [6, 9] to give an equivalence between crossed complexes and what were called there 'ω-groupoids,' and which we here call 'cubical ω-groupoids with connections.' It was also proved in [11] that crossed complexes are equivalent to what were there called '∞-groupoids,' and which we here call 'globular ω-groupoids,' following current fashions. Thus, the globular and cubical cases of ω-groupoids were known in 1981 to be equivalent, but the proof was via the category of crossed complexes.

Other equivalences with crossed complexes were established, for example with: cubical T-complexes [6, 12]; simplicial T-complexes by Ashley [3]; and polyhedral T-complexes by Jones [22]. In T-complexes, the basic concept is taken to be that of *thin elements* which determine a strengthening of the Kan extension condition. The notion of simplicial T-complex is due to Dakin [17].

Spencer observed in [24] that the methods of [15] allowed an equivalence between 2-categories and double categories with connections, using an 'up-square' construction of Bastiani and Ehresmann [4, 18], but he gave no details. The full details of this have been recently given by Brown and Mosa in [14].

The thesis of Mosa in 1987 [23] attempted to give an equivalence between crossed complexes of algebroids and cubical ω-algebroids, and while this was completed in dimension 2 even the case of dimension 3 proved hard, though some basic methods were established.

This result raised the question of an equivalence between the globular ω-categories defined in 1981 in [11] and an appropriate form of cubical ω-categories with connections, of which a definition was fairly easy to formulate as an extension of the previous definition of cubical ω-groupoid. This problem was taken up in Al-Agl's thesis of 1989 [1]. The central idea, based on the groupoid methods of [9], was to define a 'folding operation' Φ from a cubical ω-category G to the globular ω-category γG it contained. This definition was successfully accomplished, but the problem of establishing some major properties of Φ, in particular the relation with the category structures, was solved only up to dimension 3. That is, the conjectured equivalence was proved in dimension 3.

Steiner pursued the work of Al-Agl, and their joint paper [2] does prove that globular ω-categories are equivalent to cubical sets with extra structure, but, as stated in that paper, this extra structure is not described in finitary terms. Later, Steiner was stimulated by renewed interest in the cubical case coming from concurrency theory in the work of Goubault [21] and Gaucher [20], and by the publication of the 2-dimensional case by Brown and Mosa in [14]. He completed the programme given in [1] and informed Brown, who announced the result at the Aalborg 'Workshop on Geometric and Topological Methods in Concurrency' in June 1999. This paper is the result. It proves the conjecture implicit in [1] that a globular ω-category is equivalent to a cubical set with extra structure directly analogous to the structure for cubical ω-groupoids given in [6, 9].

There is considerable independent work on globular ω-categories. The thesis of Crans [16] already contains the adjoint pair (λ, γ) and also the closed monoidal structure on the category of globular ω-categories. It also seems to be the first time that the cube category (without connections) together with its ω-category realisation is explicitly defined by generators and relations.

The work in Australia by Ross Street [27–30] has an initial aim to determine a simplicial nerve NX of a globular ω-category X. This developed into finding extra structure on NX so that N gave an equivalence between ω-categories and certain structured simplicial sets, analogous to Ashley's equivalence [3] between ω-groupoids and simplicial T-complexes. It is stated in [30] that this programme has been completed by Dominic Verity, to verify the conjecture stated in [28]. Street tells us that Verity also knew the equivalence proved in the present paper, but we have no further information. We also mention that Street's paper [28] *implicitly* contains our basic proposition (3.2), namely that the cells of the n-categorical n-cube compose in such a way that they give rise to the hemispherical (i.e. globular) decomposition $\partial_1^{\pm}\Phi_n$ of the n-cube.

1. ω-CATEGORIES

An ω-category [11, 25, 27] arises when a sequence of categories $C_0, C_1, \ldots$ all have the same set of morphisms X, the various category structures commute with one another, the identities for C_p are also identities for C_q when $q > p$, and every member of X is an identity for some C_p. We write $\#_p$ for the composition in C_p. Given $x \in X$, we write $d_p^- x$ and $d_p^+ x$ for the identities of the source and target of x in C_p, so that $d_p^- x \#_p x = x \#_p d_p^+ x = x$. The structure can be expressed in terms of X, $\#_p$ and the d_p^α as follows.

DEFINITION 1.1. An ω-category is a set X together with unary operations d_p^-, d_p^+ and partially defined binary operations $\#_p$ for $p = 0, 1, \ldots$ such that the following conditions hold:

(i) $x \#_p y$ is defined if and only if $d_p^+ x = d_p^- y$;

(ii)

$$d_q^\beta d_p^\alpha x = \begin{cases} d_q^\beta x & \text{for } q<p, \\ d_p^\alpha x & \text{for } q \geqslant p; \end{cases}$$

(iii) if $x \#_p y$ is defined then

$$\begin{aligned} d_p^-(x \#_p y) &= d_p^- x, \\ d_p^+(x \#_p y) &= d_p^+ y, \\ d_q^\beta(x \#_p y) &= d_q^\beta x \#_p d_q^\beta y \quad \text{for } q \neq p; \end{aligned}$$

(iv) $d_p^- x \#_p x = x \#_p d_p^+ x = x$;

(v) $(x \#_p y) \#_p z = x \#_p (y \#_p z)$ if either side is defined;

(vi) if $p \neq q$, then

$$(x \#_p y) \#_q (x' \#_p y') = (x \#_q x') \#_p (y \#_q y')$$

whenever both sides are defined;

(vii) for each $x \in X$ there is a *dimension* $\dim x$ such that $d_p^\alpha x = x$ if and only if $p \geqslant \dim x$.

DEFINITION 1.2. An *ω-category of sets* is an ω-category X whose members are sets such that $x \#_p y = x \cup y$ whenever $x \#_p y$ is defined in X.

The theory of pasting in ω-categories [25, 28] associates ω-categories of sets $M(K)$ with simple presentations to certain complexes K; the members of $M(K)$ are subcomplexes of K. Various types of complexes have been considered, but they certainly include the cartesian products of directed paths, and we will now describe the theory in that case.

Let n be a non-negative integer. We represent a directed path of length n by the closed interval $[0, n]$; the vertices are the singleton subsets $\{0\}$, $\{1\}$, ..., $\{n\}$ and the edges are the intervals $[0, 1], [1, 2], \ldots, [n-1, n]$, where

$[m-1, m]$ is directed from $m-1$ to m. We write

$$d^{-}[m-1, m] = \{m-1\}, \qquad d^{+}[m-1, m] = \{m\}.$$

Now let $K = K_1 \times \cdots \times K_p$ be a cartesian product of directed paths. A product $\sigma = \sigma_1 \times \cdots \times \sigma_p$, where σ_i is a vertex or edge in K_i, is called a *cell* in K. We can write a cell σ in the form

$$\sigma = P_0 \times e_1 \times P_1 \times e_2 \times P_2 \times \cdots \times P_{q-1} \times e_q \times P_q,$$

where the P_j are products of vertices and the e_j are edges; the *dimension* of σ is then q. The codimension 1 faces of σ are the subsets got by replacing one edge factor e_j with $d^{-}e_j$ or $d^{+}e_j$. The faces with $d^{-}e_1$ or $d^{+}e_2$ or $d^{-}e_3$ or $\cdots$ are called *negative*, and the faces with $d^{+}e_1$ or $d^{-}e_2$ or $d^{+}e_3$ or $\cdots$ are called *positive*. The theory of pasting gives us the following result.

THEOREM 1.3. *Let K be a cartesian product of directed paths. Then there is an ω-category $M(K)$ of subsets of K with the following presentation: the generators are the cells of K; if σ is a cell of dimension q, then there are relations $d_q^{-}\sigma = d_q^{+}\sigma = \sigma$; if σ is a cell of dimension q with $q > 0$, then there are relations saying that $d_{q-1}^{-}\sigma$ and $d_{q-1}^{+}\sigma$ are the unions of the negative and positive faces of σ, respectively. Every member of $M(K)$ is an iterated composite of cells.*

We will now describe the main examples.

EXAMPLE 1.4. We write $I = [0, 1]$ and $I^n = [0, 1]^n$ for $n \geqslant 1$; for completeness we also write $I^0 = [0, 0]$. In this notation, $M(I^0) = \{I^0\}$ and $M(I) = \{I, d_0^{-}I, d_0^{+}I\}$; there are no members other than the generating cells. There are morphisms

$$\check{\partial}^{-}, \check{\partial}^{+} : M(I^0) \to M(I), \qquad \check{\varepsilon} : M(I) \to M(I^0)$$

given by

$$\check{\partial}^{\alpha}(I^0) = d_0^{\alpha} I, \qquad \check{\varepsilon}(I) = \check{\varepsilon}(d_0^{\alpha} I) = I^0.$$

EXAMPLE 1.5. The members of $M([0, 2])$ are the cells and the composite

$$[0, 2] = [0, 1] \#_0 [1, 2].$$

There are morphisms

$$\check{\imath}^{-}, \check{\imath}^{+}, \check{\mu} : M(I) \to M([0, 2])$$

given by

$$\check{\iota}^{-}(d_0^{-}I) = \{0\}, \qquad \check{\iota}^{-}(I) = [0,1], \qquad \check{\iota}^{-}(d_0^{+}I) = \{1\},$$

$$\check{\iota}^{+}(d_0^{-}I) = \{1\}, \qquad \check{\iota}^{+}(I) = [1,2], \qquad \check{\iota}^{+}(d_0^{+}I) = \{2\},$$

and

$$\check{\mu}(d_0^{-}I) = \{0\}, \qquad \check{\mu}(I) = [0,2], \qquad \check{\mu}(d_0^{+}I) = \{2\}.$$

EXAMPLE 1.6. The members of $M(I^2)$ are the cells and the composites

$$d_1^{-}I^2 = (d_0^{-}I \times I)\#_0(I \times d_0^{+}I), \qquad d_1^{+}I^2 = (I \times d_0^{-}I)\#_0(d_0^{+}I \times I).$$

There are morphisms $\check{\Gamma}^{+}, \check{\Gamma}^{-} : M(I^2) \to M(I)$ given by

$$\check{\Gamma}^{\alpha}(d_0^{-\alpha}I \times d_0^{-\alpha}I) = \check{\Gamma}^{\alpha}(d_0^{-\alpha}I \times I) = \check{\Gamma}^{\alpha}(d_0^{-\alpha}I \times d_0^{\alpha}I) = \check{\Gamma}^{\alpha}(I \times d_0^{-\alpha}I)$$

$$= \check{\Gamma}^{\alpha}(d_0^{\alpha}I \times d_0^{-\alpha}I) = d_0^{-\alpha}I,$$

$$\check{\Gamma}^{\alpha}(I^2) = \check{\Gamma}^{\alpha}(I \times d_0^{\alpha}I) = \check{\Gamma}^{\alpha}(d_0^{\alpha}I \times I) = \check{\Gamma}^{\alpha}(d_1^{-}I^2) = \check{\Gamma}^{\alpha}(d_1^{+}I^2) = I,$$

$$\check{\Gamma}^{\alpha}(d_0^{\alpha}I \times d_0^{\alpha}I) = d_0^{\alpha}I.$$

For cartesian products of members of the ω-categories that we are considering, we have the following result.

THEOREM 1.7. *Let K and L be cartesian products of directed paths, let x be a member of $M(K)$, and let y be a member of $M(L)$. Then $x \times y$ is a member of $M(K \times L)$ and*

$$d_p^{\alpha}(x \times y) = \bigcup_{i=0}^{p}(d_i^{\alpha}x \times d_{p-i}^{(-)^i\alpha}y).$$

This has the following consequence.

THEOREM 1.8. (i) *Let K, K', L, L' be cartesian products of directed graphs, and let $f : M(K) \to M(K')$ and $g : M(L) \to M(L')$ be morphisms of ω-categories. Then there is a unique morphism*

$$f \otimes g : M(K \times L) \to M(K' \times L')$$

such that

$$(f \otimes g)(x \times y) = f(x) \times g(y)$$

for $x \in M(K)$ and $y \in M(L)$.

(ii) *The assignments*

$$(M(K), M(L)) \mapsto M(K \times L), \qquad (f, g) \mapsto f \otimes g$$

form a bifunctor.

Proof. (i) From the presentation of $M(K \times L)$ and Theorem 1.7, there is a unique morphism $f \otimes g$ such that $(f \otimes g)(x \times y) = f(x) \times g(y)$ when x and y are cells. The formula then holds for a general product $x \times y$ because it is a composite of cells.

(ii) One can check bifunctoriality by considering the values of the appropriate morphisms on generators. ∎

By applying the tensor product construction, we obtain further morphisms.

EXAMPLE 1.9. Let id^r denote the identity morphism from $M(I^r)$ to itself. There are morphisms

$$\check{\partial}_i^-, \check{\partial}_i^+ : M(I^{n-1}) \to M(I^n) \qquad (1 \leqslant i \leqslant n)$$

given by

$$\check{\partial}_i^\alpha = \mathrm{id}^{i-1} \otimes \check{\partial}^\alpha \otimes \mathrm{id}^{n-i};$$

there are morphisms

$$\check{\varepsilon}_i : M(I^n) \to M(I^{n-1}) \qquad (1 \leqslant i \leqslant n)$$

given by

$$\check{\varepsilon}_i = \mathrm{id}^{i-1} \otimes \check{\varepsilon} \otimes \mathrm{id}^{n-i};$$

there are morphisms

$$\check{\imath}_i^-, \check{\imath}_i^+, \check{\mu}_i : M(I^n) \to M(I^{i-1} \times [0, 2] \times I^{n-i}) \qquad (1 \leqslant i \leqslant n)$$

given by

$$\check{\imath}_i^\alpha = \mathrm{id}^{i-1} \otimes \check{\imath}^\alpha \otimes \mathrm{id}^{n-i}, \qquad \check{\mu}_i = \mathrm{id}^{i-1} \otimes \check{\mu} \otimes \mathrm{id}^{n-i};$$

there are morphisms

$$\check{\Gamma}_i^+, \check{\Gamma}_i^- : M(I^n) \to M(I^{n-1}) \qquad (1 \leqslant i \leqslant n-1)$$

given by

$$\check{\Gamma}_i^\alpha = \mathrm{id}^{i-1} \otimes \check{\Gamma}^\alpha \otimes \mathrm{id}^{n-i-1}.$$

Most of the morphisms in Example 1.9 map generators to generators, and one can verify their existence directly from Theorem 1.3. The exceptions are the $\check{\mu}_i$ for which Theorem 1.8 is really necessary.

Remark 1.10. Suppose that K is an n-dimensional product of directed paths. Then K can be got from a family of n-cubes by gluing along $(n-1)$-dimensional faces. From the presentation of $M(K)$, one sees that it is the colimit of a corresponding diagram in which the morphisms have the form $\check{\partial}_i^\alpha : M(I^{n-1}) \to M(I^n)$. In particular, $\check{\imath}^-$ and $\check{\imath}^+$ exhibit $M([0,2])$ as the push-out of

$$M(I) \overset{\check{\partial}^+}{\leftarrow} M(I^0) \overset{\check{\partial}^-}{\rightarrow} M(I),$$

and $\check{\imath}_i^-$ and $\check{\imath}_i^+$ exhibit $M(I^{i-1} \times [0,2] \times I^{n-i})$ as the push-out of

$$M(I^n) \overset{\check{\partial}_i^+}{\leftarrow} M(I^{n-1}) \overset{\check{\partial}_i^-}{\rightarrow} M(I^n).$$

2. CUBICAL ω-CATEGORIES WITH CONNECTIONS

Suppose that X is an ω-category. There is then a sequence of sets

$$(\lambda X)_n = \mathrm{Hom}[M(I^n), X] \qquad (n = 0, 1, \cdots),$$

and the morphisms of Example 1.9 induce functions between the $(\lambda X)_n$. It turns out that the $(\lambda X)_n$ form a cubical ω-category with connections in the sense of the following definition. This definition is found in [1]. The origin is in the definition of what was called 'ω-groupoid' in [6, 9], where the justification was the equivalence with crossed complexes [6, 9] and the use in the formulation and proof of a generalised Van Kampen Theorem [7, 10]. The corresponding definition for categories arose out of the work of Spencer [24] and of Mosa [23].

Let K be a cubical set, that is, a family of sets $\{K_n;\ n \geqslant 0\}$ with for $n \geqslant 1$ face maps $\partial_i^\alpha : K_n \to K_{n-1}$ $(i = 1, 2, \cdots, n;\ \alpha = +, -)$ and degeneracy maps $\varepsilon_i : K_{n-1} \to K_n$ $(i = 1, 2, \cdots, n)$ satisfying the usual cubical relations:

(i) $$\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha \qquad (i < j),$$

(ii) $$\varepsilon_i \varepsilon_j = \varepsilon_{j+1}\varepsilon_i \qquad (i \leqslant j),$$

(iii) $$\partial_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j-1}\partial_i^\alpha & (i<j), \\ \varepsilon_j \partial_{i-1}^\alpha & (i>j), \\ \mathrm{id} & (i=j). \end{cases} \tag{2.1}$$

We say that K is a *cubical set with connections* if for $n \geqslant 0$ it has additional structure maps (called *connections*) $\Gamma_i^+, \Gamma_i^- : K_n \to K_{n+1}$ $(i = 1, 2, \cdots, n)$ satisfying the relations:

(i) $$\Gamma_i^\alpha \Gamma_j^\beta = \Gamma_{j+1}^\beta \Gamma_i^\alpha \qquad (i<j),$$

(ii) $$\Gamma_i^\alpha \Gamma_i^\alpha = \Gamma_{i+1}^\alpha \Gamma_i^\alpha,$$

(iii) $$\Gamma_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j+1}\Gamma_i^\alpha & (i<j), \\ \varepsilon_j \Gamma_{i-1}^\alpha & (i>j), \end{cases}$$

(iv) $$\Gamma_j^\alpha \varepsilon_j = \varepsilon_j^2 = \varepsilon_{j+1}\varepsilon_j,$$

(v) $$\partial_i^\alpha \Gamma_j^\beta = \begin{cases} \Gamma_{j-1}^\beta \partial_i^\alpha & (i<j), \\ \Gamma_j^\beta \partial_{i-1}^\alpha & (i>j+1), \end{cases}$$

(vi) $$\partial_j^\alpha \Gamma_j^\alpha = \partial_{j+1}^\alpha \Gamma_j^\alpha = \mathrm{id},$$

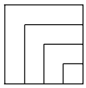
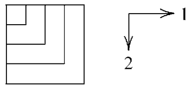
(vii) $$\partial_j^\alpha \Gamma_j^{-\alpha} = \partial_{j+1}^\alpha \Gamma_j^{-\alpha} = \varepsilon_j \partial_j^\alpha. \tag{2.2}$$

The connections are to be thought of as extra 'degeneracies.' (A degenerate cube of type $\varepsilon_j x$ has a pair of opposite faces equal and all other faces degenerate. A cube of type $\Gamma_i^\alpha x$ has a pair of adjacent faces equal and all other faces of type $\Gamma_j^\alpha y$ or $\varepsilon_j y$.) Cubical complexes with these, and other, structures have also been considered by Evrard [19].

The prime example of a cubical set with connections is the singular cubical complex KX of a space X. Here, for $n \geqslant 0$ K_n is the set of singular n-cubes in X (i.e. continuous maps $I^n \to X$) and the connection $\Gamma_i^\alpha : K_n \to K_{n+1}$ is induced by the map $\gamma_i^\alpha : I^{n+1} \to I^n$ defined by

$$\gamma_i^\alpha(t_1, t_2, \cdots, t_{n+1}) = (t_1, t_2, \cdots, t_{i-1}, A(t_i, t_{i+1}), t_{i+2}, \cdots, t_{n+1}),$$

where $A(s,t) = \max(s,t), \min(s,t)$ as $\alpha = -, +$, respectively. Given below are pictures of $\gamma_1^\alpha : I^2 \to I^1$ where the internal lines show lines of constancy of the map on I^2.

$\gamma_1^+ =$  $\gamma_1^- =$ 

The complex KX has some further relevant structures, namely the composition of n-cubes in the n different directions. Accordingly, we define a *cubical complex with connections and compositions* to be a cubical set K with connections in which each K_n has n partial compositions $\circ_j$ $(j = 1, 2, \cdots, n)$ satisfying the following axioms.

If $a, b \in K_n$, then $a \circ_j b$ is defined if and only if $\partial_j^- b = \partial_j^+ a$, and then

$$\begin{cases} \partial_j^-(a \circ_j b) = \partial_j^- a, \\ \partial_j^+(a \circ_j b) = \partial_j^+ b, \end{cases} \qquad \partial_i^\alpha(a \circ_j b) = \begin{cases} \partial_j^\alpha a \circ_{j-1} \partial_i^\alpha b & (i<j), \\ \partial_i^\alpha a \circ_j \partial_i^\alpha b & (i>j). \end{cases} \tag{2.3}$$

The interchange laws. If $i \neq j$, then

$$(a \circ_i b) \circ_j (c \circ_i d) = (a \circ_j c) \circ_i (b \circ_j d) \tag{2.4}$$

whenever both sides are defined. (The diagram

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{l} \longrightarrow i \\ \downarrow \\ j \end{array}$$

will be used to indicate that both sides of the above equation are defined and also to denote the unique composite of the four elements.)

If $i \neq j$, then

$$\varepsilon_i(a \circ_j b) = \begin{cases} \varepsilon_i a \circ_{j+1} \varepsilon_i b & (i \leqslant j), \\ \varepsilon_i a \circ_j \varepsilon_i b & (i > j), \end{cases} \tag{2.5}$$

(i) $$\Gamma_i^\alpha(a \circ_j b) = \begin{cases} \Gamma_i^\alpha a \circ_{j+1} \Gamma_i^\alpha b & (i<j), \\ \Gamma_i^\alpha a \circ_j \Gamma_i^\alpha b & (i>j), \end{cases}$$

(ii) $$\Gamma_j^+(a \circ_j b) = \begin{bmatrix} \Gamma_j^+ a & \varepsilon_j a \\ \varepsilon_{j+1} a & \Gamma_j^+ b \end{bmatrix} \begin{array}{l} \longrightarrow j \\ \downarrow \\ j+1 \end{array}$$

(iii) $$\Gamma_j^-(a \circ_j b) = \begin{bmatrix} \Gamma_j^- a & \varepsilon_{j+1} b \\ \varepsilon_j b & \Gamma_j^- b \end{bmatrix} \begin{array}{l} \longrightarrow j \\ \downarrow \\ j+1 \end{array}. \tag{2.6}$$

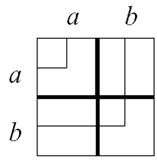
These last two equations are the *transport laws*.[3]

[3] Recall from [15] that the term *connection* was chosen because of an analogy with path connections in differential geometry. In particular, the transport law is a variation or special case of the transport law for a path connection.

It is easily verified that the singular cubical complex KX of a space X satisfies these axioms if $\circ_j$ is defined by

$$(a \circ_j b)(t_1, t_2, \cdots, t_n) = \begin{cases} a(t_1, \cdots, t_{j-1}, 2t_j, t_{j+1}, \cdots, t_n) & (t_j \leqslant \frac{1}{2}), \\ b(t_1, \cdots, t_{j-1}, 2t_j - 1, t_{j+1}, \cdots, t_n) & (t_j \geqslant \frac{1}{2}) \end{cases}$$

whenever $\partial_j^- b = \partial_j^+ a$. In this context, the transport law for $\Gamma_1^-(a \circ b)$ can be illustrated by the picture:



Definition 2.1. A *cubical ω-category with connections* $G = \{G_n\}$ is a cubical set with connections and compositions such that each $\circ_j$ is a category structure on G_n with identity elements $\varepsilon_j y$ $(y \in G_{n-1})$, and in addition

$$\Gamma_i^+ x \circ_i \Gamma_i^- x = \varepsilon_{i+1} x, \qquad \Gamma_i^+ x \circ_{i+1} \Gamma_i^- x = \varepsilon_i x. \tag{2.7}$$

For simplicity, a cubical ω-category with connections will be called a cubical ω-category in the rest of this paper.

Remark 2.2. This list is a part of the list of structure and axioms which first appears in the thesis of Mosa [23, Chapter V], in the context of cubical algebroids with connection, and appears again in the thesis of Al-Agl [1]. The rules for the connections are fairly clear extensions of the axioms given in [6, 9], given the general notion of thin structure on a double category discussed by Spencer in [24].

Note that a cubical ω-category has an underlying cubical set under its face and degeneracy operations.

It is now straightforward to construct a functor from ω-categories to cubical ω-categories. The following type of construction is well known.

Definition 2.3. The *cubical nerve* of an ω-category X is the cubical ω-category λX defined as follows:

$$(\lambda X)_n = \mathrm{Hom}[M(I^n), X],$$

and the operations ∂_i^α, ε_i, $\circ_i$, Γ_i^α are induced by $\check{\partial}_i^\alpha$, $\check{\varepsilon}_i$, $\check{\mu}_i$, $\check{\Gamma}_i^\alpha$ according to the formulae

$$\partial_i^\alpha x = x \circ \check{\partial}_i^\alpha : M(I^{n-1}) \to X$$

for $x : M(I^n) \to X$, etc.

In particular, in Definition 2.3, note that the domain of $\circ_i$ in $(\lambda X)_n \times (\lambda X)_n$ is precisely

$$\mathrm{Hom}[M(I^{i-1} \times [0,2] \times I^{n-i}), X]$$

according to Remark 1.10. To check that λX satisfies the conditions of Definition 2.1, one must check the corresponding identities for the $\check{\partial}_i^\alpha$, etc. Many relations essentially come from properties of the underlying morphisms $\check{\partial}^\alpha$, etc. The relation $\partial_i^\alpha \varepsilon_i = \mathrm{id}$, for example, comes from the easily checked relation $\check{\varepsilon} \circ \check{\partial}^\alpha = \mathrm{id}$. For relations involving composition, one must use the morphisms $\check{\imath}^\alpha : M(I) \to M([0,2])$ which present $M([0,2])$ as a push-out. Thus, to check the relation $\partial_i^-(x \circ_i y) = \partial_i^- x$, which is a relation between binary operators, one must check that

$$(\check{\imath}^-)^{-1} \check{\mu} \check{\partial}^-(\sigma) = \check{\partial}^-(\sigma)$$

and

$$(\check{\imath}^+)^{-1} \check{\mu} \check{\partial}^-(\sigma) = \emptyset$$

for every cell σ in I^0. For the associative law, one must consider morphisms from $M(I)$ to $M([0,3])$.

The functoriality of the tensor product is responsible for formulae looking like commutation rules, such as $\partial_i^\alpha \varepsilon_j = \varepsilon_{j-1} \partial_i^\alpha$ for $i<j$.

Remark 2.4. Any natural operation θ on cubical ω-categories determines an underlying homomorphism $\check{\theta}$ between ω-categories. For example, if θ maps G_n to G_m, then in particular θ maps

$$[\lambda M(I^n)]_n = \mathrm{Hom}[M(I^n), M(I^n)]$$

to $[\lambda M(I^n)]_m = \mathrm{Hom}[M(I^m), M(I^n)]$ and $\check{\theta} = \theta(\mathrm{id}) : M(I^m) \to M(I^n)$.

3. THE ω-CATEGORY ASSOCIATED TO A CUBICAL ω-CATEGORY

In this section, we construct a functor γ associating an ω-category to a cubical ω-category. The idea is to recover an ω-category from its nerve. We will use certain folding operations, which are defined as follows.

DEFINITION 3.1. Let G be a cubical ω-category. The *folding operations* are the operations

$$\psi_i, \Psi_r, \Phi_m : G_n \to G_n$$

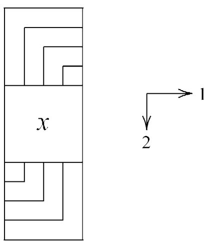
defined for $1 \leqslant i \leqslant n-1$, $1 \leqslant r \leqslant n$ and $0 \leqslant m \leqslant n$ by

$$\psi_i x = \Gamma_i^+ \partial_{i+1}^- x \circ_{i+1} x \circ_{i+1} \Gamma_i^- \partial_{i+1}^+ x,$$

$$\Psi_r = \psi_{r-1}\psi_{r-2} \cdots \psi_1,$$

$$\Phi_m = \Psi_1 \Psi_2 \cdots \Psi_m = \psi_1(\psi_2\psi_1) \cdots (\psi_{m-1} \cdots \psi_1).$$

Note in particular that Ψ_1, Φ_0 and Φ_1 are identity operations.
Here is a picture of $\psi_1 : G_2 \to G_2$:

$\psi_1(x) =$ 

The idea behind Definition 3.1 is best seen from the action of the underlying endomorphism $\check{\Phi}_n$ in the ω-category of sets $M(I^n)$.

PROPOSITION 3.2. *The endomorphism $\check{\Phi}_n : M(I^n) \to M(I^n)$ underlying the folding operation Φ_n is given by $\check{\Phi}_n(I^n) = I^n$ and*

$$\check{\Phi}_n(\sigma \times d_0^\alpha I \times I^p) = d_p^\alpha I^n$$

for any cell σ in I^{n-p-1}.

Proof. Let $\check{\psi} : M(I^2) \to M(I^2)$ be the operation underlying ψ_1 in dimension 2. The operations underlying ψ_i, Ψ_r and Φ_m in dimension n are then given by

$$\check{\psi}_i = \mathrm{id}^{i-1} \otimes \check{\psi} \otimes \mathrm{id}^{n-i-1},$$

$$\check{\Psi}_r = \check{\psi}_1 \check{\psi}_2 \cdots \check{\psi}_{r-1},$$

$$\check{\Phi}_m = \check{\Psi}_m \check{\Psi}_{m-1} \cdots \check{\Psi}_1.$$

One finds that $\check{\psi}(I^2) = I^2$, from which it follows that $\check{\psi}_i(I^n) = I^n$ and then $\check{\Phi}_n(I^n) = I^n$. One also finds that

$$\check{\psi}(\tau \times d_0^\alpha I) = d_0^\alpha I \times d_0^\alpha I$$

for any cell τ in I. For a cell σ in I^{n-p-1} it follows that

$$(\check{\Psi}_{n-p-1} \cdots \check{\Psi}_1)(\sigma \times d_0^\alpha I \times I^p) \subseteq I^{n-p-1} \times d_0^\alpha I \times I^p$$

and

$$\check{\Psi}_{n-p}(\check{\Psi}_{n-p-1} \cdots \check{\Psi}_1)(\sigma \times d_0^\alpha I \times I^p) = (d_0^\alpha I)^{n-p} \times I^p.$$

It then follows that $\check{\Phi}_n(\sigma \times d_0^\alpha I \times I^p)$ is independent of σ. It now suffices to show that

$$\check{\Phi}_n[(d_0^\alpha I)^{n-p} \times I^p] = d_p^\alpha I^n.$$

Recall that $d_{n-1}^\alpha I^n$ is the union of the $(n-1)$-cells

$$\tau_1 = d_0^\alpha I \times I^{n-1}, \qquad \tau_2 = I \times d_0^{-\alpha} I \times I^{n-2}, \cdots .$$

We see that

$$\check{\Phi}_n(\tau_2) = \check{\Phi}_n(d_0^\alpha I \times d_0^{-\alpha} I \times I^{n-2}) \subset \check{\Phi}_n(\tau_1),$$

etc., so that $\check{\Phi}(d_{n-1}^\alpha I^n) = \check{\Phi}(\tau_1)$. It follows that

$$\check{\Phi}_n(d_0^\alpha I \times I^{n-1}) = \check{\Phi}_n(\tau_1) = \check{\Phi}_n(d_{n-1}^\alpha I^n) = d_{n-1}^\alpha \check{\Phi}_n(I^n) = d_{n-1}^\alpha I^n.$$

By similar reasoning,

$$\check{\Phi}_n[(d_0^\alpha I)^2 \times I^{n-2}] = d_{n-2}^\alpha \check{\Phi}_n(d_0^\alpha I \times I^{n-1}) = d_{n-2}^\alpha d_{n-1}^\alpha I^n = d_{n-2}^\alpha I^n,$$

and so on, eventually giving

$$\check{\Phi}_n[(d_0^\alpha I)^{n-p} \times I^p] = d_p^\alpha I^n$$

as required. This completes the proof. ∎

It follows from Proposition 3.2 that $\check{\Phi}_n : M(I^n) \to M(I^n)$ is an idempotent endomorphism with image

$$F_n = \{I^n, d_{n-1}^- I^n, d_{n-1}^+ I^n, \cdots, d_0^- I^n, d_0^+ I^n\}.$$

In fact, F_n is nothing else but the n-globe. For an ω-category X, it follows that

$$\Phi_n[(\lambda X)_n] \cong \mathrm{Hom}(F_n, X).$$

Now, it is clear that F_n has a presentation with generator I^n and relations $d_n^- I^n = d_n^+ I^n = I^n$; therefore,

$$\Phi_n(\lambda X)_n \cong \{\, x \in X : d_n^- x = d_n^+ x = x\}.$$

It follows that X can be recovered from λX as the colimit of a sequence

$$\Phi_0[(\lambda X)_0] \to \Phi_1[(\lambda X)_1] \to \cdots \ .$$

We will now explain how to perform this construction for cubical ω-categories in general. We begin with some elementary relations.

PROPOSITION 3.3. *The folding operations satisfy the following relations*:

(i)

$$\psi_j \varepsilon_i = \varepsilon_i \psi_{j-1} \qquad \text{for } i<j,$$

$$\psi_j \varepsilon_j = \psi_j \varepsilon_{j+1} = \psi_j \Gamma_j^{-\alpha} = \varepsilon_j,$$

$$\psi_j \varepsilon_i = \varepsilon_i \psi_j \qquad \text{for } i > j+1,$$

$$\partial_i^\alpha \psi_j = \psi_{j-1} \partial_i^\alpha \qquad \text{for } i<j,$$

$$\partial_j^- \psi_j x = \partial_j^- x \circ_j \partial_{j+1}^+ x,$$

$$\partial_j^+ \psi_j x = \partial_{j+1}^- x \circ_j \partial_j^+ x,$$

$$\partial_{j+1}^\alpha \psi_j = \varepsilon_j \partial_j^\alpha \partial_{j+1}^\alpha,$$

$$\partial_i^\alpha \psi_j = \psi_j \partial_i^\alpha \qquad \text{for } i > j+1;$$

(ii)

$$\Psi_1 \varepsilon_1 = \varepsilon_1,$$

$$\Psi_r \varepsilon_1 = \varepsilon_1 \Psi_{r-1} \qquad \text{for } r > 1,$$

$$\Psi_r \varepsilon_i = \varepsilon_{i-1} \Psi_r \qquad \text{for } 1<i\leqslant r,$$

$$\partial_i^\alpha \Psi_r = \Psi_r \partial_i^\alpha \qquad \text{for } i > r,$$

$$\partial_r^\alpha \Psi_r = \varepsilon_1^{r-1} (\partial_1^\alpha)^r;$$

(iii)
$$\Phi_m \varepsilon_i = \varepsilon_1 \Phi_{m-1} \qquad \textit{for } 1 \leqslant i \leqslant m,$$
$$\partial_i^\alpha \Phi_m = \Phi_m \partial_i^\alpha \qquad \textit{for } i > m,$$
$$\partial_m^\alpha \Phi_m = \varepsilon_1^{m-1} (\partial_1^\alpha)^m.$$

Proof. (i) These relations are straightforward consequences of the definitions.

(ii) Since $\Psi_1 = \mathrm{id}$, we have $\Psi_1 \varepsilon_1 = \varepsilon_1$.

From part (i), if $r > 1$, then

$$\Psi_r \varepsilon_1 = (\psi_{r-1} \cdots \psi_2) \psi_1 \varepsilon_1 = (\psi_{r-1} \cdots \psi_2) \varepsilon_1 = \varepsilon_1 (\psi_{r-2} \cdots \psi_1) = \varepsilon_1 \Psi_{r-1}.$$

Also from part (i), if $1 < i \leqslant r$, then

$$\begin{aligned}
\Psi_r \varepsilon_i &= (\psi_{r-1} \cdots \psi_i) \psi_{i-1} (\psi_{i-2} \cdots \psi_1) \varepsilon_i \\
&= (\psi_{r-1} \cdots \psi_i) \psi_{i-1} \varepsilon_i (\psi_{i-2} \cdots \psi_1) \\
&= (\psi_{r-1} \cdots \psi_i) \varepsilon_{i-1} (\psi_{i-2} \cdots \psi_1) \\
&= \varepsilon_{i-1} (\psi_{r-2} \cdots \psi_{i-1}) (\psi_{i-2} \cdots \psi_1) \\
&= \varepsilon_{i-1} \Psi_{r-1}.
\end{aligned}$$

From part (i), if $i > r$, then

$$\partial_i^\alpha \Psi_r = \partial_i^\alpha (\psi_{r-1} \cdots \psi_1) = (\psi_{r-1} \cdots \psi_1) \partial_i^\alpha = \Psi_r \partial_i^\alpha.$$

It now follows that

$$\begin{aligned}
\partial_r^\alpha \Psi_r &= \partial_r^\alpha \psi_{r-1} \Psi_{r-1} \\
&= \varepsilon_{r-1} \partial_{r-1}^\alpha \partial_r^\alpha \Psi_{r-1} \\
&= \varepsilon_{r-1} \partial_{r-1}^\alpha \Psi_{r-1} \partial_r^\alpha \\
&= \cdots \\
&= \varepsilon_{r-1} \cdots \varepsilon_2 \varepsilon_1 \partial_1^\alpha \Psi_1 \partial_2^\alpha \cdots \partial_r^\alpha \\
&= \varepsilon_{r-1} \cdots \varepsilon_2 \varepsilon_1 \partial_1^\alpha \partial_2^\alpha \cdots \partial_r^\alpha \\
&= \varepsilon_1^{r-1} (\partial_1^\alpha)^r,
\end{aligned}$$

using (2.1).

(iii) From part (ii), if $1 \leqslant i \leqslant m$, then

$$\begin{aligned}
\Phi_m \varepsilon_i &= \Psi_1 (\Psi_2 \cdots \Psi_{m-i+1})(\Psi_{m-i+2} \cdots \Psi_m)\varepsilon_i \\
&= \Psi_1 (\Psi_2 \cdots \Psi_{m-i+1})\varepsilon_1 (\Psi_{m-i+1} \cdots \Psi_{m-1}) \\
&= \Psi_1 \varepsilon_1 (\Psi_1 \cdots \Psi_{m-i})(\Psi_{m-i+1} \cdots \Psi_{m-1}) \\
&= \varepsilon_1 (\Psi_1 \cdots \Psi_{m-i})(\Psi_{m-i+1} \cdots \Psi_{m-1}) \\
&= \varepsilon_1 \Phi_{m-1}.
\end{aligned}$$

Also from part (ii), if $i > m$, then

$$\partial_i^\alpha \Phi_m = \partial_i^\alpha (\Psi_1 \cdots \Psi_m) = (\Psi_1 \cdots \Psi_m)\partial_i^\alpha = \Phi_m \partial_i^\alpha.$$

It now follows that

$$\begin{aligned}
\partial_m^\alpha \Phi_m &= \partial_m^\alpha \Phi_{m-1} \Psi_m \\
&= \Phi_{m-1} \partial_m^\alpha \Psi_m \\
&= \Phi_{m-1} \varepsilon_1^{m-1} (\partial_1^\alpha)^m \\
&= \varepsilon_1^{m-1} \Phi_0 (\partial_1^\alpha)^m \\
&= \varepsilon_1^{m-1} (\partial_1^\alpha)^m. \quad \blacksquare
\end{aligned}$$

We now observe that the operators ψ_i are idempotent, and characterise their images.

PROPOSITION 3.4. *Let G be a cubical ω-category, and suppose that $1 \leqslant i \leqslant n-1$. The operator $\psi_i : G_n \to G_n$ is idempotent. An element x of G_n is in $\psi_i(G_n)$ if and only if $\partial_{i+1}^- x$ and $\partial_{i+1}^+ x$ are in* $\operatorname{Im} \varepsilon_i$.

Proof. From Proposition 3.3(i), if $x \in \psi_i(G_n)$ then $\partial_{i+1}^- x$ and $\partial_{i+1}^+ x$ are in $\operatorname{Im} \varepsilon_i$.

To complete the proof, suppose that $\partial_{i+1}^- x$ and $\partial_{i+1}^+ x$ are in $\operatorname{Im} \varepsilon_i$; it suffices to show that $\psi_i x = x$. Now,

$$\Gamma_i^{-\alpha} \partial_{i+1}^\alpha x \in \operatorname{Im} \Gamma_i^{-\alpha} \varepsilon_i = \operatorname{Im} \varepsilon_i^2 = \operatorname{Im} \varepsilon_{i+1} \varepsilon_i,$$

so that the $\Gamma_i^{-\alpha}\partial_{i+1}^{\alpha}x$ are identities for $\circ_{i+1}$. It follows that

$$\psi_i x = \Gamma_i^{+}\partial_{i+1}^{-}x \circ_{i+1} x \circ_{i+1} \Gamma_i^{-}\partial_{i+1}^{+}x = x$$

as required. This completes the proof. ∎

There is a similar result for Φ_n as follows.

PROPOSITION 3.5. *Let G be a cubical ω-category. The operator $\Phi_n : G_n \to G_n$ is idempotent. An element x of G_n is in $\Phi_n(G_n)$ if and only if $\partial_m^{\alpha}x \in \operatorname{Im}\varepsilon_1^{m-1}$ for $1 \leqslant m \leqslant n$ and $\alpha = \pm$.*

Proof. Since $\Phi_n = \Phi_m(\Psi_{m+1}\cdots\Psi_n)$, it follows from Proposition 3.3(iii) that

$$\operatorname{Im}\partial_m^{\alpha}\Phi_n \subset \operatorname{Im}\partial_m^{\alpha}\Phi_m \subset \operatorname{Im}\varepsilon_1^{m-1}.$$

Conversely, suppose that $\partial_m^{\alpha}x \in \operatorname{Im}\varepsilon_1^{m-1}$ for $1 \leqslant m \leqslant n$ and $\alpha = \pm$; it suffices to show that $\Phi_n x = x$, and for this it suffices to show that $\psi_i x = x$ for $1 \leqslant i \leqslant n-1$. But

$$\partial_{i+1}^{\alpha}x \in \operatorname{Im}\varepsilon_1^{i} = \operatorname{Im}\varepsilon_i\varepsilon_1^{i-1} \subset \operatorname{Im}\varepsilon_i$$

for $\alpha = \pm$, so that $\psi_i x = x$ by Proposition 3.4. This completes the proof. ∎

There is a useful result related to Proposition 3.5 as follows.

PROPOSITION 3.6. *If $x \in \Phi_n(G_n)$ and $1 \leqslant m \leqslant n$, then*

$$\partial_m^{\alpha}x = \varepsilon_1^{m-1}(\partial_1^{\alpha})^m x \qquad \textit{and} \qquad \varepsilon_m\partial_m^{\alpha}x = \varepsilon_1^{m}(\partial_1^{\alpha})^m x.$$

Proof. By Proposition 3.5, $\partial_m^{\alpha}x = \varepsilon_1^{m-1}x'$ for some x'. It follows that

$$x' = (\partial_1^{\alpha})^{m-1}\varepsilon_1^{m-1}x' = (\partial_1^{\alpha})^{m-1}\partial_m^{\alpha}x = (\partial_1^{\alpha})^m x,$$

so that $\partial_m^{\alpha}x = \varepsilon_1^{m-1}x' = \varepsilon_1^{m-1}(\partial_1^{\alpha})^m x$ and $\varepsilon_m\partial_m^{\alpha}x = \varepsilon_m\varepsilon_1^{m-1}(\partial_1^{\alpha})^m x = \varepsilon_1^{m}(\partial_1^{\alpha})^m x$ as required. ∎

We now deduce various closure properties for the family of sets $\Phi_n(G_n)$.

PROPOSITION 3.7. *Let G be a cubical ω-category. The family of sets $\Phi_n(G_n)$ $(n \geqslant 0)$ is closed under the ∂_i^{α} and under ε_1. The individual sets $\Phi_n(G_n)$ are closed under $\varepsilon_i\partial_i^{\alpha}$ and $\circ_i$ for $1 \leqslant i \leqslant n$.*

Proof. We use the characterisation in Proposition 3.5. We first show that the family is closed under ∂_1^α. Indeed, if $x \in \Phi_n(G_n)$, then

$$\partial_m^\beta \partial_1^\alpha x = \partial_1^\alpha \partial_{m+1}^\beta x \in \partial_1^\alpha(\operatorname{Im} \varepsilon_1^m) = \operatorname{Im} \varepsilon_1^{m-1},$$

since $\partial_1^\alpha \varepsilon_1 = \mathrm{id}$.

Next, we show that the family is closed under ε_1. Indeed, if $x \in \Phi_n(G_n)$, then $\partial_1^\beta \varepsilon_1 x \in \operatorname{Im} \varepsilon_1^0$ trivially, and for $m > 1$ we have

$$\partial_m^\beta \varepsilon_1 x = \varepsilon_1 \partial_{m-1}^\beta x \in \varepsilon_1(\operatorname{Im} \varepsilon_1^{m-2}) = \operatorname{Im} \varepsilon_1^{m-1}.$$

It now follows from Proposition 3.6 that the family is closed under ∂_i^α for all i. Similarly, $\Phi_n(G_n)$ is closed under $\varepsilon_i \partial_i^\alpha$.

It remains to show that $x \circ_i y \in \Phi_n(G_n)$ when x and y are in $\Phi_n(G_n)$ and the composite exists. Suppose that $\partial_m^\beta x = \varepsilon_1^{m-1} x'$ and $\partial_m^\beta y = \varepsilon_1^{m-1} y'$. If $m < i$, then

$$\begin{aligned}\partial_m^\beta(x \circ_i y) &= \partial_m^\beta x \circ_{i-1} \partial_m^\beta y = \varepsilon_1^{m-1} x' \circ_{i-1} \varepsilon_1^{m-1} y' \\ &= \varepsilon_1^{m-1}(x' \circ_{i-m} y') \in \operatorname{Im} \varepsilon_1^{m-1};\end{aligned}$$

if $m = i$, then $\partial_m^\beta(x \circ_i y)$ is $\partial_m^\beta x = \varepsilon_1^{m-1} x'$ or $\partial_m^\beta y = \varepsilon_1^{m-1} y'$, so $\partial_m^\beta(x \circ_i y)$ is certainly in $\operatorname{Im} \varepsilon_1^{m-1}$; and if $m > i$, then

$$\begin{aligned}\partial_m^\beta(x \circ_i y) &= \partial_m^\beta x \circ_i \partial_m^\beta y = \varepsilon_1^{m-1} x' \circ_i \varepsilon_1^{m-1} y' \\ &= \varepsilon_1^{i-1}(\varepsilon_1^{m-i} x' \circ_1 \varepsilon_1^{m-i} y') = \varepsilon_1^{i-1} \varepsilon_1^{m-i} x' \in \operatorname{Im} \varepsilon_1^{m-1}\end{aligned}$$

(note that $\varepsilon_1^{m-i} y'$ is an identity for $\circ_1$ because it lies in the image of ε_1).

This completes the proof. ∎

We now obtain the desired sequence of ω-categories.

THEOREM 3.8. *Let G be a cubical ω- category. Then there is a sequence of ω-categories and homomorphisms*

$$\Phi_0(G_0) \xrightarrow{\varepsilon_1} \Phi_1(G_1) \xrightarrow{\varepsilon_1} \Phi_2(G_2) \to \cdots$$

with the following structure on $\Phi_n(G_n)$: if $0 \leqslant p < n$, then

$$d_p^\alpha x = (\varepsilon_1)^{n-p} (\partial_1^\alpha)^{n-p} x$$

and $x \#_p y = x \circ_{n-p} y$ where defined; if $p \geqslant n$, then $d_p^\alpha x = x$ and the only composites are given by $x \#_p x = x$.

Proof. We first show that for a fixed value of n the given structure maps d_p^α and $\#_p$ make $\Phi_n(G_n)$ into an ω-category. By Proposition 3.7, $\Phi_n(G_n)$ is closed under the structure maps for $0 \leqslant p < n$, and the same result holds trivially for $p \geqslant n$.

From the identities in Section 2, if $0 \leqslant p < n$, then the triple

$$(d_p^-, d_p^+, \#_p) = (\varepsilon_{n-p}\partial_{n-p}^-, \varepsilon_{n-p}\partial_{n-p}^+, \circ_{n-p})$$

makes $\Phi_n(G_n)$ into the morphism set of a category (with $d_p^- x$ and $d_p^+ x$ the left and right identities of x and with $\#_p$ as composition), and these structures commute with one another. Trivially, the triples $(d_p^-, d_p^+, \#_p)$ for $n \geqslant p$ provide further commuting category structures. To show that these structures make $\Phi_n(G_n)$ into an ω-category, it now suffices to show that an identity for $\#_p$ is also an identity for $\#_q$ if $q > p$; in other words, it suffices to show that $d_q^\beta d_p^\alpha x = d_p^\alpha x$ for $x \in \Phi_n(G_n)$ and $q > p$. For $q \geqslant n$, this is trivial; we may therefore assume that $0 \leqslant p < q < n$. But Proposition 3.6 gives us

$$\begin{aligned} d_q^\beta d_p^\alpha x &= \varepsilon_1^{n-q}(\partial_1^\beta)^{n-q}\varepsilon_1^{n-p}(\partial_1^\alpha)^{n-p}x \\ &= \varepsilon_1^{n-q}\varepsilon_1^{q-p}(\partial_1^\alpha)^{n-p}x \\ &= \varepsilon_1^{n-p}(\partial_1^\alpha)^{n-p}x \\ &= d_p^\alpha x \end{aligned}$$

as required.

We have now shown that the $\Phi_n(G_n)$ are ω-categories. We know from Proposition 3.7 that ε_1 maps $\Phi_n(G_n)$ into $\Phi_{n+1}(G_{n+1})$, and it remains to show that this function is a homomorphism. That is to say, we must show that $\varepsilon_1 d_p^\alpha x = d_p^\alpha \varepsilon_1 x$ for $x \in \Phi_n(G_n)$, and we must show that $\varepsilon_1(x \#_p y) = \varepsilon_1 x \#_p \varepsilon_1 y$ for $x \#_p y$ a composite in $\Phi_n(G_n)$. But if $0 \leqslant p < n$, then

$$\varepsilon_1 d_p^\alpha x = \varepsilon_1 \varepsilon_{n-p}\partial_{n-p}^\alpha x = \varepsilon_{n-p+1}\partial_{n-p+1}^\alpha \varepsilon_1 x = d_p^\alpha \varepsilon_1 x$$

and

$$\varepsilon_1(x \#_p y) = \varepsilon_1(x \circ_{n-p} y) = \varepsilon_1 x \circ_{n-p+1} \varepsilon_1 y = \varepsilon_1 x \#_p \varepsilon_1 y$$

by identities in Section 2; if $p = n$ we get

$$\varepsilon_1 d_n^\alpha x = \varepsilon_1 x = \varepsilon_1 \partial_1^\alpha \varepsilon_1 x = d_n^\alpha \varepsilon_1 x$$

and

$$\varepsilon_1(x \#_n x) = \varepsilon_1 x = \varepsilon_1 x \circ_1 \varepsilon_1 x = \varepsilon_1 x \#_n \varepsilon_1 x;$$

and if $p > n$, then

$$\varepsilon_1 d_p^\alpha x = \varepsilon_1 x = d_p^\alpha \varepsilon_1 x$$

and

$$\varepsilon_1 (x \#_p x) = \varepsilon_1 x = \varepsilon_1 x \#_p \varepsilon_1 x$$

trivially. This completes the proof. ∎

We can now define a functor from cubical ω- to ω-categories.

DEFINITION 3.9. Let G be a cubical ω-category. The ω-category γG *associated to* G is the colimit of the sequence

$$\Phi_0(G_0) \xrightarrow{\varepsilon_1} \Phi_1(G_1) \xrightarrow{\varepsilon_1} \Phi_2(G_2) \to \cdots .$$

Remark 3.10. In Definition 3.9, one can identify $\Phi_n(G_n)$ with the subset of γG consisting of elements x such that $d_n^- x = d_n^+ x = x$. Indeed, the ε_1 are injective, because $\partial_1^\alpha \varepsilon_1 = \mathrm{id}$, so that $\Phi_n(G_n)$ can be identified with a subset of γG; if $x \in \Phi_n(G_n)$, then $d_n^- x = d_n^+ x = x$ by Theorem 3.8; if $x \in \Phi_m(G_m)$ with $m > n$ and $d_n^- x = d_n^+ x = x$, then

$$x = \varepsilon_{m-n} \partial_{m-n}^- x = \varepsilon_1^{m-n} (\partial_1^-)^{m-n} x$$

(Proposition 3.6) with $(\partial_1^-)^{m-n} x \in \Phi_n(G_n)$ (Proposition 3.7), and x can be identified with $(\partial_1^-)^{m-n} x$.

Remark 3.11. It is convenient to describe the ω-category γG in terms of the folding operations, but one can get a more direct description by using Proposition 3.5. The more direct description needs face maps, degeneracies and compositions, but not connections.

4. THE NATURAL ISOMORPHISM $A : \gamma\lambda X \to X$

Let X be an ω-category. From Definition 2.3 there is a cubical ω-category λX, and from Definition 3.9 there is an ω-category $\gamma\lambda X$. We will now construct a natural isomorphism $A : \gamma\lambda X \to X$.

Let F_n be the ω-category with one generator I^n and with relations $d_n^- I^n = d_n^+ I^n = I^n$. By Proposition 3.2, F_n can be realised as a sub-ω-category of $M(I^n)$, and the morphism $\check{\Phi}_n : M(I^n) \to M(I^n)$ associated to the folding operation Φ_n is an idempotent operation with image equal to F_n. Recalling that $(\lambda X)_n = \mathrm{Hom}[M(I^n), X]$, we see that

$$\Phi_n[(\lambda X)_n] = \{\, x \in \mathrm{Hom}[M(I^n), X] : x\check{\Phi}_n = x \,\}.$$

Let

$$A : \Phi_n[(\lambda X)_n] \to X$$

be the function given by

$$A(x) = x(I^n);$$

we see that A is an injection with image equal to

$$\{x \in X : d_n^- x = d_n^+ x = x\}.$$

These functions are compatible with the sequence

$$\cdots \to \Phi_n[(\lambda X)_n] \xrightarrow{\varepsilon_1} \Phi_{n+1}[(\lambda X)_{n+1}] \to \cdots \ ;$$

indeed, if $x \in \Phi_n[(\lambda X)_n]$, then

$$A(\varepsilon_1 x) = (\varepsilon_1 x)(I^{n+1}) = x\check{\varepsilon}_1(I^{n+1}) = x(I^n) = A(x).$$

The functions $A : \Phi_n[(\lambda X)_n] \to X$ therefore induce a bijection $A : \gamma\lambda X \to X$. We will now prove the following result.

THEOREM 4.1. *The functions $A : \gamma\lambda X \to X$ form a natural isomorphism of ω-categories.*

Proof. We have already shown that $A : \gamma\lambda X \to X$ is a bijection, and it is clearly natural. It remains to show that A is a homomorphism. It suffices to show that

$$A{:}\Phi_n[(\lambda X)_n] \to X$$

is a homomorphism for each n; in other words, we must show that $A(d_p^\alpha x) = d_p^\alpha A(x)$ for $x \in \Phi_n[(\lambda X)_n]$ and that $A(x \#_p y) = A(x) \#_p A(y)$ for $x \#_p y$ a composite in $\Phi_n[(\lambda X)_n]$.

Suppose that $x \in \Phi_n[(\lambda X)_n]$ and $0 \leqslant p < n$. Noting that $x = x\check{\Phi}_n$ and using Proposition 3.2, we find that

$$\begin{aligned}
A(d_p^\alpha x) &= A(\varepsilon_{n-p}\partial_{n-p}^\alpha x)\\
&= (\varepsilon_{n-p}\partial_{n-p}^\alpha x)(I^n)\\
&= x\check{\partial}_{n-p}^\alpha \check{\varepsilon}_{n-p}(I^n)\\
&= x(I^{n-p-1} \times d_0^\alpha I \times I^p)\\
&= x\check{\Phi}_n(I^{n-p-1} \times d_0^\alpha I \times I^p)\\
&= x(d_p^\alpha I^n)\\
&= d_p^\alpha x(I^n)\\
&= d_p^\alpha A(x).
\end{aligned}$$

Suppose that $x \in \Phi_n[(\lambda X)_n]$ and $p \geqslant n$. Then

$$A(d_p^\alpha x) = A(x) = x(I^n) = x(d_p^\alpha I^n) = d_p^\alpha x(I^n) = d_p^\alpha x.$$

Suppose that $x \#_p y$ is a composite in $\Phi_n[(\lambda X)_n]$ with $0 \leqslant p < n$. Let

$$(x, y) : M(I^{n-p-1} \times [0, 2] \times I^p) \to X$$

be the morphism such that $(x, y)\check{\iota}_{n-p}^- = x$ and $(x, y)\check{\iota}_{n-p}^+ = y$; then

$$A(x \#_p y) = A(x \circ_{n-p} y) = (x, y)\check{\mu}_{n-p}(I^n).$$

Let $\eta : F_n \to M(I^n)$ be the inclusion and let $\pi : M(I^n) \to F_n$ be $\check{\Phi}_n$ with its codomain restricted to F_n, so that $\check{\Phi}_n = \eta\pi$. Since x and y are in $\Phi_n[(\lambda X)_n]$, we have $x\check{\Phi}_n = x$ and $y = y\check{\Phi}_n$; we therefore get

$$(x, y)\check{\mu}_{n-p}(I^n) = (x\eta\pi, y\eta\pi)\check{\mu}_{n-p}(I^n).$$

Now let F_p be the ω-category with one generator z and with relations $d_p^- z = d_p^+ z = z$. We see that there is a factorisation

$$(x\eta\pi, y\eta\pi) = (x\eta, y\eta)(\pi, \pi)$$

through the obvious push-out of

$$F_n \leftarrow F_p \to F_n.$$

We also see that

$$(\pi,\pi)\check{\mu}_{n-p}(I^n) = \pi\check{\iota}^-_{n-p}(I^n) \#_p \pi\check{\iota}^+_{n-p}(I^n).$$

It now follows that

$$\begin{aligned}(x\eta\pi, y\eta\pi)\check{\mu}_{n-p}(I^n) &= (x\eta, y\eta)(\pi,\pi)\check{\mu}_{n-p}(I^n)\\ &= (x\eta, y\eta)[\pi\check{\iota}^-_{n-p}(I^n) \#_p \pi\check{\iota}^+_{n-p}(I^n)]\\ &= (x\eta, y\eta)\pi\check{\iota}^-_{n-p}(I^n) \#_p (x\eta, y\eta)\pi\check{\iota}^+_{n-p}(I^n)\\ &= x\eta\pi(I^n) \#_p y\eta\pi(I^n)\\ &= x(I^n) \#_p y(I^n)\\ &= A(x) \#_p A(y);\end{aligned}$$

therefore,

$$A(x \#_p y) = A(x) \#_p A(y).$$

Finally, suppose that $x \#_p y$ is a composite in $\Phi_n[(\lambda X)_n]$ with $p \geqslant n$. We must have $x = y$, and we get

$$A(x \#_p x) = A(x) = A(x) \#_p A(x).$$

This completes the proof. ∎

5. FOLDINGS, DEGENERACIES AND CONNECTIONS

According to Theorem 4.1, there are natural isomorphisms $A : \gamma\lambda X \to X$ for ω-categories X. To prove that ω-categories are equivalent to cubical ω-categories, we will eventually construct natural isomorphisms $B : G \to \lambda\gamma G$ for cubical ω-categories G. We will need properties of the folding operations, and we now begin to describe these.

We first show that the operations ψ_i behave like the standard generating transpositions of the symmetric groups (except of course that they are idempotent rather than involutory, by Proposition 3.4). There are two types of relation, the first of which is easy.

PROPOSITION 5.1. *If* $|i-j| \geqslant 2$, *then*

$$\psi_i\psi_j = \psi_j\psi_i.$$

Proof. This follows from the identities in Section 2. ∎

The next result is harder.

THEOREM 5.2. *If* $i > 1$, *then*

$$\psi_i \psi_{i-1} \psi_i = \psi_{i-1} \psi_i \psi_{i-1}.$$

Proof. Recall the matrix notation used for certain composites: if

$$(a_{11} \circ_i \cdots \circ_i a_{1n}) \circ_{i+1} \cdots \circ_{i+1} (a_{m1} \circ_i \cdots \circ_i a_{mn})$$

and

$$(a_{11} \circ_{i+1} \cdots \circ_{i+1} a_{m1}) \circ_i \cdots \circ_i (a_{1n} \circ_{i+1} \cdots \circ_{i+1} a_{mn})$$

are equal by the interchange law, then we will write

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{array}{l} \xrightarrow{\quad i} \\ \big\downarrow{\scriptstyle i+1} \end{array}.$$

for the common value. In such a matrix, we write $-$ for elements in the image of ε_i (which are the identities for $\circ_i$), and we write $|$ for elements in the image of ε_{i+1} (which are the identities for $\circ_{i+1}$).

We first compute $\psi_i \psi_{i-1} \psi_i x$. It is straightforward to check that

$$\psi_{i-1}\psi_i x = \begin{bmatrix} | & \Gamma_i^+ \partial_{i+1}^- x & \Gamma_{i-1}^- \partial_{i+1}^- x \\ \Gamma_{i-1}^+ \partial_i^- x & x & \Gamma_{i-1}^- \partial_i^+ x \\ \Gamma_{i-1}^+ \partial_{i+1}^+ x & \Gamma_i^- \partial_{i+1}^+ x & | \end{bmatrix} \begin{array}{l} \xrightarrow{\quad i} \\ \big\downarrow{\scriptstyle i+1} \end{array}.$$

It follows that

$$\begin{aligned} \Gamma_i^+ \partial_{i+1}^- \psi_{i-1} \psi_i x &= \Gamma_i^+ (\Gamma_{i-1}^+ \partial_i^- \partial_i^- x \circ_i \varepsilon_i \partial_i^- \partial_i^- x \circ_i \Gamma_{i-1}^- \partial_i^- \partial_i^- x) \\ &= \Gamma_i^+ (\Gamma_{i-1}^+ \partial_i^- \partial_i^- x \circ_i \Gamma_{i-1}^- \partial_i^- \partial_i^- x) \\ &= \Gamma_i^+ \varepsilon_{i-1} \partial_i^- \partial_i^- x \\ &= \varepsilon_{i-1} \Gamma_{i-1}^+ \partial_i^- \partial_i^- x \end{aligned}$$

and

$$\Gamma_i^- \partial_{i+1}^+ \psi_{i-1} \psi_i x = \varepsilon_{i-1} \Gamma_{i-1}^- \partial_i^+ \partial_i^+ x;$$

therefore,

$$\psi_i\psi_{i-1}\psi_i x = \varepsilon_{i-1}\Gamma^+_{i-1}\partial^-_i\partial^-_i x \circ_{i+1} \psi_{i-1}\psi_i x \circ_{i+1} \varepsilon_{i-1}\Gamma^-_{i-1}\partial^+_i\partial^+_i x.$$

Similarly, $\psi_{i-1}\psi_i\psi_{i-1}x$ is as a composite $\begin{array}{l} \xrightarrow{\qquad} i \\ \downarrow \\ i+1 \end{array}$

$$\begin{bmatrix} - & \Gamma^+_{i-1}\Gamma^+_{i-1}\partial^-_i\partial^-_i x & - & - & \Gamma^-_{i-1}\Gamma^+_{i-1}\partial^-_i\partial^-_i x \\ - & | & \Gamma^+_i\partial^-_{i+1}x & - & \Gamma^-_{i-1}\partial^-_{i+1}x \\ - & | & | & \Gamma^+_i\Gamma^-_{i-1}\partial^-_i\partial^+_i x & \Gamma^-_i\Gamma^-_{i-1}\partial^-_i\partial^+_i x \\ - & \Gamma^+_{i-1}\partial^-_i x & x & \Gamma^-_{i-1}\partial^+_i x & - \\ \Gamma^+_i\Gamma^+_{i-1}\partial^+_i\partial^-_i x & \Gamma^-_i\Gamma^+_{i-1}\partial^+_i\partial^-_i x & | & | & - \\ \Gamma^+_{i-1}\partial^+_{i+1}x & - & \Gamma^-_i\partial^+_{i+1}x & | & - \\ \Gamma^+_{i-1}\Gamma^-_{i-1}\partial^+_i\partial^+_i x & - & - & \Gamma^-_{i-1}\Gamma^-_{i-1}\partial^+_i\partial^+_i x & - \end{bmatrix}.$$

We now evaluate the rows of the matrix for $\psi_{i-1}\psi_i\psi_{i-1}x$. The first and last rows yield $\varepsilon_{i-1}\Gamma^+_{i-1}\partial^-_i\partial^-_i x$ and $\varepsilon_{i-1}\Gamma^-_{i-1}\partial^+_i\partial^+_i x$. The composite of the non-identity elements in the third row is $\varepsilon_{i+1}\Gamma^-_{i-1}\partial^-_i\partial^+_i x$, which is an identity for $\circ_{i+1}$, so the third row can be omitted. Similarly, the fifth row can be omitted. The second, fourth and sixth rows have the same values as the rows of the matrix for $\psi_{i-1}\psi_i x$. It follows that

$$\psi_{i-1}\psi_i\psi_{i-1}x = \varepsilon_{i-1}\Gamma^+_{i-1}\partial^-_i\partial^-_i x \circ_{i+1} \psi_{i-1}\psi_i x \circ_{i+1} \varepsilon_{i-1}\Gamma^-_{i-1}\partial^+_i\partial^+_i x$$

also. Therefore, $\psi_i\psi_{i-1}\psi_i x = \psi_{i-1}\psi_i\psi_{i-1}x$. This completes the proof. ∎

Remark 5.3. Proposition 5.1 and Theorem 5.2 in some sense explain the formula for Φ_m in Definition 3.1. The ψ_i behave like the generating transpositions $(i, i+1)$ in the symmetric group of permutations of $\{1, \cdots, m\}$, and, as in the symmetric group, there are $m!$ distinct composites of $\psi_1, \cdots, \psi_{m-1}$ given by

$$\Psi_{1,l(1)} \cdots \Psi_{m,l(m)}$$

for $1 \leqslant l(r) \leqslant r$, where $\Psi_{r,l(r)} = \psi_{r-1}\psi_{r-2}\cdots\psi_{l(r)}$.

The composite Φ_m corresponds to the order-reversing permutation $p \mapsto m+1-p$. In our context, we can characterise Φ_m as the zero element in the semigroup generated by $\psi_1, \cdots, \psi_{m-1}$ as follows.

THEOREM 5.4. *If* $1 \leqslant i \leqslant m-1$, *then*

$$\Phi_m \psi_i = \Phi_m.$$

Proof. If $r > 1$, then

$$\Psi_r \psi_1 = (\psi_{r-1} \cdots \psi_2)\psi_1\psi_1 = (\psi_{r-1} \cdots \psi_2)\psi_1 = \Psi_r,$$

since ψ_1 is idempotent by Proposition 3.4. For $1 < i < r$, it follows from Proposition 5.1 and Theorem 5.2 that

$$\begin{aligned}
\Psi_r \psi_i &= (\psi_{r-1} \cdots \psi_{i+1})\psi_i\psi_{i-1}(\psi_{i-2} \cdots \psi_1)\psi_i \\
&= (\psi_{r-1} \cdots \psi_{i+1})\psi_i\psi_{i-1}\psi_i(\psi_{i-2} \cdots \psi_1) \\
&= (\psi_{r-1} \cdots \psi_{i+1})\psi_{i-1}\psi_i\psi_{i-1}(\psi_{i-2} \cdots \psi_1) \\
&= \psi_{i-1}(\psi_{r-1} \cdots \psi_{i+1})\psi_i\psi_{i-1}(\psi_{i-2} \cdots \psi_1) \\
&= \psi_{i-1}\Psi_r.
\end{aligned}$$

For $1 \leqslant i \leqslant m-1$, it now follows that

$$\begin{aligned}
\Phi_m \psi_i &= \Psi_1(\Psi_2 \cdots \Psi_{m-i})\Psi_{m-i+1}(\Psi_{m-i+2} \cdots \Psi_m)\psi_i \\
&= \Psi_1(\Psi_2 \cdots \Psi_{m-i})\Psi_{m-i+1}\psi_1(\Psi_{m-i+2} \cdots \Psi_m) \\
&= \Psi_1(\Psi_2 \cdots \Psi_{m-i})\Psi_{m-i+1}(\Psi_{m-i+2} \cdots \Psi_m) \\
&= \Phi_m,
\end{aligned}$$

as required. ∎

We can now give some interactions between Φ_m, degeneracies and connections. First we have the following result.

PROPOSITION 5.5. *For all i there is a relation*

$$\psi_i \Gamma_i^\alpha = \varepsilon_i.$$

Proof. From the definitions we get

$$\begin{aligned}\psi_i\Gamma_i^+ x &= \Gamma_i^+\partial_{i+1}^-\Gamma_i^+ x \circ_{i+1} \Gamma_i^+ x \circ_{i+1} \Gamma_i^-\partial_{i+1}^+\Gamma_i^+ x\\ &= \Gamma_i^+\varepsilon_i\partial_i^- x \circ_{i+1} \Gamma_i^+ x \circ_{i+1} \Gamma_i^- x\\ &= \varepsilon_i^2\partial_i^- x \circ_{i+1} \varepsilon_i x\\ &= \varepsilon_{i+1}\varepsilon_i\partial_i^- x \circ_{i+1} \varepsilon_i x\\ &= \varepsilon_i x,\end{aligned}$$

and we similarly get $\psi_i\Gamma_i^- x = \varepsilon_i x$. ∎

We draw the following conclusions.

THEOREM 5.6. *If* $1 \leqslant i \leqslant m$, *then*

$$\Phi_m\varepsilon_i = \varepsilon_1\Phi_{m-1}.$$

If $1 \leqslant i \leqslant m-1$, *then*

$$\Phi_m\Gamma_i^\alpha = \varepsilon_1\Phi_{m-1}.$$

Proof. The first of these results was given in Proposition 3.3(iii). The second result then follows from Theorem 5.4 and Proposition 5.5: indeed, we get

$$\Phi_m\Gamma_i^\alpha = \Phi_m\psi_i\Gamma_i^\alpha = \Phi_m\varepsilon_i = \varepsilon_1\Phi_{m-1}$$

as required. ∎

6. FOLDINGS, FACE MAPS AND COMPOSITIONS

In this section, we describe interactions between the Φ_n, face maps and compositions. For face maps, the basic results are given in Proposition 3.3. For compositions, the basic results are as follows, of which the first two cases correspond to the 2-dimensional case in [14, Proposition 5.1].

PROPOSITION 6.1. *In a cubical* ω*-category*

$$\psi_i(x \circ_j y) = \begin{cases} (\psi_i x \circ_{i+1} \varepsilon_i\partial_{i+1}^+ y) \circ_i (\varepsilon_i\partial_{i+1}^- x \circ_{i+1} \psi_i y) & \text{if } j = i,\\ (\varepsilon_i\partial_i^- x \circ_{i+1} \psi_i y) \circ_i (\psi_i x \circ_{i+1} \varepsilon_i\partial_i^+ y) & \text{if } j = i+1,\\ \psi_i x \circ_j \psi_i y & \text{otherwise.}\end{cases}$$

Proof. Note that we have $\partial_j^+ x = \partial_j^- y$ for $x \circ_j y$ to be defined.

The proof for the cases $j = i$ and $j = i + 1$ consists in evaluating in two ways each of the matrices

$$\begin{bmatrix} \Gamma_i^+ \partial_{i+1}^- x & \varepsilon_i \partial_{i+1}^- x \\ \varepsilon_{i+1} \partial_{i+1}^- x & \Gamma_i^+ \partial_{i+1}^- y \\ x & y \\ \Gamma_i^- \partial_{i+1}^+ x & \varepsilon_{i+1} \partial_{i+1}^+ y \\ \varepsilon_i \partial_{i+1}^+ y & \Gamma_i^- \partial_{i+1}^+ y \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_i \varepsilon_i \partial_i^- \partial_i^- x & \Gamma_i^+ \partial_{i+1}^- x \\ \varepsilon_i \partial_i^- x & x \\ \Gamma_i^+ \partial_{i+1}^- y & \Gamma_i^- \partial_{i+1}^+ x \\ y & \varepsilon_i \partial_i^+ y \\ \Gamma_i^- \partial_{i+1}^+ y & \varepsilon_i \varepsilon_i \partial_i^+ \partial_i^+ y \end{bmatrix} \begin{array}{l} \longrightarrow i \\ \big\downarrow \\ i+1 \end{array}$$

(Note that $\varepsilon_i \varepsilon_i \partial_i^- \partial_i^- x$ and $\varepsilon_i \varepsilon_i \partial_i^+ \partial_i^+ y$ are identities for $\circ_{i+1}$ because $\varepsilon_i \varepsilon_i = \varepsilon_{i+1} \varepsilon_i$.) The other case follows from the identities in Section 2. ∎

Because of Proposition 6.1, it is convenient to regard $\varepsilon_i \partial_i^\alpha$ and $\varepsilon_i \partial_{i+1}^\alpha$ as generalisations of ψ_i. We extend this idea to Ψ_r and Φ_m, and arrive at the following definition.

DEFINITION 6.2. A generalised ψ_i is an operator of the form ψ_i or $\varepsilon_i \partial_i^\alpha$ or $\varepsilon_i \partial_{i+1}^\alpha$. A generalised Ψ_r is an operator of the form $\psi'_{r-1} \psi'_{r-2} \cdots \psi'_1$, where ψ'_i is a generalised ψ_i. A generalised Φ_m is an operator of the form $\Psi'_1 \Psi'_2 \cdots \Psi'_m$, where Ψ'_r is a generalised Ψ_r.

From (2.3) and (2.5), there are results for ε_i and ∂_i^α analogous to Proposition 6.1: $\varepsilon_i(x \circ_j y)$ is a composite of $\varepsilon_i x$ and $\varepsilon_i y$; if $j = i$, then $\partial_i^\alpha(x \circ_j y)$ is $\partial_i^\alpha x$ or $\partial_i^\alpha y$; if $j \neq i$, then $\partial_i^\alpha(x \circ_j y)$ is a composite of $\partial_i^\alpha x$ and $\partial_i^\alpha y$. From these observations and from Proposition 6.1 we immediately get the following result.

PROPOSITION 6.3. *Let ψ'_i be a generalised ψ_i. Then $\psi'_i(x^- \circ_j x^+)$ is naturally equal to a composite of factors $\psi''_i x^\alpha$ with ψ''_i a generalised ψ_i.*

Let Ψ'_r be a generalised Ψ_r. Then $\Psi'_r(x^- \circ_j x^+)$ is naturally equal to a composite of factors $\Psi''_r x^\alpha$ with Ψ''_r a generalised Ψ_r.

Let Φ'_m be a generalised Φ_m. Then $\Phi'_m(x^- \circ_j x^+)$ is naturally equal to a composite of factors $\Phi''_m x^\alpha$ with Φ''_m a generalised Φ_m.

We will eventually express a generalised Φ_n in terms of the genuine folding operators Φ_m. In order to do this, we now investigate the faces of generalised foldings.

PROPOSITION 6.4. *Let ψ'_j be a generalised ψ_j. If $i < j$, then $\partial_i^\alpha \psi'_j = \psi''_{j-1} \partial_i^\alpha$ with ψ''_{j-1} a generalised ψ_{j-1}. If $i = j$, then $\partial_i^\alpha \psi'_j x$ is naturally equal to $\partial_j^\beta x$ or*

$\partial^{\beta}_{j+1}x$ for some β, or to a composite of two such factors. If $i>j$, then $\partial^{\alpha}_i\psi'_j = \psi''_j\partial^{\beta}_i$ for some β, with ψ''_j a generalised ψ_j.

Proof. We use relations from Section 2 and Proposition 3.3.

For $i<j$, we have $\partial^{\alpha}_i\psi_j = \psi_{j-1}\partial^{\alpha}_i$ or $\partial^{\alpha}_i\varepsilon_j\partial^{\gamma}_j = \varepsilon_{j-1}\partial^{\alpha}_i\partial^{\gamma}_j = \varepsilon_{j-1}\partial^{\gamma}_{j-1}\partial^{\alpha}_i$ or $\partial^{\alpha}_i\varepsilon_j\partial^{\gamma}_{j+1} = \varepsilon_{j-1}\partial^{\alpha}_i\partial^{\gamma}_{j+1} = \varepsilon_{j-1}\partial^{\gamma}_j\partial^{\alpha}_i$.

For $i=j$, we have $\partial^{-}_j\psi_jx = \partial^{-}_jx \circ_j \partial^{+}_{j+1}x$ or $\partial^{+}_j\psi_jx = \partial^{-}_{j+1}x \circ_j \partial^{+}_jx$ or $\partial^{\alpha}_j\varepsilon_j\partial^{\gamma}_jx = \partial^{\gamma}_jx$ or $\partial^{\alpha}_j\varepsilon_j\partial^{\gamma}_{j+1}x = \partial^{\gamma}_{j+1}x$.

For $i=j+1$, we have $\partial^{\alpha}_{j+1}\psi_j = \varepsilon_j\partial^{\alpha}_j\partial^{\alpha}_{j+1}$ or $\partial^{\alpha}_{j+1}\varepsilon_j\partial^{\gamma}_j = \varepsilon_j\partial^{\alpha}_j\partial^{\gamma}_j = \varepsilon_j\partial^{\gamma}_j\partial^{\alpha}_{j+1}$ or $\partial^{\alpha}_{j+1}\varepsilon_j\partial^{\gamma}_{j+1} = \varepsilon_j\partial^{\alpha}_j\partial^{\gamma}_{j+1}$.

For $i>j+1$, we have $\partial^{\alpha}_i\psi_j = \psi_j\partial^{\alpha}_i$ or $\partial^{\alpha}_i\varepsilon_j\partial^{\gamma}_j = \varepsilon_j\partial^{\alpha}_{i-1}\partial^{\gamma}_j = \varepsilon_j\partial^{\gamma}_j\partial^{\alpha}_i$ or $\partial^{\alpha}_i\varepsilon_j\partial^{\gamma}_{j+1} = \varepsilon_j\partial^{\alpha}_{i-1}\partial^{\gamma}_{j+1} = \varepsilon_j\partial^{\gamma}_{j+1}\partial^{\alpha}_i$. ∎

For a generalised Ψ_r we get the following results.

PROPOSITION 6.5. *Let Ψ'_r be a generalised Ψ_r. If $i\geqslant r$, then $\partial^{\alpha}_i\Psi'_r = \Psi''_r\partial^{\beta}_i$ for some β, with Ψ''_r a generalised Ψ_r. If $i<r$, then $\partial^{\alpha}_i\Psi'_rx$ is naturally equal to a composite of factors $\Psi''_{r-1}\partial^{\beta}_hx$ with $h\leqslant r$ and with Ψ''_{r-1} a generalised Ψ_{r-1}.*

Proof. If $i\geqslant r$, then $\partial^{\alpha}_i\Psi'_r = \partial^{\alpha}_i(\psi'_{r-1}\cdots\psi'_1)$ with ψ'_j a generalised ψ_j, and the result is immediate from Proposition 6.4.

Now suppose that $i<r$. Then

$$\partial^{\alpha}_i\Psi'_rx = \partial^{\alpha}_i(\psi'_{r-1}\cdots\psi'_{i+1})\psi'_i\Psi'_ix,$$

with ψ'_j a generalised ψ_j and with Ψ'_i a generalised Ψ_i. By Proposition 6.4

$$\partial^{\alpha}_i\Psi'_rx = (\psi''_{r-2}\cdots\psi''_i)\partial^{\alpha}_i\psi'_i\Psi'_ix$$

with ψ''_j a generalised ψ_j. By Propositions 6.4 and 6.3, this is a composite of factors of the form

$$(\psi'''_{r-2}\cdots\psi'''_i)\partial^{\gamma}_h\Psi'_ix,$$

with $i\leqslant h\leqslant i+1\leqslant r$ and with ψ'''_j a generalised ψ_j. Since $h\geqslant i$, it follows from the case already covered that the factors can be written as

$$(\psi'''_{r-2}\cdots\psi'''_i)\Psi''_i\partial^{\beta}_hx,$$

with Ψ''_i a generalised Ψ_i. The factors now have the form $\Psi''_{r-1}\partial^{\beta}_hx$ with Ψ''_r a generalised Ψ_r, as required. ∎

By iterating Proposition 6.5, we get the following result.

PROPOSITION 6.6. *If* $i \leqslant m \leqslant n$ *and* Ψ'_r *is a generalised* Ψ_r *for* $m < r \leqslant n$, *then* $\partial_i^\alpha(\Psi'_{m+1} \cdots \Psi'_n)x$ *is naturally equal to a composite of factors* $(\Psi''_m \cdots \Psi''_{n-1})\partial_h^\beta x$ *with* $h \leqslant n$ *and with* Ψ''_r *a generalised* Ψ_r.

Proof. This follows from Propositions 6.5 and 6.3. ∎

Now let Φ'_n be a generalised Φ_n; we aim to express Φ'_n in terms of the genuine folding operators Φ_m. If $n = 0$ or 1, then necessarily $\Phi'_n = \Phi_n$ already. In general, we use an inductive process; the inductive step is as follows.

PROPOSITION 6.7. *Let* Φ'_n *be a generalised* Φ_n *which is distinct from* Φ_n. *Then* $\Phi'_n x$ *is naturally a composite of factors* $\varepsilon_1 \Phi'_{n-1}\partial_h^\beta x$ *with* Φ'_{n-1} *a generalised* Φ_{n-1}.

Proof. By considering the first place where Φ'_n and Φ_n differ, we see that

$$\Phi'_n x = [\Phi_{m-1}(\psi_{m-1} \cdots \psi_{j+1})\varepsilon_j][\partial_i^\alpha \Psi'_j(\Psi'_{m+1} \cdots \Psi'_n)x]$$

for some m and j such that $1 \leqslant j < m \leqslant n$, with $i = j$ or $i = j + 1$ and with Ψ'_r a generalised Ψ_r. Since $j \leqslant m - 1$, it follows from Proposition 3.3 that

$$\Phi_{m-1}(\psi_{m-1} \cdots \psi_{j+1})\varepsilon_j = \Phi_{m-1}\varepsilon_j(\psi_{m-2} \cdots \psi_j) = \varepsilon_1 \Phi_{m-2}(\psi_{m-2} \cdots \psi_j);$$

since $j \leqslant i \leqslant m$, it follows from Propositions 6.5, 6.6 and 6.3 that

$$\partial_i^\alpha \Psi'_j(\Psi'_{m+1} \cdots \Psi'_n)x$$

is a composite of factors $\Psi''_j(\Psi''_m \cdots \Psi''_{n-1})\partial_h^\beta x$ with Ψ''_r a generalised Ψ_r. By Proposition 6.3, $\Phi'_n x$ is then a composite of factors of the form

$$\varepsilon_1 \Phi'_{m-2}(\psi'_{m-2} \cdots \psi'_j)\Psi''_j(\Psi''_m \cdots \Psi''_{n-1})\partial_h^\beta x$$

with Φ'_{m-2} a generalised Φ_{m-2} and with ψ'_k a generalised ψ_k. These factors have the form $\varepsilon_1 \Phi'_{n-1}\partial_h^\beta x$ with Φ'_{n-1} a generalised Φ_{n-1}, as required. ∎

We can now describe the interaction of Φ_n with compositions and face maps in general terms as follows.

PROPOSITION 6.8. *If a composite* $x^- \circ_i x^+$ *exists, then* $\Phi_n(x^- \circ_i x^+)$ *is naturally equal to a composite of factors* $\varepsilon_1^{n-m}\Phi_m D x^\alpha$ *with* D *an* $(n - m)$*-fold product of face operators. If* $i \leqslant n$, *then* $\partial_i^\alpha \Phi_n x$ *is naturally equal to a composite of factors* $\varepsilon_1^{n-m-1}\Phi_m D x$ *with* D *an* $(n - m)$*- fold product of face operators.*

Proof. The result for $\Phi_n(x^- \circ_i x^+)$ comes from Proposition 6.3 by iterated application of Proposition 6.7; recall from (2.5) that $\varepsilon_1(y^- \circ_j y^+)$ is a composite of $\varepsilon_1 y^-$ and $\varepsilon_1 y^+$.

Now suppose that $i \leqslant n$. By Proposition 3.3(iii)

$$\partial_i^\alpha \Phi_n x = \partial_i^\alpha \Phi_i(\Phi_{i+1} \cdots \Phi_n)x = \varepsilon_1^{i-1}(\partial_1^\alpha)^i(\Phi_{i+1} \cdots \Phi_n)x.$$

From Proposition 6.6, this is a composite of factors $\varepsilon_1^{i-1}\Phi'_{n-i}D'x$ with Φ'_{n-i} a generalised Φ_{n-i} and with D' an i-fold product of face operators. By repeated application of Proposition 6.7, there is a further decomposition into factors $\varepsilon_1^{n-m-1}\Phi_m Dx$ with D an $(n-m)$-fold product of face operators.

This completes the proof. ∎

We will now specify the composites in Proposition 6.8 more precisely. Let G be a cubical ω-category, and consider $\Phi_n(x^- \circ_i x^+)$, where $x^- \circ x^+$ is a composite in G_n. The factors $\varepsilon_1^{n-m}\Phi_m Dx^\alpha$ lie in $\Phi_n(G_n)$ (see Proposition 3.7), and their composite can be regarded as a composite in the ω-category $\Phi_n(G_n)$ (see Theorem 3.8). To identify the composite, we take the universal case

$$G = \lambda M(I^{i-1} \times [0,2] \times I^{n-i});$$

we may then identify $\Phi_n(G_n)$ with $M(I^{i-1} \times [0,2] \times I^{n-i})$ by Theorem 4.1. The universal elements

$$x^\alpha \in [\lambda M(I^{i-1} \times [0,2] \times I^{n-i})]_n = \mathrm{Hom}[M(I^n), M(I^{i-1} \times [0,2] \times I^{n-i})]$$

are the inclusions i_i^α representing $M(I^{i-1} \times [0,2] \times I^{n-i})$ as a push-out. Evaluating $\Phi_n(x^- \circ_i x^+)$ and the corresponding composite on I^n, and using Proposition 3.2, we see that $\Phi_n(x^- \circ x^+)$ gives us $I^{i-1} \times [0,2] \times I^{n-i}$ and the factors give us cells in $I^{i-1} \times [0,2] \times I^{n-i}$. The composite for $\Phi_n(x^- \circ_i x^+)$ in Proposition 6.8 is an ω-category formula expressing $I^{i-1} \times [0,2] \times I^{n-i}$ as a composite of cells. All such formulae are equivalent in all ω-categories because of the presentation of $M(I^{i-1} \times [0,2] \times I^{n-i})$ in Theorem 1.3. The formula uses $\#_p$ only for $0 \leqslant p < n$ (see Theorem 3.8). Similarly, the formula for $\partial_i^\alpha \Phi_n x$ is an ω-category formula expressing $d_{n-i}^\alpha I^n$ as a composite of cells (see Propositions 3.6 and 3.2).

In order to state these results more clearly, we introduce the following notation.

Definition 6.9. Let σ be a cell in I^n, and let the dimension of σ be m. Then $\partial_\sigma : G_m \to G_n$ is the cubical ω-category operation of the form

$\partial_{i(1)}^{\alpha(1)} \cdots \partial_{i(n-m)}^{\alpha(n-m)}$ such that the underlying homomorphism

$$\check{\partial}_\sigma : M(I^m) \to M(I^n)$$

sends I^m to σ.

Note that ∂_σ is uniquely determined by σ because of relation 2.1(i).
In this notation, we can state the following theorem.

THEOREM 6.10.

(i) *Let f be a formula expressing $I^{i-1} \times [0,2] \times I^{n-i}$ as a $(\#_0, \cdots, \#_{n-1})$-composite of cells $\check{\imath}_i^-(\sigma)$ and $\check{\imath}_i^+(\sigma)$, where*

$$\check{\imath}_i^\alpha : M(I^n) \to M(I^{i-1} \times [0,2] \times I^{n-i})$$

are the inclusions expressing $M(I^{i-1} \times [0,2] \times I^{n-i})$ as a push-out. Let $x^- \circ_i x^+$ be a composite in a cubical ω-category. Then $\Phi_n(x \circ_i y)$ can be got from f by replacing $\check{\imath}_i^\alpha(\sigma)$ with $\varepsilon_1^{n-m}\Phi_m\partial_\sigma x^\alpha$, where $m = \dim\sigma$, and by replacing $\#_p$ with $\circ_{n-p}$.

(ii) *Let g be a formula expressing $d_{n-i}^\alpha I^n$ as a $(\#_0, \cdots, \#_{n-2})$-composite of cells, where $1 \leqslant i \leqslant n$. In a cubical ω-category, $\partial_i^\alpha \Phi_n x$ can be got from g by replacing σ with $\varepsilon_1^{n-m-1}\Phi_m\partial_\sigma x$, where $m = \dim\sigma$, and by replacing $\#_p$ with $\circ_{n-p}$.*

7. THE NATURAL HOMOMORPHISM $B: G \to \lambda\gamma G$

Let G be a cubical ω-category. We will now use Theorem 6.10 to construct a natural homomorphism $B: G \to \lambda\gamma G$. Let x be a member of G_n. We must define

$$B(x) \in (\lambda\gamma G)_n = \mathrm{Hom}[M(I^n), \gamma G].$$

Now, $M(I^n)$ is generated by the cells in I^n (see Theorem 1.3), and γG is the colimit of the sequence

$$\Phi_0(G_0) \xrightarrow{\varepsilon_1} \Phi_1(G_1) \xrightarrow{\varepsilon_1} \Phi_2(G_2) \to \cdots$$

(see Definition 3.9). We can therefore define $B(x)$ by giving a suitable value to $[B(x)](\sigma)$ for σ a cell in I^n; these values must lie in the $\Phi_m(G_m)$, and a value $\varepsilon_1^s y$ can be identified with y. The precise result is as follows.

THEOREM 7.1. *There is a natural homomorphism $B\colon G \to \lambda\gamma G$ for G a cubical ω-category given by*

$$[B(x)](\sigma) = \Phi_m \partial_\sigma x$$

for σ a cell in I^n, where $m = \dim \sigma$.

Proof. We first show that the values prescribed for the $[B(x)](\sigma)$ really define a homomorphism on $M(I^n)$; in other words, we must show that they respect the relations given in Theorem 1.3. Let σ be an m-dimensional cell in I^n. We must show that $d_m^\alpha(\Phi_m \partial_\sigma x) = \Phi_m \partial_\sigma x$; if $m > 0$ we must also show that $d_{m-1}^\alpha(\Phi_m \partial_\sigma x)$ is the appropriate composite of the $\Phi_l \partial_\tau x$, where $\tau \subset \sigma$.

The first of these equations, $d_m^\alpha(\Phi_m \partial_\sigma x) = \Phi_m \partial_\sigma x$, is an immediate consequence of Theorem 3.8.

For the second equation, let σ be a cell of positive dimension m. By Theorem 3.8,

$$d_{m-1}^\alpha(\Phi_m \partial_\sigma x) = \varepsilon_1 \partial_1^\alpha \Phi_m \partial_\sigma x,$$

which may be identified with $\partial_1^\alpha \Phi_m \partial_\sigma x$. By Theorem 6.10, this is the appropriate composite of the $\Phi_l \partial_\tau x$, as required.

We have now constructed functions $B\colon G_n \to (\lambda\gamma G)_n$, and we must show that these functions form a homomorphism of cubical ω-categories. We must therefore show that $B(\partial_i^\alpha x) = \partial_i^\alpha B(x)$, that $B(\varepsilon_i x) = \varepsilon_i B(x)$, that $B(x^- \circ_i x^+) = B(x^-) \circ_i B(x^+)$, and that $B(\Gamma_i^\alpha x) = \Gamma_i^\alpha B(x)$.

First we consider $B(\partial_i^\alpha x)$, where $x \in G_n$. Let σ be a cell in I^{n-1} of dimension m, and let $\tau = \check{\partial}_i^\alpha(\sigma)$. We then have $\tau = \check{\partial}_i^\alpha \check{\partial}_\sigma(I^m)$, so $\check{\partial}_\tau = \check{\partial}_i^\alpha \check{\partial}_\sigma$ and $\partial_\tau = \partial_\sigma \partial_i^\alpha$. It follows that

$$[B(\partial_i^\alpha x)](\sigma) = \Phi_m \partial_\sigma \partial_i^\alpha x = \Phi_m \partial_\tau x$$

and

$$[\partial_i^\alpha B(x)](\sigma) = [B(x)][\check{\partial}_i^\alpha(\sigma)] = [B(x)](\tau) = \Phi_m \partial_\tau x;$$

therefore, $[B(\partial_i^\alpha x)](\sigma) = [\partial_i^\alpha B(x)](\sigma)$ as required.

Next we consider $B(\varepsilon_i x)$, where $x \in G_n$. Let σ be a cell in I^{n+1} of dimension m. From Definition 2.1, we see that $\partial_\sigma \varepsilon_i$ has the form $\mathrm{id}\, \partial_\tau$ or $\varepsilon_j \partial_\tau$. Let $l = \dim \tau$, so that $l = m$ in the first case and $l = m - 1$ in the second case. Let $\theta\colon G_l \to G_m$ be id or $\varepsilon_j\colon G_l \to G_m$ as the case may be, and let $\check{\theta}\colon M(I^m) \to M(I^l)$ be the underlying ω-category homomorphism. We now see that $\partial_\sigma \varepsilon_i = \theta \partial_\tau$ and $\check{\varepsilon}_i \check{\partial}_\sigma = \check{\partial}_\tau \check{\theta}$ with $\check{\theta}(I^m) = I^l$. It follows that

$$\check{\varepsilon}_i(\sigma) = \check{\varepsilon}_i \check{\partial}_\sigma(I^m) = \check{\partial}_\tau \check{\theta}(I^m) = \check{\partial}_\tau(I^l) = \tau.$$

Using Theorem 5.6, we also see that $\Phi_m\theta = \varepsilon_1^{m-l}\Phi_l$. We now get

$$[B(\varepsilon_i x)](\sigma) = \Phi_m \partial_\sigma \varepsilon_i x = \Phi_m \theta \partial_\tau x = \varepsilon_1^{m-l}\Phi_l \partial_\tau x = \Phi_l \partial_\tau x$$

(recall that $\varepsilon_1^s y$ is to be identified with y) and

$$[\varepsilon_i B(x)](\sigma) = [B(x)][\check{\varepsilon}_i(\sigma)] = [B(x)](\tau) = \Phi_l \partial_\tau x,$$

so that $[B(\varepsilon_i x)](\sigma) = [\varepsilon_i B(x)](\sigma)$ as required.

Next we consider $B(x^- \circ_i x^+)$, where $x^- \circ_i x^+$ is a composite in G_n. Let σ be a cell in I^n of dimension m. From Definition 2.1,

$$[B(x^- \circ_i x^+)](\sigma) = \Phi_m \partial_\sigma (x^- \circ_i x^+)$$

is equal to $\Phi_m \partial_\sigma x^-$ or $\Phi_m \partial_\sigma x^+$ or to $\Phi_m(\partial_\sigma x^- \circ_j \partial_\sigma x^+)$ for some j. In any case, using Theorem 6.10 if necessary, we see that $[B(x^- \circ_i x^+)](\sigma)$ is a composite of factors $\Phi_l \partial_\tau x^\alpha$ such that $\check{\mu}_i(\sigma)$ is the corresponding composite of the $\check{\imath}_i^\alpha(\tau)$, where

$$\check{\imath}_i^-, \check{\imath}_i^+ : M(I^n) \to M(I^{i-1} \times [0,2] \times I^{n-i})$$

are the functions expressing $M(I^{i-1} \times [0,2] \times I^{n-i})$ as a push-out. Let

$$(B(x^-), B(x^+)) : M(I^{i-1} \times [0,2] \times I^{n-i}) \to \gamma G$$

be the function such that

$$(B(x^-), B(x^+))\check{\imath}_i^\alpha = B(x^\alpha);$$

we see that

$$[B(x^- \circ_i x^+)](\sigma) = (B(x^-), B(x^+))\check{\mu}_i(\sigma) = [B(x^-) \circ_i B(x^+)](\sigma)$$

as required.

Finally, we consider $B(\Gamma_i^\alpha x)$, where $x \in G_n$. Let σ be a cell in I^{n+1} of dimension m. From Definition 2.1, $\partial_\sigma \Gamma_i^\alpha$ has the form ∂_τ or $\varepsilon_i \partial_\tau$ or $\Gamma_i^\alpha \partial_\tau$. We can now use the same argument as for $B(\varepsilon_i x)$, noting that $\check{\Gamma}_i^\alpha(I^m) = I^{m-1}$ and that $\Phi_m \Gamma_i^\alpha = \varepsilon_1 \Phi_{m-1}$ by Theorem 5.6.

This completes the proof. ∎

8. THE NATURAL ISOMORPHISM $B: G \to \lambda\gamma G$

In Theorem 4.1, we have constructed a natural isomorphism $A: \gamma\lambda X \to X$ for X an ω-category. In Theorem 7.1, we have constructed a natural homomorphism $B: G \to \lambda\gamma G$ for G a cubical ω-category. We will now show that ω-categories and cubical ω-categories are equivalent by showing that B is an isomorphism.

We begin with the following observation.

PROPOSITION 8.1. *Let G be a cubical ω-category. Then $\gamma B : \gamma G \to \gamma\lambda\gamma G$ is an isomorphism.*

Proof. Consider the composite

$$A \circ (\gamma B) : \gamma G \to \gamma G.$$

By Theorem 4.1, A is an isomorphism; it therefore suffices to show that the composite $A \circ (\gamma B)$ is the identity. This amounts to showing that $AB(x) = x$ for $x \in \Phi_n(G_n)$. Now, from the definitions of A and B, we find that

$$AB(x) = [B(x)](I^n) = \Phi_n x;$$

since $x \in \Phi_n(G_n)$ and Φ_n is idempotent (Proposition 3.5), it follows that $AB(x) = x$ as required. This completes the proof. ∎

Because of Proposition 8.1, to show that B is an isomorphism it suffices to show that a cubical ω-category G is determined by the ω-category γG. Because of Remark 3.10, this is the same as showing that G is determined by the $\Phi_n(G_n)$. We will work inductively, showing that an element x of G_n is determined by $\Phi_n x$ and by its faces. To handle the family of faces of x, we will use the following terminology.

DEFINITION 8.2. Let G be a cubical ω-category and let n be a positive integer. An *n-shell* in G is an ordered $(2n)$-tuple:

$$z = (z_1^-, z_1^+, \cdots, z_n^-, z_n^+)$$

of members of G_{n-1} such that $\partial_i^\alpha z_j^\beta = \partial_{j-1}^\beta z_i^\alpha$ whenever $i < j$. The set of n-shells is denoted $\square\ G_{n-1}$.

Remark 8.3. This construction is used in [9, Section 5] to construct a coskeleton functor from $(n-1)$-truncated cubical ω-groupoids to n-truncated ω-groupoids determined by

$$(G_0, G_1, \cdots, G_{n-1}) \mapsto (G_0, G_1, \cdots, G_{n-1}, \square G_{n-1}),$$

and the same construction clearly works for the category case. It follows that the folding operations are also defined on $\square G_{n-1}$. In the following, we take a slightly more direct route.

First, by Definition 2.1, it is easy to check the following result.

PROPOSITION 8.4. *Let G be a cubical ω-category and let n be a positive integer. There is a boundary map* $\partial : G_n \to \square G_{n-1}$ *given by*

$$\partial x = (\partial_1^- x, \partial_1^+ x, \cdots, \partial_n^- x, \partial_n^+ x).$$

Now we define folding operations on shells directly.

PROPOSITION 8.5. *Let G be a cubical ω-category. For* $1 \leqslant j \leqslant n-1$, *the cubical structure of* $(G_0, \cdots, G_{n-1})$ *yields a natural function* $\psi_j : \square G_{n-1} \to \square G_{n-1}$ *such that*

$$\psi_j \partial = \partial \psi_j : G_n \to \square G_{n-1}.$$

Proof. Let $z = (z_i^\alpha)$ be an n-shell. Guided by Proposition 3.3(i), we let $\psi_i z$ be the $(2n)$-tuple $w = (w_i^\alpha)$ such that

$$w_i^\alpha = \begin{cases} \psi_{j-1} z_i^\alpha & \text{for } i < j, \\ z_j^- \circ_j z_{j+1}^+ & \text{for } (\alpha, i) = (-, j), \\ z_{j+1}^- \circ_j z_j^+ & \text{for } (\alpha, i) = (+, j), \\ \varepsilon_j \partial_j^\alpha z_{j+1}^\alpha & \text{for } i = j+1, \\ \psi_j z_i^\alpha & \text{for } i > j+1. \end{cases}$$

From Proposition 3.1(i) and the identities in Section 2, it is straightforward to check that ψ_j is a well-defined function from $\square G_{n-1}$ to itself, and it is easy to see that $\psi_j \partial = \partial \psi_j$. ∎

We will now show that the n-dimensional elements ($n > 0$) in a cubical ω-category are determined by the lower-dimensional elements and by the image of Φ_n.

THEOREM 8.6. *Let G be a cubical ω-category, let n be a positive integer, and let* $\Phi_n : \square G_{n-1} \to \square G_{n-1}$ *be the function given by*

$$\Phi_n = \psi_1(\psi_2\psi_1)(\psi_3\psi_2\psi_1)\cdots(\psi_{n-1}\cdots\psi_1).$$

Then there is a bijection $x \mapsto (\partial x, \Phi_n x)$ *from* G_n *to the pull-back*

$$\square G_{n-1} \times_{G_n} \Phi_n(G_n) = \{ (z, y) \in \square G_{n-1} \times \Phi_n(G_n) : \Phi_n z = \partial y \}.$$

Proof. This amounts to showing that

$$\Phi_n : \partial^{-1}(z) \to \partial^{-1}(\Phi_n z)$$

is a bijection for each z in $\square G_{n-1}$. Since Φ_n is a composite of operators ψ_j, it suffices to show that

$$\psi_j{:}\partial^{-1}(z) \to \partial^{-1}(\psi_j z)$$

is a bijection for each z in $\square G_{n-1}$.

Given $y \in \partial^{-1}(\psi_j z)$, it is straightforward to check that there is a composite

$$\theta y = (\varepsilon_j z_j^- \circ_{j+1} \Gamma_j^+ z_{j+1}^+) \circ_j y \circ_j (\Gamma_j^- z_{j+1}^- \circ_{j+1} \varepsilon_j z_j^+),$$

and that $\theta y \in \partial^{-1}(z)$. We will carry out the proof by showing that $\theta\psi_j x = x$ for $x \in \partial^{-1}(z)$ and that $\psi_j \theta y = y$ for $y \in \partial^{-1}(\psi_j z)$.

Let x be a member of $\partial^{-1}(z)$. Then

$$\theta\psi_j x \begin{bmatrix} | & \Gamma_j^+ z_{j+1}^- & \Gamma_j^- z_{j+1}^- \\ \varepsilon_j z_j^- & x & \varepsilon_j z_j^+ \\ \Gamma_j^+ z_{j+1}^+ & \Gamma_j^- z_{j+1}^+ & | \end{bmatrix} \begin{array}{l} \xrightarrow{\quad} j \\ \downarrow \\ j+1 \end{array}.$$

The first and third rows are in the image of ε_{j+1} by (2.5) and (2.7), so they are identities for $\circ_{j+1}$ and can therefore be omitted. This leaves the second row in which $\varepsilon_j z_j^-$ and $\varepsilon_j z_j^+$ are identities for $\circ_j$. It follows that $\theta\psi_j x = x$.

Now let y be a member of $\partial^{-1}(\psi_j z)$. By (2.2)(vi) and (2.1)(ii), $\varepsilon_j \partial_j^- \Gamma_j^+ = \varepsilon_j \varepsilon_j \partial_j^- = \varepsilon_{j+1} \varepsilon_j \partial_j^-$, so

$$\begin{aligned} \Gamma_j^+ \partial_{j+1}^- \theta y &= \Gamma_j^+ z_{j+1}^- \\ &= \varepsilon_j \partial_j^- \Gamma_j^+ z_{j+1}^- \circ_j \varepsilon_j \partial_j^- \Gamma_j^+ z_{j+1}^- \circ_j \Gamma_j^+ z_{j+1}^- \\ &= \varepsilon_{j+1} \varepsilon_j \partial_j^- z_{j+1}^- \circ_j \varepsilon_{j+1} \varepsilon_j \partial_j^- z_{j+1}^- \circ_j \Gamma_j^+ z_{j+1}^-. \end{aligned}$$

Similarly,

$$\Gamma_j^- \partial_{j+1}^+ \theta y = \Gamma_j^- z_{j+1}^+ \circ_j \varepsilon_{j+1} \varepsilon_j \partial_j^+ z_{j+1}^+ \circ_j \varepsilon_{j+1} \varepsilon_j \partial_j^+ z_{j+1}^+.$$

It follows that

$$\begin{aligned}\psi_j\theta y &= \Gamma_j^+\partial_{j+1}^-\theta y \circ_{j+1} \theta y \circ_{j+1} \Gamma_j^-\partial_{j+1}^+\theta y \\ &= \begin{bmatrix} | & | & \Gamma_j^+ z_{j+1}^- \\ \varepsilon_j z_j^- \circ_{j+1} \Gamma + -_j z_{j+1}^+ & y & \Gamma_j^- z_{j+1}^- \circ_{j+1} \varepsilon_j z_j^+ \\ \Gamma_j^- z_{j+1}^+ & | & | \end{bmatrix} \begin{array}{l} \xrightarrow{\quad} j \\ \downarrow \\ j+1 \end{array}.\end{aligned}$$

By (2.7) and (2.5), the first and third columns are in the image of ε_j, so they are identities for $\circ_j$ and can be omitted. This leaves the second column so that $\psi_j\theta y = y$.

This completes the proof. ∎

From Theorem 8.6, we deduce the following result.

THEOREM 8.7. *Let $f: G \to H$ be a morphism of cubical ω-categories such that $\gamma f : \gamma G \to \gamma H$ is an isomorphism. Then f is an isomorphism.*

Proof. By Remark 3.10, f induces isomorphisms from $\Phi_n(G_n)$ to $\Phi_n(H_n)$. Since Φ_0 is the identity operation, f induces a bijection from G_0 to H_0. By an inductive argument using Theorem 8.6, f induces a bijection from G_n to H_n for all n. Therefore, f is an isomorphism. ∎

It follows from Proposition 8.1 and Theorem 8.7 that $B: G \to \lambda\gamma G$ is a natural isomorphism for cubical ω-categories G. From Theorem 4.1, $A: \gamma\lambda X \to X$ is a natural isomorphism for ω-categories X. We draw the following conclusion.

THEOREM 8.8. *The categories of ω-categories and of cubical ω-categories are equivalent under the functors λ and γ.*

9. THIN ELEMENTS AND COMMUTATIVE SHELLS IN A CUBICAL ω-CATEGORY

In this section,we use the equivalence of Theorem 8.8 to clarify two concepts in the theory of cubical ω-categories: thin elements and commutative shells. Thin elements (sometimes called hollow elements) were introduced in the thesis of Dakin [17], and were developed in the cubical ω-groupoid context by Brown and Higgins [6, 9, 10]. They are used by Ashley [3] and by Street [27]. In the cubical nerve of an ω-category they arise as follows.

Throughout this section, let G be a cubical ω-category. Whenever convenient, we will identify G with the nerve of γG; in other words, each element x of G_n is identified with a homomorphism $x : M(I^n) \to \gamma G$.

First, we deal with thin elements. Intuitively, an element is thin if its real dimension is less than its apparent dimension. In the nerve of an ω-category we can make this precise as follows.

DEFINITION 9.1. Let x be a member of G_n. Then x is *thin* if

$$\dim x(I^n) < n.$$

Given an element x of G_n, we can identify $x(I^n)$ with $\Phi_n x$ by Theorem 7.1. By Remark 3.10, $\dim \Phi_n x < n$ if and only if $\Phi_n x$ is in the image of ε_1. We therefore have the following characterisation.

PROPOSITION 9.2. *Let x be a member of G_n. Then x is thin if and only if $\Phi_n x$ is in the image of* ε_1.

There is also a less obvious characterisation in more elementary cubical terms: the thin elements of G_n are those generated by the G_m with $m < n$. The precise statement is as follows.

THEOREM 9.3. *Let x be a member of G_n. Then x is thin if and only if it is a composite of elements of the forms $\varepsilon_i y$ and $\Gamma_j^\alpha z$ for various values of i, j, α, y, z.*

Proof. Suppose that x is a composite of elements of the forms $\varepsilon_i y$ and $\Gamma_j^\alpha z$. Then $\Phi_n x$ is in the image of ε_1 by Theorems 5.6 and 6.10, so x is thin by Proposition 9.2.

Conversely, suppose that x is thin. It follows from the proof of Theorem 8.5 that x is a composite of $\Phi_n x$ with elements of the forms $\varepsilon_i y$ and $\Gamma_j^\alpha z$. By Proposition 9.2, $\Phi_n x$ is in the image of ε_1, so x is itself a composite of elements of the forms $\varepsilon_i y$ and $\Gamma_j^\alpha z$. ∎

Next, we deal with commutative shells. There is an obvious concept of commutative square, or commutative 2-shell; we want commutative n-shells for arbitrary positive n. Now an n-shell z in G can be identified with a homomorphism $z : M(d_{n-1}^- I^n \cup d_{n-1}^+ I^n) \to \gamma G$, and we must obviously define a commutative n-shell as follows.

DEFINITION 9.4. For $n > 0$ an n-shell z in G is *commutative* if

$$z(d_{n-1}^- I^n) = z(d_{n-1}^+ I^n).$$

By Theorem 6.10(ii), if z is an n-shell with $n > 0$, then $z(d^{\alpha}_{n-1})$ can be identified with $(\Phi_n z)^{\alpha}_1$, the $(\alpha, 1)$ face of the n-shell $\Phi_n z$ as in Theorem 8.6. We can therefore describe commutative n-shells in cubical terms as follows.

PROPOSITION 9.5. *For $n > 0$ an n-shell z in G is commutative if and only if*

$$(\Phi_n z)^{-}_1 = (\Phi_n z)^{+}_1.$$

10. MONOIDAL CLOSED STRUCTURES

In [2], Al-Agl and Steiner constructed a monoidal closed structure on the category $\omega\text{-}\mathsf{Cat}^{\bigcirc}$ of (globular) ω-categories by using a cubical description of that category. Now that we have a more explicit cubical description we can give a more explicit description of the monoidal closed structure; we modify the construction which is given by Brown and Higgins [13] for the case of a single connection and for groupoids rather than categories. Following the method there, we first define the closed structure on the category $\omega\text{-}\mathsf{Cat}^{\square}$ of cubical ω-categories using a notion of n-fold left homotopy which we outline below, and then obtain the tensor product as the adjoint to the closed structure. This gives:

THEOREM 10.1. *The category $\omega\text{-}\mathsf{Cat}^{\square}$ admits a monoidal closed structure with an adjoint relationship*

$$\omega\text{-}\mathsf{Cat}^{\square}(G \otimes H, K) \cong \omega\text{-}\mathsf{Cat}^{\square}(G, \omega\text{-}\mathsf{CAT}^{\square}(H, K))$$

in which $\omega\text{-}\mathsf{CAT}^{\square}(H, K)_0$ is the set of morphisms $H \to K$, while for $n \geqslant 1$ $\omega\text{-}\mathsf{CAT}^{\square}\ (H, K)_n$ is the set of n-fold left homotopies $H \to K$.

The proof is given below.

Because of the equivalence between $\omega\text{-}\mathsf{Cat}^{\square}$ and the category $\omega\text{-}\mathsf{Cat}^{\circ}$ of ω-categories we have:

COROLLARY 10.2. *The category $\omega\text{-}\mathsf{Cat}^{\bigcirc}$ admits a monoidal closed structure with an adjoint relationship*

$$\omega\text{-}\mathsf{Cat}^{\bigcirc}(X \otimes Y, Z) \cong \omega\text{-}\mathsf{Cat}^{\bigcirc}(X, \omega\text{-}\mathsf{CAT}^{\bigcirc}(Y, Z))$$

in which $\omega^{\bigcirc}\text{-}\mathsf{CAT}(Y, Z)_0$ is the set of morphisms $Y \to Z$, while for $n \geqslant 1$ $\omega\text{-}\mathsf{CAT}^{\bigcirc}(Y, Z)_n$ is the set of n-fold left homotopies $Y \to Z$ corresponding to the cubical homotopies.

The tensor product in Corollary 10.2 is an extension of the tensor product in Theorem 1.8.

Note that by Remark 3.11, the set $\omega\text{-}\mathsf{CAT}^{\bigcirc}(Y, Z)_n$ of globular n-fold left homotopies may be thought of as an explicitly described subset of the set of cubical n-fold left homotopies $\lambda Y \to \lambda Z$. Because of the complications of the folding operations, explicit descriptions of the globular monoidal closed structure are not so easy, but have been partly accomplished by Steiner [26]. See also Crans [16].

We now give details of these cubical constructions, following directly the methods of [13].

Let H be a cubical ω-category and n be a non-negative integer. We can construct a cubical ω-category $P^n H$ called the *n-fold* (*left*) *path cubical ω-category* of H as follows: $(P^n H)_r = H_{n+r}$; the operations ∂_i^α, ε_i, Γ_i^α and $\circ_i$ of $P^n H$ are the operations ∂_{n+i}^α, ε_{n+i}, Γ_{n+i}^α and $\circ_{n+i}$ of H. The operations $\partial_1^\alpha, \cdots, \partial_n^\alpha$ not used in $P^n H$ give us morphisms of cubical ω-categories from $P^n H$ to $P^{n-1} H$, etc., and we get an internal cubical ω-category

$$\mathsf{P}H = (H, P^1 H, P^2 H, \cdots)$$

in the category $\omega\text{-}\mathsf{Cat}^\square$.

For any cubical ω-categories G, H we now define

$$\omega\text{-}\mathsf{CAT}^\square(G, H) = \omega\text{-}\mathsf{Cat}^\square(G, \mathsf{P}H);$$

that is, $\omega\text{-}\mathsf{CAT}_m^\square(G, H) = \omega\text{-}\mathsf{Cat}^\square(G, P^m H)$, and the cubical ω-category structure on $\omega\text{-}\mathsf{CAT}_m^\square(G, H)$ is induced by the internal cubical ω-category structure on $\mathsf{P}H$. Ultimately, this means that the operations ∂_i^α, etc. on $\omega\text{-}\mathsf{CAT}_m^\square(G, H)$ are induced by the similarly numbered operations on H. In dimension 0, $\omega\text{-}\mathsf{CAT}^\square(G, H)$ consists of all morphisms $G \to H$, while in dimension n it consists of *n-fold* (*left*) *homotopies* $G \to H$. We make ω-$\mathsf{CAT}^\square(G, H)$ a functor in G and H (contravariant in G) in the obvious way.

The definition of tensor product of cubical ω-categories is harder. We require that $-\otimes G$ be left adjoint to $\omega\text{-}\mathsf{CAT}^\square(G, -)$ as a functor from ω-$\mathsf{Cat}^\square$ to $\omega\text{-}\mathsf{Cat}^\square$, and this determines $\otimes$ up to natural isomorphism. Its existence, that is, the representability of the functor $\omega\text{-}\mathsf{Cat}^\square(F, \omega\text{-}\mathsf{CAT}^\square(G, -))$, can be asserted on general grounds. Indeed, $\omega\text{-}\mathsf{Cat}^\square$ is an equationally defined category of many sorted algebras in which the domains of the operations are defined by finite limit diagrams, and general theorems on such algebraic categories imply that $\omega\text{-}\mathsf{Cat}^\square$ is complete and cocomplete.

We can also specify the tensor product cubical ω-category by a *presentation*; that is, we give a set of generators in each dimension and a set of relations of the form $u = v$, where u, v are well-formed formulae of the same dimension made from generators and the operators $\partial_i^\alpha, \varepsilon_i, \Gamma_i^\alpha, \circ_i$. This is analogous to the standard tensor product of modules over a ring, and the universal property of the presentation gives the required adjointness.

The details are as follows.

DEFINITION 10.3. Let F, G be cubical ω-categories. Then $F \otimes G$ is the cubical ω-category generated by elements in dimension $n \geqslant 0$ of the form $x \otimes y$ where $x \in F_p,\ y \in G_q$ and $p + q = n$, subject to the following defining relations (plus, of course, the laws for cubical ω-categories):

(i)

$$\partial_i^\alpha (x \otimes y) = \begin{cases} (\partial_i^\alpha x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\partial_{i-p}^\alpha y) & \text{if } p + 1 \leqslant i \leqslant n; \end{cases}$$

(ii) $\varepsilon_i (x \otimes y) = \begin{cases} (\varepsilon_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p + 1, \\ x \otimes (\varepsilon_{i-p} y) & \text{if } p + 1 \leqslant i \leqslant n + 1; \end{cases}$

(iii) $\Gamma_i^\alpha (x \otimes y) = \begin{cases} (\Gamma_i^\alpha x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\Gamma_{i-p}^\alpha y) & \text{if} p + 1 \leqslant i \leqslant n; \end{cases}$

(iv) $(x \circ_i x') \otimes y = (x \otimes y) \circ_i (x' \otimes y)$ if $1 \leqslant i \leqslant p$, and $x \circ_i x'$ is defined in F;

(v) $x \otimes (y \circ_j y') = (x \otimes y) \circ_{p+j} (x \otimes y')$ if $1 \leqslant j \leqslant q$, and $y \circ_j y'$ is defined in G;

we note that the relation

(vi) $(\varepsilon_{p+1} x) \otimes y = x \otimes (\varepsilon_1 y)$ follows from (ii).

An alternative way of stating this definition is to define a *bimorphism* $(F, G) \to A$, where F, G, A are cubical ω-categories, to be a family of maps $F_p \times G_q \to A_{p+q}$ $(p, q \geqslant 0)$, denoted by $(x, y) \mapsto \chi(x, y)$ such that

(a) for each $x \in F_p$, the map $y \mapsto \chi(x, y)$ is a morphism of cubical ω-categories $G \to P^p A$;

(b) for each $g \in G_q$ the map $x \mapsto \chi(x, y)$ is a morphism of cubical ω-categories $F \to TP^q TA$,

where the cubical ω-category TX has the same elements as X but its cubical operations, connections and compositions are numbered in reverse order. The cubical ω-category $F \otimes G$ is now defined up to natural isomorphisms by the two properties:

(i) the map $(x, y) \mapsto x \otimes y$ is a bimorphism $(F, G) \to F \otimes G$;

(ii) every bimorphism $(F, G) \to A$ is uniquely of the form $(x, y) \mapsto \sigma(x \otimes y)$ where $\sigma : F \otimes G \to A$ is a morphism of cubical ω-categories.

In the definition of a bimorphism $(F, G) \to A$, condition (a) gives maps $F_p \to \omega\text{-}\mathsf{CAT}^{\square}_p(G, A)$ for each p, and condition (b) states that these combine to give a morphism of cubical ω-categories $F \to \omega\text{-}\mathsf{CAT}^{\square}(G, A)$. This observation yields a natural bijection between bimorphisms $(F, G) \to A$ and morphisms $F \to \omega\text{-}\mathsf{CAT}^{\square}(G, A)$. Since we also have a natural bijection between bimorphisms $(F, G) \to A$ and morphisms $F \otimes G \to A$, we have

PROPOSITION 10.4. *The functor* $-\otimes G$ *is left adjoint to the functor* $\omega\text{-}\mathsf{CAT}^{\square}(G, -)$ *from* $\omega\text{-}\mathsf{Cat}^{\square}$ to $\omega\text{-}\mathsf{Cat}^{\square}$.

PROPOSITION 10.5. *For cubical* ω*-categories* F, G, H, *there are natural isomorphisms of cubical* ω*-categories*

(i) $(F \otimes G) \otimes H \cong F \otimes (G \otimes H)$, *and*

(ii) $\omega\text{-}\mathsf{CAT}^{\square}(F \otimes G, H) \cong \omega\text{-}\mathsf{CAT}^{\square}(F, \omega\text{-}\mathsf{CAT}^{\square}(G, H))$ *giving* $\omega\text{-}\mathsf{Cat}^{\square}$ *the structure of a monoidal closed category.*

Proof. (i) This isomorphism may be proved directly, or, as is well known, be deduced from the axioms for a monoidal closed category.

(ii) In dimension r there is by adjointness a natural bijection

$$\begin{aligned}
\omega\text{-}\mathsf{CAT}^{\square}_r(F \otimes G, H) &= \omega\text{-}\mathsf{Cat}^{\square}(F \otimes G, P^r H)\\
&\cong \omega\text{-}\mathsf{Cat}^{\square}(F, \omega\text{-}\mathsf{CAT}^{\square}(G, P^r H))\\
&= \omega\text{-}\mathsf{Cat}^{\square}(F, P^r(\omega\text{-}\mathsf{CAT}^{\square}(G, H)))\\
&= \omega\text{-}\mathsf{CAT}^{\square}_r(F, \omega\text{-}\mathsf{CAT}^{\square}(G, H)).
\end{aligned}$$

These bijections combine to form the natural isomorphism (ii) of cubical ω-categories because, on both sides, the cubical ω-category structures are induced by the corresponding operators $\partial^{\alpha}_i, \varepsilon_j$, etc. in H. ∎

We can also relate the construction to the category of cubical sets, which we denote Cub. The underlying cubical set functor $U : \omega\text{-}\mathsf{CAT}^{\square} \to \mathsf{Cub}$ has a left adjoint $\sigma : \mathsf{Cub} \to \omega\text{-}\mathsf{Cat}^{\square}$, and we call $\sigma(K)$ the *free cubical* ω*-category on the cubical set* K. The category Cub has a monoidal closed structure in the same way as $\omega\text{-}\mathsf{Cat}^{\square}$ (see [13]); the internal hom CUB is given by $\mathsf{CUB}(L, M)_r = \mathsf{Cub}(L, P^r M)$ where P^r is now the n-fold path functor on cubical sets. We have the following results.

PROPOSITION 10.6. *For a cubical set* L *and cubical* ω*-category* G, *there is a natural isomorphism of cubical sets*

$$U(\omega\text{-}\mathsf{CAT}^{\square}(\sigma(L), G)) \cong \mathsf{CUB}(L, UG).$$

Proof. The functor $\sigma\colon \mathsf{Cub} \to \omega\text{-}\mathsf{Cat}^{\square}$ is left adjoint to $U\colon \omega\text{-}\mathsf{Cat}^{\square} \to \mathsf{Cub}$, and this is what the proposition says in dimension 0. In dimension r, we have a natural bijection

$$\begin{aligned}
\omega\text{-}\mathsf{CAT}^{\square}_r(\sigma(L), G) &= \omega\text{-}\mathsf{Cat}^{\square}(\sigma(L), P^r G)\\
&\cong \mathsf{Cub}(L, UP^r G)\\
&= \mathsf{CUB}_r(L, UG)
\end{aligned}$$

and these bijections are compatible with the cubical operators. ∎

PROPOSITION 10.7. *If K, L are cubical sets, there is a natural isomorphism of cubical ω-categories*

$$\sigma(K) \otimes \sigma(L) \cong \sigma(K \otimes L).$$

Proof. For any cubical ω-category G, there are natural isomorphisms of cubical sets

$$\begin{aligned}
U(\omega\text{-}\mathsf{CAT}^{\square}(\sigma(K) \otimes \sigma(L), G)) &\cong U(\omega\text{-}\mathsf{CAT}^{\square}(\sigma(K), \omega\text{-}\mathsf{CAT}^{\square}(\sigma(L), G)))\\
&\cong \mathsf{CUB}(K, U(\omega\text{-}\mathsf{CAT}^{\square}(\sigma(L), G)))\\
&\cong \mathsf{CUB}(K, \mathsf{CUB}(L, UG))\\
&\cong \mathsf{CUB}(K \otimes L, UG)\\
&\cong U(\omega\text{-}\mathsf{CAT}^{\square}(\sigma(K \otimes L), G).
\end{aligned}$$

The proposition follows from the information in dimension 0, namely

$$\omega\text{-}\mathsf{Cat}^{\square}(\sigma(K \otimes L), G) \cong \omega\text{-}\mathsf{Cat}^{\square}(\sigma(K) \otimes \sigma(L), G)). \quad ∎$$

The ω-categories $M(I^n)$ of Section 1 can be fitted into this framework if one regards them as cubical ω-categories. Indeed, as a cubical ω-category, $M(I^n)$ is freely generated by one element in dimension n; therefore, $M(I^n) = \sigma(\mathbb{I}^n)$ where $\mathbb{I}^n$ is the cubical set freely generated by one element in dimension n. Calculations with cubical sets show that $\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$, and we get the following result.

COROLLARY 10.8. *These are natural isomorphisms of cubical ω-categories*

$$M(I^m) \otimes M(I^n) \cong M(I^{m+n}). \quad ∎$$

PROPOSITION 10.9.

(i) $M(I^n) \otimes -$ *is left adjoint to* $P^n : \omega\text{-}\mathsf{Cat}^{\square} \to \omega\text{-}\mathsf{Cat}^{\square}$.

(ii) $- \otimes M(I^n)$ *is left adjoint to* $\omega\text{-}\mathsf{CAT}^{\square}(M(I^n), -)$.

(iii) $\omega\text{-}\mathsf{CAT}^{\square}(M(I^n), -)$ *is naturally isomorphic to* TP^nT.

Proof. (i) There are natural bijections

$$\begin{aligned} \omega\text{-}\mathsf{Cat}^{\square}(M(I^n) \otimes H, K) &\cong \omega\text{-}\mathsf{Cat}^{\square}(M(I^n), \omega\text{-}\mathsf{CAT}^{\square}(H, K)) \\ &\cong \omega\text{-}\mathsf{CAT}^{\square}_n(H, K) \\ &= \omega\text{-}\mathsf{Cat}^{\square}(H, P^n K). \end{aligned}$$

(ii) This is a special case of Proposition 10.4.

(iii) It follows from (i) that $TP^nT : \omega\text{-}\mathsf{Cat}^{\square} \to \omega\text{-}\mathsf{Cat}^{\square}$ has left adjoint $T(M(I^n) \otimes T(-)) \cong - \otimes TM(I^n)$. But the obvious isomorphism $T\mathbb{I} \to \mathbb{I}$ induces an isomorphism $TM(I^n) \cong M(I^n)$, so $- \otimes TM(I^n)$ is naturally isomorphic to $- \otimes M(I^n)$. The result now follows from (ii). ∎

The free cubical ω-category on a cubical set is important in applications to concurrency theory. The data for a concurrent process can be given as a cubical set K, and the evolution of the data can be reasonably described by the free cubical ω-category $\sigma(K)$; indeed, $\sigma(K)$ is the higher-dimensional analogue of the path category on a directed graph. The idea is pursued by Gaucher in [20].

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