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The Ariki–Terasoma–Yamada tensor space and the blob algebra

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ABSTRACT

We show that the Ariki–Terasoma–Yamada tensor module and its permutation submodules $M(\lambda)$ are modules for the blob algebra when the Ariki–Koike algebra is a Hecke algebra of type B. We show that $M(\lambda)$ and the standard modules $\Delta(\lambda)$ have the same dimensions, the same localization and similar restriction properties and are equal in the Grothendieck group. Still we find that the universal property for $\Delta(\lambda)$ fails for $M(\lambda)$, making $M(\lambda)$ and $\Delta(\lambda)$ different modules in general. Finally, we prove that $M(\lambda)$ is isomorphic to the dual Specht module for the Ariki–Koike algebra.

1. Introduction

In this paper we combine the representation theories of the Ariki–Koike algebra and of the blob algebra. The link between the two theories is the tensor space module $V^{\otimes n}$ for the Ariki–Koike algebra defined in [ATY] by Ariki, Terasoma and Yamada.

The blob algebra $b_n = b_n(q,m)$ was defined by Martin and Saleur [MS] as a generalization of the Temperley–Lieb algebra by introducing periodicity in the statistical mechanics model. The blob algebra is also sometimes called the Temperley–Lieb algebra of type B, or the one-boundary Temperley–Lieb algebra, and indeed it has a diagram calculus generalizing the Temperley–Lieb diagram calculus. Our work treats the non-semisimple representation theory of b_n.

There is a natural embedding $b_n \subset b_{n+1}$ which gives rise to restriction and induction functors between the module categories. These functors are part of a powerful category theoretical formalism on the representation theory of the entire tower of algebras. It also involves certain localization and globalization functors F and G between the categories of b_n-modules for different n. We denote it the localization/globalization formalism.

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The formalism is closely related to the fact that b_n is quasi-hereditary in the sense of Cline, Parshall and Scott [CPS] (when $q+q^{-1} \neq 0$). Its parameterizing poset is $\Lambda_n := \{n, n-2, \ldots, -n\}$. The standard modules $\Delta_n(\lambda)$, $\lambda \in \Lambda_n$, can be defined by a diagram basis and have dimensions equal to certain binomial coefficients.

A main point of our work is the existence of a surjection π from the Hecke algebra $H(n,2) = H_n(q, \lambda_1, \lambda_2)$ of type B_n to the blob algebra b_n, for appropriate choices of the parameters. It makes it possible to pullback b_n-modules to $H(n,2)$-modules and in this way the category of b_n-modules may be viewed as a subcategory of the $H(n,2)$-modules.

Since $H(n,2)$ is a special case of an Ariki–Koike algebra it has a tensor module $V^{\otimes n}$ as described in [ATY]. As a first result we prove that $V^{\otimes n}$ and its 'permutation' submodules $M_n(\lambda)$ are b_n-module when $\dim V = 2$. We are then in position to apply the localization/globalization formalism to the module $M_n(\lambda)$, and to compare it to the standard module $\Delta_n(\lambda)$.

In our main results we show that the two modules have the same dimensions, share the same localization properties and even are equal in the Grothendieck group of b_n-modules. They also have related behaviors under restriction from b_n to b_{n-1}. Even so we find that $M_n(\lambda)$ and $\Delta_n(\lambda)$ are different modules in general. We show this by demonstrating that the universal property for $\Delta_n(\lambda)$ fails for $M_n(\lambda)$. To be more precise, we show that in general $GFM_n(\lambda) \ncong M_n(\lambda)$ whereas it is known that $GF\Delta_n(\lambda) \cong \Delta_n(\lambda)$ (when $\lambda \neq \pm n$).

This rises the question whether $M_n(\lambda)$ may be identified with another 'known' module. We settle this question by considering the Specht module $S(n_1, n_2)$ for $H(n,2)$, where (n_1, n_2) is a two-line bipartition associated with λ. We show that this module is the pullback of a b_n-module, also denoted $S(n_1, n_2)$, and that $M_n(\lambda)$ is isomorphic to the contragredient dual of $S(n_1, n_2)$.

We find that, somewhat surprisingly, neither of the b_n-modules $M_n(\lambda)$, $S(n_1, n_2)$ nor their duals identify with the standard module $\Delta_n(\lambda)$ for b_n.

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2. Preliminaries

In this section we shall briefly recall the results of [MW] and [ATY], the two main sources of inspiration for the present paper. Let us start out by considering the work of Martin and Woodcock [MW]. Among other things they realize the blob algebra b_n as a quotient of the Ariki–Koike algebra $H(n,2)$ by the ideal generated by the idempotents associated with certain irreducible representations of $H(2,2)$. It then turns out that this ideal has a simple description in terms of the $H(n,2)$-generators. Let us explain all this briefly.

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}, \lambda_1, \lambda_2]$. Let $H(n,2) = H(n, q, \lambda_1, \lambda_2)$ be the unital $\mathcal{A}$-algebra generated by $\{X, g_1, \ldots, g_{n-1}\}$ with relations

$$g_i g_{i\pm1} g_i = g_{i\pm1} g_i g_{i\pm1}, \qquad [g_i, g_j] = 0, \quad i \neq j \pm 1,$$
$$g_1 X g_1 X = X g_1 X g_1, \qquad [X, g_j] = 0, \quad j > 1,$$
$$(g_i - q)\bigl(g_i + q^{-1}\bigr) = 0,$$
$$(X - \lambda_1)(X - \lambda_2) = 0.$$

It is the $d = 2$ case of the Ariki–Koike algebra $H(n,d)$ or the cyclotomic Hecke algebra of type $G(d,1,n)$, see [AK] and [BM]. For $\lambda_1 = -\lambda_2^{-1}$ it is the Hecke algebra of type B_n. Note that there is a canonical embedding $H(n,2) \subset H(n+1,2)$.

As usual, if k is an $\mathcal{A}$-algebra we write $H_k(n,2) := H(n,2) \otimes_{\mathcal{A}} k$ for the specialized algebra.

Recall the concept of cellular algebras, that was introduced by Graham and Lehrer in [GL] in order to provide a common framework for many algebras that appear in non-semisimple representation theory. It is shown in [GL] that the Ariki–Koike algebra is cellular for general parameters n, d. In our case $d = 2$ it also follows from [DJM].

Let k be a field and suppose that k is made into an $\mathcal{A}$-algebra by mapping q, λ_1, λ_2 to nonzero elements q, λ_1, λ_2 of k. Assume that $q^4 \neq 1$, $\lambda_1 \neq \lambda_2$ and $\lambda_1 \neq q^2\lambda_2$. Then there are formulas for $e^{-1}, e^{-2} \in H_k(2,2)$, the primitive central idempotents corresponding to the two one-dimensional cell representations given by $(1^2, \emptyset)$, $(\emptyset, 1^2)$, see [MW] for a more precise statement concerning the actual cell modules that we are referring to and for the details. Let $I \subset H_k(n,2)$ be the ideal in $H_k(n,2)$ generated by e^{-1}, e^{-2}. Using the mentioned formulas, it is shown in (27) of [MW] that I is generated by either of the elements

$$\big(X_1 + X_2 - (\lambda_1 + \lambda_2)\big)(g_1 - q),$$
$$(X_1X_2 - \lambda_1\lambda_2)(g_1 - q)$$

where as usual $X_1 := X$, $X_i := g_{i-1}X_{i-1}g_{i-1}$ for $i = 2, 3, \ldots$.

Let $m \in \mathbb{Z}$ and assume that n is a positive integer. The blob algebra $b_n = b_n(q,m)$ is the unital k-algebra on generators $\{U_0, U_1, \ldots, U_{n-1}\}$ and relations

$$U_iU_{i\pm1}U_i = U_i, \qquad U_i^2 = -[2]U_i, \qquad U_0^2 = -[m]U_0, \qquad U_1U_0U_1 = [m-1]U_1$$

for $i > 0$ and commutativity between the generators otherwise. As usual $[m]$ is here the Gaussian integer $[m] := \frac{q^m - q^{-m}}{q - q^{-1}}$. The blob algebra was introduced in [MS] via a basis of decorated Temperley–Lieb algebras, which explains its name. We shall however mostly need the above presentation of it. This is only one of several different presentations of b_n, the one used in [MW].

Let $H^{\mathcal{D}}(n,2)$ be the quotient $H_k(n,2)/I$ and choose

$$\lambda_1 = \frac{q^m}{q - q^{-1}} \quad \text{and} \quad \lambda_2 = \frac{q^{-m}}{q - q^{-1}}.$$

Using the above description of I, it is then shown in Proposition (4.4) of [MW] that the map φ given by $\varphi : g_i - q \mapsto U_i$, $X - \lambda_1 \mapsto U_0$ induces a k-algebra isomorphism

$$\varphi : H^{\mathcal{D}}(n,2) \cong b_n(q,m). \tag{1}$$

We finish this section by recalling the construction of the tensor representation of the Ariki–Koike algebra $H(n,d)$ found by Ariki, Terasoma and Yamada [ATY]. It is an extension to the Ariki–Koike case of Jimbo's classical tensor representation of the Hecke algebra, [J], and therefore basically amounts to the extra action of X factorizing through the relations. On the other hand, this action is quite non-trivial and is for example not local in the sense of [MW].

The [ATY] construction works for all Ariki–Koike algebras $H(n,d)$, but we shall only need the $d = 2$ case, which we now explain. Let V be a free $\mathcal{A}$-module of rank two and let v_1, v_2 be a basis. Let $R \in \mathrm{End}_{\mathcal{A}}(V \otimes V)$ be given by

$$\left\{ \begin{array}{c} R(v_i \otimes v_j) = q v_i \otimes v_j \quad \text{if } i = j \\ R(v_2 \otimes v_1) = v_1 \otimes v_2 \\ R(v_1 \otimes v_2) = v_2 \otimes v_1 + \big(q - q^{-1}\big) v_1 \otimes v_2 \end{array} \right\}.$$

Then the $H(n,2)$ generator g_i acts on $V^{\otimes n}$ through

$$T_{i+1} := Id^{\otimes i-1} \otimes R \otimes Id^{\otimes n-i-1}.$$

The g_i generate a subalgebra of $H(n,d)$ isomorphic to the Iwahori–Hecke algebra of type A and the above action is the $\dim V = 2$ case of the one found by Jimbo in [J]. The maximal quotient of it acting faithfully on $V^{\otimes n}$ is the Temperley–Lieb algebra TL_n.

For $j = 2, 3, \dots, n$ we shall need the $\mathcal{A}$-linear map $S_j \in \mathrm{End}_{\mathcal{A}}(V^{\otimes n})$, that by definition acts on $v = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_{j-1}} \otimes v_{i_j} \otimes \cdots \otimes v_{i_n}$ through

$$S_j(v) = \begin{cases} qv & \text{if } i_{j-1} = i_j, \\ v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_j} \otimes v_{i_{j-1}} \otimes \cdots \otimes v_{i_n} & \text{otherwise.} \end{cases}$$

Let $\theta := S_n S_{n-1} \cdots S_2$ and let $\varpi \in \mathrm{End}_{\mathcal{A}}(V^{\otimes n})$ be the map given by

$$v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \mapsto \lambda_{\delta(1)} v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$$

where $\delta(1) = 1$ if $i_1 = 1$ and $\delta(1) = 2$ if $i_1 = 2$. Then $\theta\varpi$ is given by

$$\theta\varpi : v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes \cdots \otimes v_{i_n} \mapsto \lambda_{\delta(1)} q^{a-1} v_{i_2} \otimes v_{i_3} \otimes \cdots \otimes v_{i_n} \otimes v_{i_1}$$

where a is the number of i_k such that $i_k = i_1$. Now [ATY] defines the action of $X \in H(n, 2)$ by the formula

$$T_1 := T_2^{-1} T_3^{-1} \cdots T_n^{-1} \theta\varpi.$$

As mentioned in [ATY], the proof that the $T_1, T_2, \dots, T_{n-1}$ satisfy the Ariki–Koike relations works in specializations as well. One of the steps of their proof is the following lemma, which we shall need later on.

Lemma 1. *Let $Y_{j,p}$ be the $\mathcal{A}$-submodule of $V^{\otimes n}$ generated by basis elements $v = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$ such that $i_p \geqslant j$. Then if $v \in Y_{j,p}$ we have that*

$$T_{p+1}^{-1} T_{p+2}^{-1} \cdots T_n^{-1} S_n S_{n-1} \cdots S_{p+1} v = v \mod Y_{j+1,p}.$$

3. The Ariki–Terasoma–Yamada tensor space as blob algebra module

From now on we assume that k is an algebraically closed field, such that $q, \lambda_1, \lambda_2 \in k$ and $q^4 \neq 1$, $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq q^2 \lambda_2$. We moreover assume that $\lambda_1 = \frac{q^m}{q-q^{-1}}$ and $\lambda_2 = \frac{q^{-m}}{q-q^{-1}}$ where m is an integer. With these assumptions the results of the previous section are valid.

In this section we prove that the Ariki–Koike action given by the above construction factors through the blob algebra. Let V, T_i be as in the previous section. Then we have

Theorem 1. $(T_1 T_2 T_1 T_2 - \lambda_1 \lambda_2)(T_2 - q) = 0$ *in* $\mathrm{End}_k(V^{\otimes n})$.

Proof. We start by noting that by the Ariki–Koike relations

$$(T_1 T_2 T_1 T_2 - \lambda_1 \lambda_2)(T_2 - q) = (T_2 - q)(T_1 T_2 T_1 T_2 - \lambda_1 \lambda_2).$$

We show that $(T_1 T_2 T_1 T_2 - \lambda_1 \lambda_2)(T_2 - q) = 0$ on all basis elements of $V^{\otimes n}$. It clearly holds for $v = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$ where $i_1 = i_2$, so we assume $i_1 \neq i_2$. If $i_1 = 2$ and $i_2 = 1$ we get by Lemma 1 that the action of T_1 on v is multiplication by λ_2. But then $T_2 T_1 T_2$ acts on v through

$$\begin{aligned} T_2 T_1 T_2 (v_2 \otimes v_1 \otimes \cdots \otimes v_{i_n}) &= T_2 T_2^{-1} T_3^{-1} \cdots T_n^{-1} S_n S_{n-1} \cdots S_2 \varpi T_2 (v_2 \otimes v_1 \otimes \cdots \otimes v_{i_n}) \\ &= \lambda_1 T_3^{-1} \cdots T_n^{-1} S_n S_{n-1} \cdots S_3 (v_2 \otimes v_1 \otimes \cdots \otimes v_{i_n}) \\ &= \lambda_1 (v_2 \otimes v_1 \otimes \cdots \otimes v_{i_n}) \mod Y_{2,2} \end{aligned}$$

by Lemma 1 once again. Actually, since $T_3^{-1} \cdots T_n^{-1} S_n S_{n-1} \cdots S_3$ does not change the first coordinate of v we can even calculate modulo the subspace Y_2 of $V^{\otimes n}$ generated by $v_2 \otimes v_2 \otimes v_{i_3} \otimes \cdots \otimes v_{i_n}$. We conclude that $(T_1T_2T_1T_2 - \lambda_1\lambda_2)v \in Y_2$. But clearly $T_2 - q$ kills Y_2 and we are done in this case.

On the other hand, we have that

$$V^{\otimes n} = \ker(T_2 - q) + \operatorname{span}_k\{v_2 \otimes v_1 \otimes v_{i_3} \otimes \cdots \otimes v_{i_n} \mid i_j = 1, 2 \text{ for } j \geqslant 3\}$$

and hence $V^{\otimes n}$ is also equal to

$$\ker(T_2 - q)(T_1T_2T_1T_2 - \lambda_1\lambda_2) + \operatorname{span}_k\{v_2 \otimes v_1 \otimes v_{i_3} \otimes \cdots \otimes v_{i_n} \mid i_j = 1, 2 \text{ for } j \geqslant 3\}.$$

Combining with the above, the theorem follows. □

Remark 1. The formula of the theorem is easy to implement in a computer system and amusing to verify.

Corollary 1. *$V^{\otimes n}$ is a $b_n(q, m)$-module with U_i, $i \geqslant 1$, acting through $T_{i+1} - q$ and U_0 through $T_1 - \lambda_1$.*

Proof. Using that $\lambda_1 = \frac{q^m}{q-q^{-1}}$ and $\lambda_2 = \frac{q^{-m}}{q-q^{-1}}$ (and the other assumptions on the parameters) this follows from the theorem and Proposition (4.4) of [MW]. □

4. Localization and globalization

The main results of our paper depend on a category theoretical approach to the representation theory of b_n that we shall now briefly explain. It was introduced by J.A. Green in the Schur algebra setting, [G], but has turned out to be useful in the context of diagram algebras as well, see e.g. [CDM, MR]. In the case of the blob algebra b_n, a good reference to the formalism is [MW1], see also [CGM].

Recall first that $[2] \neq 0$ in k so that we can define $e := -\frac{1}{[2]}U_{n-1}$. This is an idempotent of b_n and we have that $eb_ne \cong b_{n-2}$, see [MW1]. Hence it gives rise to the exact localization functor

$$F : b_n\text{-mod} \to b_{n-2}\text{-mod}, \quad M \mapsto eM.$$

It has a left-adjoint, the globalization functor

$$G : b_{n-2}\text{-mod} \to b_n\text{-mod}, \quad M \mapsto b_ne \otimes_{eb_ne} M$$

which is right exact. Let $\Lambda_n := \{n, n-2, \ldots, -n+2, -n\}$. Under our assumption $[2] \neq 0$, the category b_n-mod is quasi-hereditary with labeling poset $(\Lambda_n, \prec)$, where $\lambda \prec \mu \Leftrightarrow |\lambda| > |\mu|$. Hence for all $\lambda \in \Lambda_n$ we have a standard module $\Delta_n(\lambda)$, a costandard module $\nabla_n(\lambda)$, a simple module $L_n(\lambda)$, a projective module $P_n(\lambda)$ and an injective module $I_n(\lambda)$. The simple module $L_n(\lambda)$ is the unique simple quotient of $\Delta_n(\lambda)$. In general $\Delta_n(\lambda)$ and $L_n(\lambda)$ are different.

One can find in [MW1] a diagrammatical description of $\Delta_n(\lambda)$. We shall however first of all need the following category theoretical properties of $\Delta_n(\lambda)$. Assume first that $n \geqslant 3$ to avoid b_n for $n \leqslant 0$ that we have not defined. Then we have

$$F\Delta_n(\lambda) \cong \begin{cases} \Delta_{n-2}(\lambda) & \text{if } \lambda \in \Lambda_n \setminus \{\pm n\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$G \circ F\Delta_n(\lambda) \cong \begin{cases} \Delta_n(\lambda) & \text{if } \lambda \in \Lambda_n \setminus \{\pm n\}, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where the second isomorphism is the adjointness map of the pair F and G. Note that the second statement is false if $\Delta_n(\lambda)$ is replaced by $\nabla_n(\lambda)$. Together with

$$\Delta_n(\pm n) \cong L_n(\pm n) \cong \nabla_n(\pm n)$$

and

$$FL_n(\mu) \cong \begin{cases} L_{n-2}(\mu) & \text{if } \mu \in \Lambda_n \setminus \{\pm n\}, \\ 0 & \text{otherwise} \end{cases}$$

these properties give the universal property for $\Delta_n(\lambda)$. For assume that N is a b_n-module with $[N : L_n(\lambda)] = 1$ satisfying $[N : L_n(\mu)] \neq 0$ only if $\mu \prec \lambda$. Then applying a sequence of functors F until arriving at $L_{|\lambda|}(\lambda)$ followed by a similar sequence of functors G, we obtain a nonzero homomorphism $\Delta_n(\lambda) \to N$. In other words, $\Delta_n(\lambda)$ is projective in the category of b_n-modules whose simple factors are all of the form $L_n(\mu)$ with $\mu \preccurlyeq \lambda$.

Let us now return to the tensor space module $V^{\otimes n}$ for b_n from the previous section. For $\lambda \in \Lambda_n$, we denote by $M(\lambda) = M_n(\lambda)$ the 'permutation' module. By definition, its basis vectors are $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}$ satisfying

$$\lambda = \#\{j \mid i_j = 1\} - \#\{j \mid i_j = 2\}.$$

It is clear from the previous section that it is a b_n-submodule of $V^{\otimes n}$.

We shall frequently make use of the *sequence* notation that was introduced in [MR] for the basis vectors of $V^{\otimes n}$. Under it 112 corresponds to $v_1 \otimes v_1 \otimes v_2$ and so on. As in [MR] the set of sequences of 1s and 2s of length n is denoted seq_n. The subset of these sequences with 1 appearing n_1 times is denoted $\mathrm{seq}_n^{n_1}$. With this notation $M_n(\lambda)$ has basis seq_n^a where $a = \frac{\lambda+n}{2}$. Its dimension is given by the binomial coefficient $\binom{n}{a}$. This is also the dimension of $\Delta_n(\lambda)$.

We shall also need the underline notation from [MR]. It is useful for doing calculations in FM where M is a submodule of $V^{\otimes n}$. In the present setup it is given by $\underline{12} := q^{-1}12 - 21$ for $n = 2$ and extended linearly to higher n. For example, for $n = 3$, $\lambda = 1$ we get the following identities in $FM_n(\lambda) = eM_n(\lambda)$

$$1\underline{12} = [2]e(112) = -U_2(112) = -(T_3 - q)(112) = -\big(121 - q^{-1}112\big).$$

Since $M_n(\lambda)$ and $\Delta_n(\lambda)$ have the same dimension one might guess that they are isomorphic b_n-modules. To see whether this is true one would have to verify for $M_n(\lambda)$ the category theoretical properties given in (2). The following theorem shows that the first of these indeed holds.

Theorem 2. *For $n \geqslant 3$ there is an isomorphism of b_{n-2}-modules*

$$FM_n(\lambda) \cong \begin{cases} M_{n-2}(\lambda) & \text{if } \lambda \in \Lambda_n \setminus \{\pm n\}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The theorem is easy to verify for $\lambda = \pm n$ so let us assume that $\lambda \in \Lambda_n \setminus \{\pm n\}$. Let $f : M_{n-2}(\lambda) \to FM_n(\lambda)$ be the k-linear map given by

$$i_1 i_2 \cdots i_{n-2} \mapsto i_1 i_2 \cdots i_{n-2}\underline{12} := q^{-1} i_1 i_2 \cdots i_{n-2}12 - i_1 i_2 \cdots i_{n-2}21.$$

We show that f is a b_{n-2}-linear isomorphism.

But by Lemma 1 of [MR] we already know that f is a vector space isomorphism and that it is linear with respect to the Temperley–Lieb action. Hence we must show that f is linear with respect to the action of X. Here X acts on the left-hand side through the restriction to $M_{n-2}(\lambda)$ of $T_1 \in$

$\mathrm{End}_k(V^{\otimes n-2})$ whereas it acts on the right-hand side through the restriction to $FM_n(\lambda)$ of $\frac{-1}{[2]}(T_n - q)T_1\frac{-1}{[2]}(T_n - q) \in \mathrm{End}_k(V^{\otimes n})$. Since we assume $n \geqslant 3$ the factors of the product commute. Noting furthermore that $\frac{-1}{[2]}(T_n - q)$ acts through the identity on $FM_n(\lambda)$, we get that the action of X on the right-hand side is nothing but the restriction of $T_1 \in \mathrm{End}_k(V^{\otimes n})$ to $FM_n(\lambda)$.

It is now enough to show that f is linear with respect to $T_1 \in \mathrm{End}_k(V^{\otimes n-2})$ and $T_1 \in \mathrm{End}_k(V^{\otimes n})$, in other words that

$$f\left(T_2^{-1}\cdots T_{n-2}^{-1}S_{n-2}\cdots S_2\varpi v\right) = T_2^{-1}\cdots T_{n-1}^{-1}T_n^{-1}S_nS_{n-1}\cdots S_2\varpi f(v)$$

for all $v \in M_{n-2}(\lambda)$. For this we first note that f clearly commutes with $T_2, \dots, T_{n-2}$, $S_2, \dots, S_{n-2}$, and ϖ. Since these are all invertible, we are reduced to proving that

$$f(v) = T_{n-1}^{-1}T_n^{-1}S_nS_{n-1}f(v) \quad \text{for all } v \in M_{n-2}(\lambda). \tag{3}$$

This equation only involves the last three factors of $f(v)$ so we may assume that $n = 3$. But for $n = 3$, the cases $\lambda = \pm 3$ of (3) are trivially fulfilled, leaving us the $\lambda = \pm 1$ cases.

If $\lambda = 1$ we have that

$$\mathrm{Im}\, f = eM_3(1) = \mathrm{span}_k\{1\underline{12}\} = \mathrm{span}_k\{112 - q121\}$$

and we must prove that $T_2^{-1}T_3^{-1}S_3S_2(112 - q121) = 112 - q121$ or

$$S_3S_2(112 - q121) = T_3T_2(112 - q121). \tag{4}$$

The left-hand side of this equation is $q(121 - q211)$ whereas the right-hand side is

$$\begin{aligned} T_3T_2(112 - q121) &= T_3\left(q112 - q\left(211 + \left(q - q^{-1}\right)121\right)\right) \\ &= T_3\left(q112 - q211 - \left(q^2 - 1\right)121\right) \\ &= q121 + \left(q^2 - 1\right)112 - q^2211 - \left(q^2 - 1\right)112 = q121 - q^2211 \end{aligned}$$

as claimed.

If $\lambda = -1$ we have that

$$\mathrm{Im}\, f = eM_3(-1) = \mathrm{span}_k\{212 - q221\}$$

and so Eq. (4) corresponds to

$$S_3S_2(212 - q221) = T_3T_2(212 - q221).$$

The left-hand side of this is $q(112 - q212)$, and the right-hand side is

$$T_3T_2(212 - q221) = T_3\left(122 - q^2221\right) = q122 - q^2212$$

as claimed. The theorem is proved. □

We now go on to consider the analogue for $M_n(\lambda)$ of the second category theoretical property for $\Delta_n(\lambda)$ in (2). It turns out *not* to hold for $M_n(\lambda)$. Let us be more precise. Let $\mathrm{seq}_n^{n_1}$ be the basis for $M_n(\lambda)$ as above and define $n_2 := n - n_1$ such that $\lambda = n_1 - n_2$. We then have the following result.

Lemma 2. *Let $n \geqslant 3$ and suppose that q is an lth primitive root of unity, where l is odd. Suppose $\lambda \in \Lambda_n \setminus \{\pm n\}$. Then we have*:

(a) *The adjointness map $\varphi_\lambda : G \circ FM_n(\lambda) \to M_n(\lambda)$ is surjective if and only if $n_2 \neq m \mod l$.*
(b) *The adjointness map $\varphi_\lambda : G \circ FM_n(\lambda) \to M_n(\lambda)$ is injective iff $n_2 \neq m \mod l$.*
(c) *The adjointness map $\varphi_\lambda : G \circ FM_n(\lambda) \to M_n(\lambda)$ is an isomorphism iff $n_2 \neq m \mod l$.*

Proof. Part (c) obviously follows by combining (a) and (b). Let us now prove (a). Assume first that $n_2 \neq m \mod l$ and suppose that φ_λ is not surjective.

Note first that for $w \in \mathrm{seq}_{n-2}^{n_1-1}$ and $(i_{n-1}, i_n) = (1,2)$ or $(2,1)$ we have that $e(wi_{n-1}i_n) = cw\underline{12}$ for some scalar $c \in k^\times$. Recall next from [MR] that b_ne is generated as an eb_ne right module by the set

$$\mathcal{G} := \{U_{n-1}, U_{n-2}U_{n-1}, \ldots, U_0 \cdots U_{n-2}U_{n-1}\}$$

and that $\varphi_\lambda : G \circ FM_n(\lambda) \to M_n(\lambda)$ is the multiplication map

$$b_ne \otimes_{eb_ne} eM_n(\lambda) \to M_n(\lambda), \quad U \otimes m \mapsto Um.$$

Suppose that $w = i_1i_2 \cdots i_{n-2}$. A key point, used in [MR] as well, is now that for $j \geqslant 1$ the multiplication of $U_jU_{j+1} \cdots U_{n-1} \in \mathcal{G}$ on $w\underline{12}$ shifts the underline to position $(j, j+1)$ in the following sense

$$U_jU_{j+1} \cdots U_{n-1}w\underline{12} = -[2]i_1i_2 \cdots i_{j-1}\underline{12}i_{j+2} \cdots i_n$$

as follows easily from the definitions. Using it we get that $im\varphi_\lambda$ is the span of

$$I_1 = \left\{(X - \lambda_1)\underline{12}x \;\middle|\; x \in \mathrm{seq}_{n-2}^{n_1-1}\right\}$$

together with

$$I_2 = \left\{v_1\underline{12}v_2 \;\middle|\; v_1 \in \mathrm{seq}_k^{l_1},\ v_2 \in \mathrm{seq}_{n-2-k}^{n_1-l_1-1},\ k \leqslant n-2,\ l_1 \leqslant n_1 - 1\right\}.$$

Let $N_2 := \mathrm{span}_k\{w \mid w \in I_2\}$. Then $Q := M_n(\lambda)/N_2$ is a vector space of dimension one since the elements of I_2 can be viewed as straightening rules that allow us to rewrite any element of $M_n(\lambda)/N_2$ as a scalar multiple of $1^{n_1}2^{n_2}$ (or $2^{n_2}1^{n_1}$). Indeed, by the definition of $\underline{12}$ we have the following identity, valid in Q

$$v_112v_2 = qv_121v_2 \quad \text{for } v_1 \in \mathrm{seq}_k^{l_1},\ v_2 \in \mathrm{seq}_{n-2-k}^{n_1-l_1-1}. \tag{5}$$

But $N_2 \subseteq im\varphi_\lambda$ and so we conclude $im\varphi_\lambda = N_2$ since φ_λ is not surjective.

But then Q is a b_n-module. It has dimension one and hence the action of X on Q is given by a scalar, which we shall work out. Notice first that if $i \geqslant 2$ then T_i^{-1} acts through the constant q^{-1} on Q, since U_i acts as zero for $i > 0$.

Set $v = 1^{n_1}2^{n_2} \in Q$. Since X acts through $T_2^{-1}T_3^{-1} \cdots T_n^{-1}\theta\varpi$ we get that

$$\begin{aligned} Xv &= \lambda_1q^{n_1-1}q^{-n_1-n_2+1}1^{n_1-1}2^{n_2}1 = \lambda_1q^{n_1-1}q^{-n_1-n_2+1}q^{-n_2}1^{n_1}2^{n_2} \\ &= \lambda_1q^{-2n_2}1^{n_1}2^{n_2} = \lambda_1q^{-2n_2}v \end{aligned}$$

using the straightening rules (5). Hence the scalar in question is $\lambda_1q^{-2n_2}$.

Set now $w = 2^{n_2}1^{n_1} \in Q$. Then we get the same way

$$Xw = \lambda_2 q^{n_2-1} q^{-n_1-n_2+1} 2^{n_2-1} 1^{n_1} 2 = \lambda_2 q^{n_2-1} q^{-n_1-n_2+1} q^{n_1} 2^{n_2} 1^{n_1} = \lambda_2 w.$$

The two scalars must be same, that is $\lambda_1 q^{-2n_2} = \lambda_2$ and hence $\lambda_1/\lambda_2 = q^{2m} = q^{2n_2}$. Since l is odd, this implies that $n_2 = m \mod l$, which is the desired contradiction.

To prove the other implication we assume that $n_2 = m \mod l$ and must show that φ_λ is not surjective. We show that $I_1 \subseteq N$ or equivalently $(N_1 + N_2)/N_2 = 0$ where $N_1 := \operatorname{span}_k\{w \mid w \in I_1\}$.

Since the actions of X and U_i commute for $i = 3, \dots, n-1$, we get for any $w \in \operatorname{seq}_{n-2}^{n_1-1}$ that

$$(X - \lambda_1)\underline{12}w = cX\underline{12}1^{n_1-1}2^{n_2-1} \mod N_2$$

where $c \in k^\times$. We go on calculating modulo N_2 and find

$$\begin{aligned} X\underline{12}1^{n_1-1}2^{n_2-1} &= Xq^{-1}121^{n_1-1}2^{n_2-1} - X211^{n_1-1}2^{n_2-1} \\ &= q^{-n_2-1}\lambda_1 21^{n_1-1}2^{n_2-1}1 - \lambda_2 q^{-n_1}1^{n_1}2^{n_2} \\ &= q^{-2n_2-n_1}\lambda_1 1^{n_1}2^{n_2} - \lambda_2 q^{-n_1}1^{n_1}2^{n_2} = 0 \end{aligned}$$

because $\lambda_1 q^{-2n_2} = \lambda_2$. This finishes the proof of (a). Note that for this last implication we do not need l to be odd.

We proceed to prove (b). We use the same principle for proving injectivity as in the proofs of Theorem 1 and Proposition 8 of [MR], although the combinatorial setup is different.

Since $\mathcal{G}$ generates $b_n e$ as a right $eb_n e$-module it induces a generating set of $G \circ FM_n(\lambda)$ as a vector space

$$\mathcal{M} := \mathcal{G} \otimes_{eb_n e} \operatorname{seq}_{n-2}^{n_1-1}\, \underline{12}.$$

We then have $I := \varphi_\lambda(\mathcal{M}) = I_1 \cup I_2$, where I_1 and I_2 are as above. Let us say that the elements of I_1 are of TL-type. The elements of I are not independent: there are trivial relations between the TL-type elements as follows

$$(\mathit{Triv}_1) \quad q^{-1}w_1 12 w_2 \underline{12} w_3 - w_1 21 w_2 \underline{12} w_3 = q^{-1} w_1 \underline{12} w_2 12 w_3 - w_1 \underline{12} w_2 21 w_3$$

for w_1, w_2, w_3 words in 1 and 2, i.e. belonging to appropriate seq_r^s.

There are also certain trivial relations involving the first element $U_{0,\dots,n-1} := U_0 U_1 \cdots U_{n-1}$ of $\mathcal{G}$ and the TL-elements. To handle these define first $U_{0,\dots,n-1}^{\lambda_1} := (U_0 + \lambda_1)U_1 \cdots U_{n-1}$ and replace then $U_{0,\dots,n-1}$ by

$$\mathcal{U}_{0,\dots,n-1} = (U_{n-1} + q)(U_{n-2} + q) \cdots (U_1 + q) U_{0,\dots,n-1}^{\lambda_1}$$

in $\mathcal{G}$. By this, $\mathcal{G}$ remains a generating set of b_n as $eb_n e$-module, since the expansion of $\mathcal{U}_{0,\dots,n-1}$ gives $U_{0,\dots,n-1}$ plus a linear combination of the other elements of $\mathcal{G}$ modulo $eb_n e$.

Now $U_0 + \lambda_1 = X$ and $U_i = T_{i+1} - q$ and so we get

$$\varphi_\lambda(\mathcal{U}_{0,\dots,n-1} \otimes_{eb_n e} i_1 i_2 \cdots i_{n-2}\underline{12}) = S_{n-1}S_{n-2} \cdots S_2 \varpi\, \underline{12} i_1 i_2 \cdots i_{n-2}.$$

Let us denote these elements by $\underline{1}i_1 i_2 \cdots i_{n-2}\underline{2}$. They are

$$\underline{1}i_1 i_2 \cdots i_{n-2}\underline{2} := -\lambda_2 q^{n_2-1} 1 i_1 \cdots i_{n-2} 2 + \lambda_1 q^{n_1-2} 2 i_1 \cdots i_{n-2} 1.$$

The trivial relations between the $\underline{1}i_1i_2\cdots i_{n-2}\underline{2}$ and the TL-type elements are then

$$(Triv_2)\quad q^{-1}\underline{1}w_1 12 w_2\underline{2} - \underline{1}w_1 21 w_2\underline{2} = -\lambda_2 q^{n_2-1} 1 w_1 \underline{12} w_2 2 + \lambda_1 q^{n_1-2} 2 w_1 \underline{12} w_2 1$$

where w_1, w_2 are words in 1 and 2 belonging to appropriate seq_r^s.

To get a better understanding of these trivial relations we now consider $w_1\underline{12}w_2$, $\underline{1}w_3\underline{2}$ as symbols and define

$$W_1 := \mathrm{span}_k\bigl\{w_1\underline{12}w_2, \underline{1}w_3\underline{2} \bigm| w_1 \in \mathrm{seq}_k^l,\ w_2 \in \mathrm{seq}_{n-k}^{l-n_1},\ w_3 \in \mathrm{seq}_n^{n_1}\bigr\}$$

and $W := W_1/\,\mathrm{span}_k\{R \mid R \in Triv_1 \cup Triv_2\}$. One checks on the relations that there is a linear map $\psi_\lambda : W \to G \circ FM_n(\lambda)$ defined by

$$w_1\underline{12}w_2 \mapsto U_iU_{i+1}\cdots U_{n-1} \otimes_{eb_ne} w_1w_2\underline{12},$$
$$\underline{1}w_3\underline{2} \mapsto \mathcal{U}_{0,\ldots,n-1} \otimes_{eb_ne} w_3\underline{12}.$$

Using the relations $Triv_1$ and $Triv_2$, it is straightforward to check that the elements $22\ldots11\ldots11\underline{12}i_ki_{k+1}\cdots i_n$ (with no 12 before the underline) and $\underline{1}222\ldots111\underline{2}$ generate W. We show that these elements map to a basis of $M_n(\lambda)$ under $\varphi_\lambda \circ \psi_\lambda$ which implies that φ_λ is injective.

We have that

$$\varphi_\lambda \circ \psi_\lambda(22\ldots111\underline{12}i_k\cdots i_n) = 22\ldots111\underline{12}i_k\cdots i_n \in M_n(\lambda),$$
$$\varphi_\lambda \circ \psi_\lambda(\underline{1}222\ldots111\underline{2}) = \underline{1}222\ldots111\underline{2} \in M_n(\lambda).$$

The first kind of elements (of TL-type) were shown to be linearly independent in [MW1]. To show that $\underline{1}222\ldots111\underline{2}$ is independent of these, it is enough to show that it is nonzero modulo the TL-type elements. Calculating modulo the TL elements, we have $12 = q21$ and so we find that $\underline{1}222\ldots111\underline{2}$ is equal to

$$\underline{1}2^{n_2-1}1^{n_1-1}\underline{2} = -\lambda_2 q^{n_2-1} 1 2^{n_2-1} 1^{n_1-1} 2 + \lambda_1 q^{n_1-2} 2^{n_2} 1^{n_1}$$
$$= \bigl(-\lambda_2 q^{2n_2+n_1-2} + \lambda_1 q^{n_1-2}\bigr) 2^{n_2+1} 1^{n_1+1}.$$

By the assumption of the lemma this is nonzero since $\lambda_1/\lambda_2 = q^{2m}$.

Finally the other implication of (b) follows also from the last calculation since ψ_λ is surjective. We have proved the lemma. □

A consequence of the lemma is that $M_n(\lambda)$ is not isomorphic to $\Delta_n(\lambda)$ in general. Moreover, we shall later in Section 5 explain how the above proof can be used to deduce that $M_n(\lambda)$ is also not isomorphic to $\nabla_n(\lambda)$ in general.

On the other hand, we now prove by induction that $M_n(\lambda)$ and $\Delta_n(\lambda)$ are equal in the Grothendieck group of b_n-modules. The next lemma is the induction basis.

Lemma 3. *For $n \geqslant 1$ we have the following isomorphisms of b_n-modules*

$$\text{(a) } M_n(n) \cong \Delta_n(n), \qquad \text{(b) } M_n(-n) \cong \Delta_n(-n), \qquad \text{(c) } M_2(0) \cong \Delta_2(0).$$

Proof. The parts (a) and (b) of the lemma are easy to check since all the involved b_n-modules are one-dimensional and have trivial U_i actions for $i \geqslant 1$. One then just needs to verify that $U_0 = X - \lambda_1$ acts the right way.

In order to prove part (c) we first get for $n = 2$ by direct calculations that the matrices of U_1 and X with respect to the basis $\{12, 21\}$ of $M_2(0)$ are given by

$$U_1 = \begin{pmatrix} -q^{-1} & 1 \\ 1 & -q \end{pmatrix}, \qquad X = \begin{pmatrix} \lambda_1 & 0 \\ -\lambda_1(q - q^{-1}) & \lambda_2 \end{pmatrix},$$

and hence the matrix of $U_0 = X - \lambda_1$ is

$$U_0 = \begin{pmatrix} 0 & 0 \\ -\lambda_1(q - q^{-1}) & -[m] \end{pmatrix}$$

since $[m] = \lambda_1 - \lambda_2$. The ket basis of $\Delta_2(0)$, see [MW1], modulo multiplication by nonzero scalars, is given by $\{\cup, U_0\cup\}$. Define φ by

$$\varphi : \underline{12} = q^{-1}12 - 21 \mapsto \cup, \quad U_0\underline{12} \mapsto U_0\cup.$$

This is the desired b_n-isomorphism provided that $U_0\underline{12}$ is nonzero and is an eigenvector of U_0 with eigenvalue $-[m]$. But by the above

$$U_0\underline{12} = q^{-1}\big(-\lambda_1\big(q - q^{-1}\big) + q[m]\big)21.$$

The coefficient is nonzero iff $\lambda_1(q - q^{-1}) \neq q[m]$, which by $\lambda_1 = \frac{q^m}{q - q^{-1}}$ is equivalent to $q^{2m} \neq q^2$, which holds by the assumptions on q given in the beginning of Section 3. But then $\underline{12}$ is automatically an eigenvector for U_0 of the right eigenvalue. □

Theorem 3. *Assume that $n \geqslant 1$. Then $[\Delta_n(\lambda) : L_n(\mu)] = [M_n(\lambda) : L_n(\mu)]$ for all $\lambda, \mu \in \Lambda_n$.*

Proof. We prove the theorem by induction on n. The induction basis $n = 1$ and $n = 2$ is provided by the above lemma. We assume the theorem to hold for all n' strictly smaller than n and prove it for n. Recall once again that the simple b_n-modules $L_n(\mu)$ satisfy that

$$FL_n(\mu) \cong \begin{cases} L_{n-2}(\mu) & \text{if } \mu \in \Lambda_n \setminus \{\pm n\}, \\ 0 & \text{otherwise.} \end{cases}$$

By induction, exactness of F, the category theoretical property for $\Delta_n(\lambda)$ stated in (2) and Theorem 2, we then get for $\mu \in \Lambda_n \setminus \{\pm n\}$ that

$$\begin{aligned}\big[\Delta_n(\lambda) : L_n(\mu)\big] &= \big[F\Delta_n(\lambda) : FL_n(\mu)\big] = \big[\Delta_{n-2}(\lambda) : L_{n-2}(\mu)\big] \\ &= \big[M_{n-2}(\lambda) : L_{n-2}(\mu)\big] = \big[FM_n(\lambda) : FL_n(\mu)\big] = \big[M_n(\lambda) : L_n(\mu)\big]\end{aligned}$$

and we need now only to prove $[\Delta_n(\lambda) : L_n(\pm n)] = [M_n(\lambda) : L_n(\pm n)]$.

But X acts semisimply in any b_n-module and so we obtain the following $k[X]$-module decompositions

$$\Delta_n(\lambda) = \bigoplus_{\mu \in \Lambda_n} L_n(\mu)^{d_{\lambda\mu}}, \qquad M_n(\lambda) = \bigoplus_{\mu \in \Lambda_n} L_n(\mu)^{e_{\lambda\mu}}$$

where $d_{\lambda\mu} = [\Delta_n(\lambda) : L_n(\mu)]$ and $e_{\lambda\mu} = [M_n(\lambda) : L_n(\mu)]$. On the other hand, the only possible eigenvalues for X are λ_1 and λ_2 and we just saw that $d_{\lambda\mu} = e_{\lambda\mu}$ for $\mu \in \Lambda_n \setminus \{\pm n\}$. Hence it is enough to show that $\Delta_n(\lambda)$ and $M_n(\lambda)$ are isomorphic $k[X]$-modules to deduce $d_{\lambda\mu} = e_{\lambda\mu}$ for the remaining $\mu \in \Lambda_n$ and so finish the proof. Indeed $L_n(n)$ and $L_n(-n)$ are both one-dimensional, generated by eigenvectors for X of eigenvalues λ_1 and λ_2 respectively (recall $\lambda_1 \neq \lambda_2$ by our assumptions).

Now $\Delta_n(\lambda) \cong M_n(\lambda)$ as $k[X]$-modules if and only if the eigenspace multiplicities with respect to X are equal, so we show that this is the case.

For this we observe that the Bratteli diagram or Pascal triangle of restriction rules from b_n to b_{n-1} given in [MW1] can be used to determine the eigenvalues of X on $\Delta_n(\lambda)$ in the following way: A diagram of the diagram basis of $\Delta_n(\lambda)$ is an eigenvector for $X = U_0 + \lambda_1$ of eigenvalue λ_2 iff its first line is marked with a filled blob. This induces the following Pascal triangle pattern of multiplicities of the eigenvalue λ_2.

$$\begin{array}{lccccccccc}
n=1 & & & & 1 & & 0 & & & \\
n=2 & & & 1 & & 1 & & 0 & & \\
n=3 & & 1 & & 2 & & 1 & & 0 & \\
n=4 & 1 & & 3 & & 3 & & 1 & & 0.
\end{array}$$

For example, the first number 3 says that $\Delta_4(-2)$ has 3 diagrams with first line marked and hence λ_2 has multiplicity 3 in $\Delta_4(-2)$.

We must compare this pattern with the λ_2-multiplicity of X in $M_n(\lambda)$. We have with the usual notation $\lambda = n_1 - n_2$ a basis of $M_n(\lambda)$ consisting of $B := \mathrm{seq}_n^{n_1}$. Define B_1 as the sequences from $\mathrm{seq}_n^{n_1}$ that begin with a 1 and B_2 as $\mathrm{seq}_n^{n_1} \setminus B_1$. Put an order on B such that the elements of B_2 come before the elements of B_1. Then by Lemma 1 the action of X is upper triagonal with λ_2 in the first $|B_2|$ diagonal elements and with λ_1 in the last $|B_1|$ diagonal elements. Hence the λ_2-multiplicity of X is $|B_2|$. But the numbers B_2 satisfy the same Pascal triangle recursion as the above. The theorem is proved. □

5. Specht modules and duality

In this section we shall relate the results of the previous sections to the $H_k(n,2)$-module $\tilde{S}^\lambda$ introduced in [DJM] for bipartitions $\lambda = (\tau, \mu)$ of n. The module $\tilde{S}^\lambda$ is a cell module for a certain cellular structure on $H_k(n,2)$. Following modern terminology as used in for example [Ma], we shall therefore denote it the *Specht module* for $H_k(n,2)$, although it is rather an analogue of the dual Specht module, and for $\lambda = (\tau, \mu)$ we shall accordingly use the notation $S(\lambda)$ or $S(\tau, \mu)$ for it. If $\lambda = ((n_1), (n_2))$ is a two-line bipartition of n, that is $n_1, n_2 \geqslant 0$ such that $n_1 + n_2 = n$, we shall also write $S(n_1, n_2)$ for $S(\lambda)$. Similarly, if $\lambda = ((1^{n_1}), (1^{n_2}))$ is a two-column bipartition, we shall write $S(1^{n_1}, 1^{n_2})$ for $S(\lambda)$.

In this section we show that the Specht module $S(n_1, n_2)$ as well as its contragredient dual $S(n_1, n_2)^{\circledast}$ are modules for b_n. We moreover establish a b_n-isomorphism between $S(n_1, n_2)^{\circledast}$ and $M_n(\lambda)$ where $\lambda = n_1 - n_2$. Finally, we prove an analogue of Lemma 2 for $M_n(\lambda)^{\circledast}$ and as a consequence we get that, somewhat surprisingly, neither of the modules $S(n_1, n_2)$, $S(n_1, n_2)^{\circledast}$, $M_n(\lambda)$, $M_n(\lambda)^{\circledast}$ is the pullback of the standard module $\Delta_n(\lambda)$ for b_n in general.

On the other hand, the pullback of the simple b_n-module $L_n(\lambda)$ to $H_k(n,2)$ certainly *is* a simple $H_k(n,2)$-module. Thus, the statements of the previous paragraph are apparently not compatible with the statement of Theorem 3 on equality in the Grothendieck groups, since the dominance order on bipartitions does not induce the quasi-hereditary order $\prec$ on Λ_n. But note that the bipartitions $(\tau, \mu) = ((n_1), (n_2))$ are only Kleshchev (= restricted) in 'small' cases and therefore, apart from these small cases, $L_n(\lambda)$ is not the simple module associated with the bipartition $((n_1), (n_2))$ when viewed as $H_k(n,2)$-module, see [AJ]. In fact, it would be interesting to know which is the Kleshchev bipartition corresponding to $L_n(\lambda)$. (In the recent preprint [RH] we have solved this problem.)

Let us now recall the combinatorial description of the permutation module $M_H(\tau, \mu)$ and the Specht module $S(\tau, \mu)$ for $H_k(n,2)$ given in [DJM] and [DJMa]. Since these references use right modules rather than left modules and since they moreover use a slightly different presentation of $H_k(n,2)$, the following formulas vary slightly from theirs.

Let (τ, μ) be a bipartition of n. Then a (τ, μ)-bitableau t is a pair (t^1, t^2) where t^1 is a τ-tableau and t^2 is a μ-tableau and where tableaux means fillings with the numbers $I_n = \{\pm 1, \pm 2, \ldots, \pm n\}$, where either i or $-i$ occurs exactly once. Two (τ, μ)-bitableaux (t^1, t^2) and (s^1, s^2) are said to be row equivalent if the tableaux obtained by taking absolute values in t^1 and s^1 are row equivalent in the usual sense, and if t^2 and s^2 are row equivalent. The equivalence class of the bitableau t is called a tabloid and is written $\{t\}$.

The permutation module $M_H(\tau, \mu)$ for $H_k(n, 2)$ is now

$$M_H(\tau, \mu) := \operatorname{span}_k\big\{\{t_1, t_2\} \bigm| (t_1, t_2) \text{ is a row standard } (\tau, \mu)\text{-bitableaux}\big\}$$

where the action can be read off from Lemmas 3.9, 3.10 and 3.11 of [DJMa].

The Specht module $S_H(\tau, \mu)$ is now the quotient $M_H(\tau, \mu)/N_H(\tau, \mu)$ for $N_H(\tau, \mu)$ a certain submodule of $M_H(\tau, \mu)$. The standard tabloids induce a basis for $S(\tau, \mu)$

$$[t_1, t_2] := \{t_1, t_2\} + N_H(\tau, \mu)$$

where standard means that all entries are positive, and that each component is row standard and column standard.

We shall be especially concerned with the case of two-line bipartitions $(\tau, \mu) = ((n_1), (n_2))$. In that case, standard bitableaux are just row standard tableaux with positive entries and so the formulas for the action of $H_k(n, 2)$ on $M_H(\tau, \mu)$ induce the following formulas for the action on $[t] = [t_1, t_2] \in S(\tau, \mu)$

$$g_i[t] = \begin{cases} \sigma_i[t] & \text{if } (i \in t^1,\ i+1 \in t^2), \\ \sigma_i[t] + (q - q^{-1})[t] & \text{if } (i+1 \in t^1,\ i \in t^2), \\ q[t] & \text{if } (i, i+1 \in t^1) \text{ or } (i, i+1 \in t^2) \end{cases} \tag{6}$$

where the transposition $\sigma_i = (i, i+1)$ acts by permuting the entries. The action of X can only partially be made explicit. We consider first the action of X_i. Let $t^{\tau,\mu}$ be the (τ, μ)-bitableau with $\{1, \ldots, n\}$ positioned increasingly from left to right. For example, in the case $n_1 = 5$, $n_2 = 6$ we have

$$t^{\tau,\mu} = \left(\boxed{1|2|3|4|5}\,,\ \boxed{6|7|8|9|10|11}\right).$$

Then by [DJM] we have

$$X_i\big[t^{\tau,\mu}\big] = \begin{cases} \lambda_1 q^{2(i-1)}[t^{\tau,\mu}] & \text{if } i = 1, \ldots, n_1, \\ \lambda_2 q^{2(i-n_1-1)}[t^{\tau,\mu}] & \text{if } i = n_1 + 1, \ldots, n. \end{cases}$$

To get the action on the other standard tableaux, one has to use the commutation rules of $H_n(n, 2)$. This implicit description is enough to prove the following theorem. Although it is a main philosophical idea of [MW], a formal proof was not given.

Theorem 4. *$S(\tau, \mu)$ is a module for b_n when $(\tau, \mu) = ((n_1), (n_2))$.*

Proof. By the isomorphism theorem (1) we must verify that

$$(X_1 X_2 - \lambda_1 \lambda_2)(g_1 - q) = 0 \tag{7}$$

in $\operatorname{End}_k(S(n_1, n_2))$. Let therefore $[t] = [t_1, t_2]$ be the class of a standard bitableau for the bipartition $((n_1), (n_2))$. If 1, 2 both belong to t_1 or t_2 the statement is clear by (6). Using (6) once again, we have that

$$S(n_1,n_2)=\ker(g_1-q)+\operatorname{span}_k\bigl\{[t_1,t_2]\bigm| 1\in t_1,\ 2\in t_2\bigr\}$$

and we are left with the case $1\in t_1$, $2\in t_2$. But then we can find $w=\sigma_{i_1}\cdots\sigma_{i_r}\in\langle\sigma_i\mid i=2,\dots,n-1\rangle$ such that $wt^{\tau,\mu}=(t_1,t_2)$ and so we have $X_1[t_1,t_2]=\lambda_1[t_1,t_2]$ since $X=X_1$ commutes with all $g_2,\dots,g_{n-1}$.

We then consider the action of X_2 on $[t_1,t_2]$. Let t^{12} be the bitableau with $1\in t^1$, $2\in t^2$ and the other entries increasing from left to right. For example, if $n_1=5$ and $n_2=6$, it is

$$t^{12}=\bigl(\boxed{1}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\,,\,\boxed{2}\boxed{7}\boxed{8}\boxed{9}\boxed{10}\boxed{11}\bigr).$$

Then any $t=(t_1,t_2)$ with $1\in t_1$ and $2\in t_2$ is of the form $t=wt^{12}$ where $w=\sigma_{i_1}\cdots\sigma_{i_r}\in\langle\sigma_i\mid i=3,\dots,n-1\rangle$. We claim that $X_2[t^{12}]=\lambda_2[t^{12}]$ modulo a linear combination of elements $[(t^1,t^2)]$ all satisfying $1,2\in t^1$. Believing this, we would also get that $X_2[t]=\lambda_2[t]$ modulo a similar linear combination of elements $[(t^1,t^2)]$, since $X_2=g_1Xg_1$ and g_i commute for $i=3,\dots,n$. From this (7) would follow.

To prove the claim for t^{12} we first use (6) to write

$$g_2g_3\cdots g_{n_1-1}g_{n_1}\bigl\{t^{\tau\mu}\bigr\}=\bigl\{t^{12}\bigr\}.$$

Since $X_{n_1+1}^{-1}\{t^{\tau\mu}\}=\lambda_2^{-1}\{t^{\tau\mu}\}$ and $X_{n_1+1}=g_{n_1}\cdots g_1X_1g_1\cdots g_{n_1}$ we deduce that

$$X_2\bigl\{t^{12}\bigr\}=\lambda_2g_2^{-1}\cdots g_{n_1}^{-1}\bigl\{t^{\tau\mu}\bigr\}.$$

The claim now follows. $\square$

Recall that the contragredient dual $M^{\circledast}$ of an $H_k(n,2)$-module M is the linear dual $\operatorname{Hom}_k(M,k)$ equipped with the $H_k(n,2)$ action $(hf)(m):=f(h^*m)$ for $*$ the antiinvolution of $H_k(n,2)$ given by $g_i^*:=g_i$ and $X^*:=X$.

Let $H'_k(n,2)$ be the Ariki–Koike algebra $H_k(-q^{-1},\lambda_2,\lambda_1)$. There is a k-algebra isomorphism $\theta: H_k(n,2)\to H'_k(n,2)$ given by

$$X\mapsto X,\quad g_i\mapsto g_i.$$

Following [Ma] and [F], we define $S'(\tau,\mu)$ as the pullback under θ of the Specht module $S(\tau,\mu)$ for $H'_k(n,2)$. Now Mathas proved in [Ma] the following result.

Theorem 5. *As $H_k(n,2)$-modules we have $S(\tau,\mu)^{\circledast}\cong S'(\mu',\tau')$ where τ' and μ' are the usual conjugate partitions of τ and μ.*

In the case $(\tau,\mu)=((n_1),(n_2))$, the isomorphism of the theorem will also be an isomorphism of b_n-modules, since $*$ induces the usual antiinvolution $*$ of b_n that appears in the definition of contragredient duality in b_n-mod. Specially, $S'(1^{n_2},1^{n_1})$ will be a b_n-module as well.

The standard basis for $S(\mu',\tau')=S'(1^{n_2},1^{n_1})$ consists of the classes of bitableaux $t=(t_1,t_2)$ of the bipartition $((1^{n_2}),(1^{n_1}))$. We get for g_i the same action rules as before:

$$g_i[t]=\begin{cases}\sigma_i[t] & \text{if } (i\in t^1,\ i+1\in t^2),\\ \sigma_i[t]+(q-q^{-1})[t] & \text{if } (i+1\in t^1,\ i\in t^2),\\ q[t] & \text{if } (i,i+1\in t^1) \text{ or } (i,i+1\in t^2).\end{cases}\tag{8}$$

As before, we have a special standard bitableau $t^{\mu',\tau'}$, this time with the numbers $1,\dots,n$ filled in increasingly first down the first column, then down the second column. The action of X_i on this $[t^{\mu',\tau'}]$ is given by

$$X_i\big[t^{\mu',\tau'}\big]=\begin{cases}\lambda_2 q^{2(i-1)}[t^{\mu',\tau'}] & \text{if } i=1,\dots,n_2,\\ \lambda_1 q^{2(i-n_2-1)}[t^{\mu',\tau'}] & \text{if } i=n_2+1,\dots,n.\end{cases}$$

We are now in position to prove the following result

Theorem 6. *Let as before $\lambda=n_1-n_2$. Then there is an isomorphism of b_n-modules $M_n(\lambda)\cong S(n_1,n_2)^{\circledast}$.*

Proof. We had by Mathas's theorem that $S(n_1,n_2)^{\circledast}\cong S'(1^{n_2},1^{n_1})$. We then define a linear map $\varphi: S'(1^{n_2},1^{n_1})\to M_n(\lambda)$ by

$$\varphi\big([t_1,t_2]\big)=i_1 i_2\cdots i_n \quad \text{where } i_j=1 \text{ iff } j\in t_2.$$

It is easily checked that φ is linear with respect to g_i. On the other hand, we have that $\varphi(t^{\mu',\tau'})=2^{n_2}1^{n_1}$. Using the next lemma we see that X_i acts through the same constant on $[t^{\mu',\tau'}]$ as on $2^{n_2}1^{n_1}$. This is enough to complete the proof by the commutation rules for $H_k(n,2)$. □

Lemma 4. *Let $w=2^{n_2}1^{n_1}\in M_n(\lambda)$. Then*

$$X_i w=\begin{cases}\lambda_2 q^{2(i-1)} w & \text{if } i=1,\dots,n_2,\\ \lambda_1 q^{2(i-n_2-1)} w & \text{if } i=n_2+1,\dots,n.\end{cases}$$

Proof. By Lemma 1 the action of $X=X_1$ on w is multiplication by λ_2, hence the action of $X_2=T_2X_1T_2$ is multiplication by $q^2\lambda_2$ and so on until we reach X_{n_2}.

To calculate the action of X_{n_2+1} we write

$$X_{n_2+1}=T_{n_2+2}^{-1}\cdots T_n^{-1}S_n\cdots S_2\varpi T_2\cdots T_{n_2+1}$$

and so

$$\begin{aligned}X_{n_2+1}w&=T_{n_2+2}^{-1}\cdots T_n^{-1}S_n\cdots S_2\varpi T_2\cdots T_{n_2+1}2^{n_2}1^{n_1}=\lambda_1 T_{n_2+2}^{-1}\cdots T_n^{-1}S_n\cdots S_2 12^{n_2}1^{n_1-1}\\ &=q^{n_1-1}\lambda_1 T_{n_2+2}^{-1}\cdots T_n^{-1}2^{n_2}1^{n_1}=\lambda_1 2^{n_2}1^{n_1}=\lambda_1 w\end{aligned}$$

and the action is multiplication by λ_1. This implies that X_{n_2+2} acts by $\lambda_1 q^2$ and so on. □

We can now finally prove the result alluded to in the previous section.

Corollary 2. *Let $n\geqslant 3$ and suppose that q is an lth primitive root of unity, where l is odd. Suppose $\lambda\in\Lambda_n\setminus\{\pm n\}$. Then the adjointness map $\psi_\lambda: G\circ FM_n(\lambda)^{\circledast}\to M_n(\lambda)^{\circledast}$ is an isomorphism iff $n_1=m \bmod l$.*

Proof. By the actions rules given above and Theorem 6 the actions on $M_n(\lambda)^{\circledast}$ and $M_n(\lambda)$ are the same, except that λ_1 and λ_2 are interchanged as are n_1 and n_2. We then repeat the argument of Lemma 2 and get that φ_λ is an isomorphism iff $\lambda_2/\lambda_1=(-q)^{-2n_1}$, which is equivalent to $n_1=m \bmod l$ as claimed. □

Combining the corollary with Lemma 2 we deduce that neither $M_n(\lambda)$ nor $M_n(\lambda)^{\circledast}$ is the standard module $\Delta_n(\lambda)$ for b_n in general. And then, combining this with the above theorem, we get the same statement for the Specht module $S_n(n_1,n_2)$ and for $S_n(n_1,n_2)^{\circledast}$.

6. Alcove geometry

We already saw that although $M_n(\lambda)$ does not identify with the standard module $\Delta_n(\lambda)$ for b_n in general, the two modules still have many features in common. In this section we shall further pursue this point, by considering the behavior of the restriction functor $\mathrm{res}_{b_{n-1}}^{b_n}$ from b_n-mod to b_{n-1}-mod on $M_n(\lambda)$.

It is known from [MW1] and [CGM] that the representation theory of b_n is governed by an alcove geometry on $\mathbb{Z}$ where l determines the alcove length and m the position of the fundamental alcove. The associated Weyl group is the affine Weyl group for $\mathfrak{sl}_2$ and there is a linkage principle controlled by this. In the case where the characteristic of k is zero the decomposition numbers are calculated in [MW1], they are given by the corresponding Kazhdan–Lusztig polynomials. In [GL1] the standard modules for b_n are shown to be related to certain standard modules for the extended affine Hecke algebra of type A, namely those given by two-step nilpotent matrices. From this it follows that the decomposition numbers for b_n also give rise to certain decomposition numbers for the affine Hecke algebra. Finally, we mention the case of positive characteristic where the decomposition numbers are calculated in [CGM].

Let us now set up some exact sequences that arise from restriction from b_n-mod to b_{n-1}-mod. Let $\lambda \in \Lambda_n \setminus \{\pm n\}$. As a TL_{n-1}-module the restricted module $\mathrm{res}_{b_{n-1}}^{b_n} M_n(\lambda)$ is isomorphic to the direct sum

$$M_{n-1}(\lambda+1) \oplus M_{n-1}(\lambda-1).$$

This is however not automatically the case when $\mathrm{res}_{b_{n-1}}^{b_n} M_n(\lambda)$ is considered as a b_{n-1}-module since X acts differently as element of b_n and of b_{n-1}. But the following statement always holds.

Lemma 5. *Assume $\lambda \in \Lambda_n \setminus \{\pm n\}$. Then there is a short exact sequence of b_{n-1}-modules*

$$0 \to M_{n-1}(\lambda-1) \to \mathrm{res}_{b_{n-1}}^{b_n} M_n(\lambda) \to M_{n-1}(\lambda+1) \to 0.$$

Proof. We identify $M_{n-1}(\lambda-1)$ with the span of the sequences of the form $v_1 v_2 \cdots v_{n-1}1$. Since for all $x \in \mathrm{seq}_{n-2}$ we have that $T_n^{-1} S_n(x11) = x11$ and

$$T_n^{-1} S_n(x21) = T_n^{-1}(x12) = x21,$$

we get that $M_{n-1}(\lambda-1)$ in this way is a b_{n-1}-submodule of $\mathrm{res}_{b_{n-1}}^{b_n} M_n(\lambda)$.

The quotient of $\mathrm{res}_{b_{n-1}}^{b_n} M_n(\lambda)$ by $M_{n-1}(\lambda-1)$ is now generated by the classes of the sequences that end in 2. It can be identified with $M_{n-1}(\lambda+1)$ since for $x \in \mathrm{seq}_{n-2}$ we have $T_n^{-1} S_n(x22) = x22$ and

$$T_n^{-1} S_n(x12) = T_n^{-1}(x21) = x12 \mod M_{n-1}(\lambda-1).$$

The lemma now follows. □

One observes that these sequences are very similar to the sequences for $\mathrm{res}_{b_{n-1}}^{b_n} \Delta_n(\lambda)$ given in Lemma 4.5 of [MW1]. The only difference is that in [MW1] the appearances of $\lambda-1$ and $\lambda+1$ are interchanged when λ is negative. But $M_n(\lambda)$ is not the pullback of $\Delta_n(\lambda)$, as we already pointed out several times, and it seems to be a difficult task to compare the two systems of exact sequences.

We finish the paper by showing that the sequences of the lemma are split when λ is not a wall of the alcove geometry. This result could also have been obtained using Theorem 3 and the linkage principle for b_n-mod, but we here deduce it from the machinery we have set up. We use central elements.

It is known, see for example the appendix of [MW], that the symmetric polynomials in the X_i are central elements of $H(n,2)$ and hence also of b_n. We consider $z := X_1 X_2 \cdots X_n$ as an element of the center $Z(b_n)$ of b_n and work out the action of it on $M_n(\lambda)$.

Lemma 6. *Recall that $\lambda = n_1 - n_2$. Then the action of z on $M_n(\lambda)$ is diagonal, given by the constant*

$$\lambda_1^{n_1} \lambda_2^{n_2} q^{n_1(n_1-1)} q^{n_2(n_2-1)}.$$

Proof. As a b_n-module $M(\lambda)$ is generated by $2^{n_2} 1^{n_1}$. Since z is central, it is therefore enough to prove the assertion on that element. Recall that the X_i commute. By Lemma 4 we find that $X_1 X_2 \cdots X_{n_2}$ acts by

$$\lambda_2^{n_2} q^{0+2+4+\cdots 2(n_2-1)} = \lambda_2^{n_2} q^{n_2(n_2-1)}.$$

Once again by Lemma 4, we have that $X_{n_2+1} \cdots X_n$ acts by

$$\lambda_1^{n_1} q^{0+2+4+\cdots 2(n_1-1)} = \lambda_1^{n_1} q^{n_1(n_1-1)}.$$

The lemma now follows by combining. □

We can now prove the promised splitting.

Theorem 7. *Assuming $\lambda \neq -m \bmod l$, the exact sequences from Lemma 5 are split.*

Proof. If the sequence were nonsplit, any preimage in $\operatorname{res}_{b_{n-1}}^{b_n} M_n(\lambda)$ of the $M_n(\lambda+1)$ generator $w = 2^{n_2} 1^{n_1}$ would generate a submodule $M \subset \operatorname{res}_{b_{n-1}}^{b_n} M_n(\lambda)$ nonisomorphic to $M_n(\lambda+1)$. Moreover M would map surjectively onto $M_n(\lambda+1)$ and would have a composition factor in common with $M_{n-1}(\lambda-1)$. But then z would act through the same constant on $M_n(\lambda+1)$ and $M_n(\lambda-1)$.

Let $\lambda = n_1 - n_2$. The action of z on $M_{n-1}(\lambda-1)$ is

$$\lambda_1^{n_1-1} \lambda_2^{n_2} q^{(n_1-1)(n_1-2)} q^{n_2(n_2-1)}$$

and the action of z on $M_{n-1}(\lambda+1)$ is

$$\lambda_1^{n_1} \lambda_2^{n_2-1} q^{n_1(n_1-1)} q^{(n_2-1)(n_2-2)}.$$

Equating, we get

$$\lambda_2 q^{2(n_2-1)} = \lambda_1 q^{2(n_1-1)}$$

which implies that $\frac{\lambda_1}{\lambda_2} = q^{2m} = q^{2(n_2-n_1)}$ and the theorem follows. □

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