Unitary similarity classes within the cospectral-congruence class of a matrix

Susana Furtado\textsuperscript{a,*,1}, Charles R. Johnson\textsuperscript{b}

\textsuperscript{a}Faculdade de Economia, Universidade do Porto, Rua Dr. Roberto Frias, 4200 Porto, Portugal
\textsuperscript{b}Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA

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Abstract

Matrix $B \in M_n(\mathbb{C})$ is C-S equivalent (resp. C-E equivalent) to $A \in M_n(\mathbb{C})$ if $B$ is both congruent and similar to (resp. cospectral with) $A$. We are concerned with the number (typically one or infinitely many) of unitary similarity classes in the C-S (resp. C-E) equivalence class of a given matrix. The case $n = 2$ and the general normal case are fully understood for C-S equivalence. Also, the singular case may generally be reduced to the nonsingular case.

The present work includes four main results. (1) If 0 lies in the interior of the field of values of a nonsingular $A \in M_n$, $n \geq 3$, then the C-E equivalence class contains infinitely many unitary similarity classes. (2) When 0 is not in the interior, general sufficient conditions are given for the C-E class (and thus the C-S class) to contain only one unitary class. (3) When $n = 3$, these conditions are also necessary and a classification of all C-E and C-S classes is given. (4) For $n \geq 3$, it is shown that the matrices for which the C-S class contains infinitely many unitary similarity classes are dense among all matrices.

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\* Corresponding author.
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1. Introduction

Matrix \( B \in M_n \) (the \( n \)-by-\( n \) complex matrices) is said to be congruent to \( A \in M_n \) if there is a nonsingular \( C \in M_n \) such that \( B = C^*AC \). Of course, congruence is an equivalence relation on \( M_n \). Matrices \( A, B \in M_n \) are said to be cospectral if they have the same eigenvalues, counting multiplicities (or, equivalently, the same characteristic polynomial). Cospectrality and, of course, similarity as well are equivalence relations on \( M_n \). Thus, the intersection of the congruential equivalence classes with either the similarity or the cospectrality classes also forms an equivalence relation.

In [13], study of the former equivalence relation was begun (see [14] for a summary); \( B \in M_n \) is said to be C-S equivalent to \( A \in M_n \) if \( B \) is both congruent and similar to \( A \); let \( CS(A) \) denote the C-S equivalence class of \( A \). Indeed, study of C-S equivalence is intriguing and raises challenging questions. Thus far, the central one has been to understand the “number” of unitary similarity classes in each C-S equivalence class; initial results and a complete understanding of the normal case may be found in [5].

Here, we also consider a second, coarser, equivalence relation: \( B \in M_n \) is said to be C-E equivalent to \( A \in M_n \) if \( B \) is both cospectral with and congruent to \( A \); let \( CE(A) \) denote the C-E equivalence class of \( A \). Of course, if \( A \) has distinct eigenvalues, C-E and C-S equivalence coincide, and, in general, there may be several (but, of course, only a finite number of) C-S equivalence classes in a C-E equivalence class (though the two always coincide in the 2-by-2 case). It should be noted that, typically, within one cospectrality class (similarity class), there will be many C-E equivalence classes (C-S equivalence classes).

We are again concerned with the number of unitary similarity classes, now also in a given C-E equivalence class. As before, there are two possibilities of interest. In general, we say that a nonvoid set \( S \subseteq M_n \) is mono-unitary if \( S \) intersects only one unitary similarity class; if \( S \) intersects infinitely many unitary similarity classes, we call \( S \) multi-unitary. We apply either term to C-E or C-S equivalence classes, but, in these cases, intersection with a unitary similarity class implies containment of the entire unitary similarity class. No possibilities, other than mono- or multi-unitary, for C-S or C-E equivalence classes are known, and we suspect that they are the only two, and, in fact, that there is always a continuum of unitary classes in the multi-unitary case. In general, either a C-E or C-S equivalence class is unitary similarity invariant, so that, in order to study \( CE(A) \) or \( CS(A) \), \( A \) may be put in whatever special form we like via a unitary similarity. In fact, to determine if \( A \) and \( B \) are C-E or C-S equivalent, \( A \) and \( B \) may independently be subjected to unitary similarities. Of course, if \( CE(A) \) is mono-unitary, then \( CS(A) \) is as well, but the converse need not hold; \( CE(A) \) may be multi-unitary, while \( CS(A) \) is mono-unitary. It can occur that a matrix in \( CE(A) \) with Jordan form different from that of \( A \) has a multi-unitary C-S equivalence class.
For completeness, we mention here the extension, to the case of C-E equivalence, of some basic facts already known for C-S equivalence [5,13]. Reduction of the singular to the nonsingular case and the 2-by-2 case are rather similar.

If \( A \in M_n \) is singular and is unitarily similar to a matrix of the form \( 0 \oplus B \), for some nonsingular \( B \in M_k \), then \( CE(A) \) (resp. \( CS(A) \)) has as “many” unitary similarity classes as \( CE(B) \) (resp. \( CS(B) \)). If the singular \( A \) is not unitarily similar to a matrix of the latter form, then \( A \) is unitarily similar to a matrix of the form

\[
A' = \begin{bmatrix}
0 & e \\
0 & B
\end{bmatrix}
\]

for some \( B \in M_{n-1} \) and \( e \neq 0 \). For each positive real number \( r \), let \( C_r = [r] \oplus I_{n-1} \). Clearly, \( C_r^* A C_r \in CS(A') = CS(A) \) and, as \( r \) varies, the Frobenius norm of \( C_r^* A C_r \) runs over a continuum. Thus, \( CS(A) \) and \( CE(A) \) are multi-unitary.

Interestingly, in the 2-by-2 case, \( CS(A) \) is mono-unitary, unless \( A \) is rank 1 and not normal [5,13]. In fact, \( CE(A) \) is mono-unitary, unless \( A \) is rank 1 and not normal.

If there are two equal eigenvalues, for one of the two possible Jordan structures, \( A \) is a scalar matrix and \( CE(A) \) consists of only one matrix. (For the other Jordan structure, if nonsingular, \( CE(B) = CS(B) \) is mono-unitary). This, however, is not indicative of the general case. For \( n > 2 \), it seems common (though not universal) that \( CS(A) \), and then \( CE(A) \), is multi-unitary.

Recall that the field of values of \( A \in M_n \) is defined by

\[
F(A) \equiv \{x^* Ax : x \in \mathbb{C}^n, x^* x = 1\};
\]

the angular version (angular field of values) is defined by

\[
F'(A) \equiv \{x^* Ax : 0 \neq x \in \mathbb{C}^n\}.
\]

The former is a compact, convex subset of \( \mathbb{C} \) [10], while the latter is the smallest angular sector of \( \mathbb{C} \) that contains the former. The field of values (resp. angular field of values) of any principal submatrix of \( A \) lies in \( F(A) \) (resp. \( F'(A) \)). Congruent matrices have the same angular field of values, though their fields of values may vary. Whether or not \( 0 \in F(A) \) or \( 0 \in F'(A) \) is a congruential invariant, and for many questions, including those considered herein, the location of \( 0 \) relative to \( F(A) \) is very important. Of course, we may have \( 0 \notin F(A) \), \( 0 \in \partial F(A) \) (the boundary of \( F(A) \)), or \( 0 \in \text{int} F(A) \) (the interior of \( F(A) \)). Note that interior here means relative to \( \mathbb{C} \). If \( F(A) \) is a point or a line segment (i.e. \( A = \alpha I + \beta B \), with \( \alpha, \beta \in \mathbb{C} \) and \( B \) Hermitian), then \( F(A) \) has no interior; in all other cases it will have. In case \( F(A) \) is a segment of a line passing through the origin, \( e^{i\theta} A \) is Hermitian for some angle \( \theta \), and we say that \( A \) is rotationally Hermitian. By the result of [5] (see also [13,14]), such matrices are always mono-unitary (in the C-S sense), and this will be generalized here. In [5], it was shown that, for normal matrices \( A \), \( CS(A) \) is multi-unitary if and only if \( 0 \in \text{int} F(A) \). Here we generalize this by showing that for \( n \geq 3 \), whenever \( 0 \in \text{int} F(A) \), with \( A \) nonsingular, \( CE(A) \) is multi-unitary (Section 3). Then, we identify a class of matrices, for any size, that is mono-unitary in the C-E, and thus C-S, sense (Section 4); of course, for these matrices, \( 0 \in \partial F(A) \) or
0 ∉ F(A). However, these matrices form a rather thin set. In the 3-by-3 case, we are able to determine whether CE(A) and CS(A) are mono- or multi-unitary for any matrix A (Section 5). The multi-unitary case is common even if 0 ∉ int F(A), and, indeed, we close by showing that the matrices A for which CS(A) is multi-unitary are dense in Mn for all n ≥ 3 (Section 6).

2. Background and preliminaries

As it is both a congruence and a similarity, we may transform $A ∈ M_n$, a swirl, by unitary similarity, without changing CS(A). In particular, by Schur’s Theorem [9], we may assume that $A$ is upper triangular. We, of course, may also change $A$ by congruences that preserve the eigenvalues (similarity class) without changing CE(A) (CS(A)). A primary strategy, to show that $A$ is multi-unitary, is to demonstrate an infinite number of matrices in CS(A) (or CE(A)), each with different Frobenius norm (∥∥F). (A similar strategy was used in some cases in [13,14].) Different norms imply different unitary similarity classes. One explicit way to do this is to partition (upper triangular) $A ∈ M_n$ as

$$
\begin{bmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{bmatrix},
$$

with $A_{11} ∈ M_k$, $k < n$, and apply a congruence via

$$
C = \begin{bmatrix}
C_1 & 0 \\
0 & I_{n-k}
\end{bmatrix}
$$

in such a way that $C_1^*A_{11}C_1 = A_{11}$ but $∥C_1^*A_{12}∥_F ≠ ∥A_{12}∥_F$.

The (congruential) automorphism group of $B ∈ M_n$ is defined and denoted by

$$\text{Aut}(B) = \{ R ∈ M_n : R^*BR = B \text{ and } R \text{ is nonsingular} \}.$$ 

Then,

$$\text{Aut}^*(B) = \{ R^* : R ∈ \text{Aut}(B) \}.$$ 

The above strategy requires an understanding of Aut($A_{11}$), which was studied in [7], and a nonzero block whose Frobenius norm can be changed by an element of Aut($A_{11}$). Fortunately, when this happens, there is always an infinite number of possibilities. Here our purpose is to record, and adapt, some facts from prior work [7,8] to provide machinery to demonstrate when a C-E or C-S class is multi-unitary. Before beginning, we mention some notation/terminology, and some elementary facts.

We say that $x ∈ C^n$ is invariant for $A ∈ M_n$ if the subspace generated by $x$ is an invariant subspace for every member of Aut($A$); note that 0 is always an invariant vector. Since Aut($B$) = $C^{-1}$Aut($A$)$C$ if $B = C^*AC$, $C ∈ M_n$ nonsingular, $C^*x$ is invariant for $B$ if and only if $x$ is invariant for $A$. We will use these facts throughout, without further comment. Even if $x$ is invariant for $A$, we may be able to change the
norm of $x$ with an automorphism of $A$, but, at least, when $A$ is a 2-by-2 nonnormal matrix, we will show that the norm of any noninvariant $x$ may be changed. This will be used in the proof of some general results.

In [12] an $A \in M_n$ is called unitoid if it is diagonalizable by congruence, and, then, the arguments of the nonzero diagonal entries of a diagonal matrix congruent to $A$ are shown to be canonical (congruentially invariant) and called canonical angles; any 0 diagonal entries are called degenerate canonical angles, and their number is also canonical. If $0 \notin \rho(A)$ then $A$ is automatically unitoid [3], and, generally, an invertible $A$ is unitoid if and only if $A^{-1}A^*$ is similar to a unitary matrix [3,8].

We denote the list of eigenvalues, spectrum, of $A \in M_n$ by $\lambda_1, \ldots, \lambda_n$.

Here, multiple eigenvalues are listed as many times as their multiplicity, so that $\lambda_1, \ldots, \lambda_n$ is, technically, a multi-set. As usual, we denote a principal submatrix of $A \in M_n$, lying in rows and columns indexed by $S \subseteq \{1, \ldots, n\}$, by $A[S]$; if $S = \{i_1, \ldots, i_k\}$, we just write $A[i_1, \ldots, i_k]$.

We recall standard facts [9] about unitary matrices. If $x, y \in \mathbb{C}^n$ are such that $\|y\| = \|x\|$, then there is a unitary matrix $U \in M_n$ such that $y = Ux$. Moreover, $U \in M_n$ is unitary if and only if $\|Ux\| = \|x\|$ for all $x \in \mathbb{C}^n$. Here, as throughout, $\|\|$ denotes the Euclidean norm on vectors.

First, we mention a cancellation principle for nonsingular matrices that follows from the work on congruential canonical forms in [8]. It seems not to be well known and may be of independent interest.

**Lemma 1.** Suppose that $A \in M_k$ and $B_1, B_2 \in M_{n-k}$ are all nonsingular. If $A \oplus B_1$ is congruent to $A \oplus B_2$ then $B_1$ is congruent to $B_2$.

We will use Lemma 1 only when $k = 1$.

We next record facts about automorphism groups that we often use.

**Lemma 2.** If

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

then each matrix in $\text{Aut}^+(A)$ has equal eigenvalues that lie on the unit circle. Unless the matrix is a scalar multiple of $I$, the common eigenvalues have geometric multiplicity 1 and $[1 \quad 1]^T$ is a basis for the eigenspace.

**Proof.** Suppose that $R \in \text{Aut}(A)$. Then $R^{-1}A^{-1}A^*R = A^{-1}A^*$, or, equivalently, $R^*A(A^*)^{-1} = A(A^*)^{-1}R^*$, which implies

$$R^* = \begin{bmatrix} w - 2z & z \\ -z & w \end{bmatrix}$$

for some $w, z \in \mathbb{C}$. Clearly, $w - z$ is an eigenvalue of $R^*$ with multiplicity 2. Moreover, since $\det(R^*)$ has unit modulus, also $w - z$ has unit modulus. If $z \neq 0$, $R^*$ is
nonscalar and \( w - z \) has geometric multiplicity 1. Because \([1 \ 1]^T\) is an eigenvector of \( R^* \), the claim follows. □

We note that the first (spectral) conclusion of Lemma 2 could also be deduced from proposition 1 of [2], where special forms under congruence for matrices \( A \) with \( 0 \in \partial F(A) \) are considered.

Lemma 3. If

\[
A = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix},
\]

with \( a, b, x \in \mathbb{C} \setminus \{0\} \), then we have the following:

(a) if \( 0 \notin F(A) \), then for every \( y \in \mathbb{C}^2 \), \( ||R^*y|| \) runs through a continuum as \( R \) runs through \( \text{Aut}(A) \), unless \( y \) is invariant for \( A \);

(b) if \( 0 \in \partial F(A) \), then for every \( y \in \mathbb{C}^2 \) and every positive integer \( n \), there is \( R \in \text{Aut}(A) \) such that \( ||R^*y|| > n \), unless \( y \) is invariant for \( A \); and

(c) if \( 0 \in \text{int} F(A) \) then, for every \( 0 \neq y \in \mathbb{C}^2 \) and positive integer \( n \), there is \( R \in \text{Aut}(A) \) such that \( ||R^*y|| > n \).

Proof. (a) Suppose \( 0 \notin F(A) \). In this event \( A \) is unitoid. Let \( C \in M_2 \) be a non-singular matrix such that \( A' = C^*AC \) is diagonal unitary. Note that, since \( A \) is not rotationally Hermitian, the canonical angles for \( A' \) do not lie on the same line through the origin. According to the work in [7], \( \text{Aut}(A') \) is the group of the 2-by-2 unitary diagonal matrices. Then \( \text{Aut}(A) = \{CDC^{-1} : D \in M_2 \text{ diagonal unitary} \} \) and if \( R \in \text{Aut}(A) \) is nonscalar, the eigenspaces of \( R^* \) (associated with unit modulus eigenvalues) are

\[
E_1 = \left\{ (C^*)^{-1} \begin{bmatrix} z \\ 0 \end{bmatrix} : z \in \mathbb{C} \right\}
\]

and

\[
E_2 = \left\{ (C^*)^{-1} \begin{bmatrix} 0 \\ z \end{bmatrix} : z \in \mathbb{C} \right\}.
\]

Let \( u = [u_1 \ u_2]^T \in E_1 \setminus \{0\} \) and \( v = [v_1 \ v_2]^T \in E_2 \setminus \{0\} \). Then \((u, v)\) is a basis of eigenvectors of any element in \( \text{Aut}^*(A) \). Clearly, if \( y \in E_1 \cup E_2 \) then \( ||R^*y|| = ||y|| \) for all \( R \in \text{Aut}(A) \). Now suppose that \( y \notin E_1 \cup E_2 \). Let \( \lambda_1, \lambda_2 \in \mathbb{C} \) be such that \( y = \lambda_1 u + \lambda_2 v \). Clearly, \( \lambda_1 \) and \( \lambda_2 \) are nonzero. For \( \theta \in [0, 2\pi[, \) let

\[
R^*_\theta = (C^*)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} C^* \in \text{Aut}^*(A).
\]

Then

\[
R^*_\theta y = \lambda_1 u + \lambda_2 e^{i\theta} v.
\]
and \(|R^* y||^2 = \lambda_1 \Re u^* u + \lambda_2 \Re v^* v + 2\Re(\lambda_1 \lambda_2 u^* v)\). Note that \(u^* v \neq 0\). In fact, \(u^* v = 0\) is equivalent to
\[
\bar{c}_1 t c_{12} + \bar{c}_2 t c_{22} = 0, 
\]
for \((C^*)^{-1} = [c_{ij}]\). Because the \((2,1)\) entry of \(A = (C^*)^{-1} A'C^{-1}\) is zero and the \((1,2)\) entry is nonzero, a calculation shows that \((1)\) would imply \(A'\) scalar, which does not occur. Thus, \(||R^* y||\) runs through a continuum as \(R^*\) runs through Aut\((A)\).

Note that any element in Aut\(^*(A)\) is of the form \(e^{i\gamma} R^*_p\), for some \(\gamma, \theta \in [0, 2\pi]\).

(b) Suppose \(0 \in \partial F(A)\). Note that in this case \(A\) cannot be unitoid. According to [6,8], there is \(C \in M_2\) such that
\[
A' = C^* A C = e^{i\gamma} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}
\]
for some \(\gamma \in \mathbb{R}\). Since Aut\(^*(A) = (C^*)^{-1} \text{Aut}^* (A') C^*\), it follows from Lemma 2 that the eigenvalues of any element in Aut\(^*(A)\) are unit modulus. Then, if \(y = [y_1, y_2]^T\) is invariant for \(A\), \(||R^* y|| = ||y||\) for all \(R \in \text{Aut}(A)\). Now suppose that \(R \in \text{Aut}(A)\) and \(y \neq 0\) is not an eigenvector of \(R^*\). Because \(R^*\) is nonscalar, bearing in mind Lemma 2, there is a nonsingular \(P \in M_2\) such that
\[
R^* = e^{i\theta} P^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} P
\]
for some \(\theta \in \mathbb{R}\). For
\[
P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},
\]

\[(R^k \hat{y}) = e^{ik\theta} \begin{bmatrix} y_1 + \frac{P_{21} y_{1} + P_{22} y_{2}}{P_{11} P_{22} - P_{12} P_{21}} P_{22 k} \\ y_2 - \frac{P_{21} y_{1} + P_{22} y_{2}}{P_{11} P_{22} - P_{12} P_{21}} P_{21 k} \end{bmatrix}
\]
for each positive integer \(k\). Since \(y\) is not an eigenvector of \(R^*\), then \(p_{21} y_{1} + p_{22} y_{2} \neq 0\). Also \(p_{21} \neq 0\) (and \(p_{22} \neq 0\), otherwise \(R\) would be a nonscalar triangular matrix, which, it is easy to see, cannot be an element of Aut\((A)\). Clearly, \(k\) can be chosen such that \(||(R^k \hat{y})\||\) is arbitrarily large.

(c) Suppose \(0 \in \text{int} F(A)\). Also in this case \(A\) cannot be unitoid. Let \(R = A^{-1} A^*\). Note that \(R, R^{-1} \in \text{Aut}(A)\). According to [8], \(R\), and thus \(R^*\), has one eigenvalue with modulus greater than 1 and one eigenvalue with modulus less than 1. If \(y\) is an eigenvector of \(R^*\), \(y\) is an eigenvector associated with an eigenvalue of modulus greater than 1 of either \(R^*\) or \((R^{-1})^*\). Then, either the norm of \((R^k \hat{y})\) or the norm of \((R^{-k}) \hat{y}\) is unbounded, as \(k\) runs through the positive integers. Now suppose that \(y \neq 0\) is not an eigenvector of \(R^*\). Let \((u, v)\) be a basis of eigenvectors of \(R^*\) associated with the eigenvalues \((t_1, t_2)\), with \(|t_1| > 1\) (and \(|t_2| < 1\), and \(\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}\) be such that \(y = \lambda_1 u + \lambda_2 v\). For each positive integer \(k\), we have
\[
(R^k \hat{y}) = \lambda_1 t_1^k u + \lambda_2 t_2^k v.
\]
Clearly, since \(|t_1|\) goes to infinity and \(|t_2|\) goes to 0, as \(k\) goes to infinity, \(||(R^k \hat{y})||\) is unbounded as \(k\) runs through the positive integers, completing the proof. □
Lemma 4. Let
\[ A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix} \in M_3 \]
be nonsingular, with \( z \neq 0 \). Suppose that \( a \in \text{int} F'(A[2, 3]) \). Then, \( A \) is congruent to
\[ B = \begin{bmatrix} a & x' & y' \\ 0 & b & z' \\ 0 & 0 & c \end{bmatrix}, \]
with \( z' \neq 0 \) and \([x' \ y'] \neq 0\). Moreover, if \( 0 \in \text{int} F(A[2, 3]) \) then \( 0 \in \text{int} F(B[2, 3]) \).

Proof. Because \( z \neq 0 \), \( b, c \in \text{int} F'(A[2, 3]) \). Since \( a \in \text{int} F'(A[2, 3]) \), according to [11], there is a nonsingular \( C_1 \in M_2 \) such that
\[ Q = C_1^* A[2, 3] C_1 = \begin{bmatrix} a & k \\ 0 & b' \end{bmatrix} \]
for some \( b' \neq 0 \). Then \( F'(Q) = F'(A[2, 3]) \) and \( \text{arg}(ab') = \text{arg}(bc) \). Moreover, if \( 0 \in \text{int} F(A[2, 3]) \) then \( 0 \in \text{int} F(Q) \). Because \( a \in \text{int} F'(Q) \), \( k \neq 0 \). With an auxiliary diagonal unitary similarity, we may assume \( k > 0 \). Choose \( d \in ]0, k[ \) close enough to \( k \) so that either (i) \( 0 \in \text{int} F(A[2, 3]) \) and \( 0 \in \text{int} F(Q') \) (in this event \( F'(Q') \) is the entire complex plane), with
\[ Q' = \begin{bmatrix} a & d \\ 0 & b' \end{bmatrix}, \]
or (ii) \( 0 \notin \text{int} F(A[2, 3]) \) and \( b, c \in \text{int} F'(Q') \). Let \( U \in M_2 \) be a unitary matrix such that
\[ U^* \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} \sqrt{k^2 - d^2} \\ d \end{bmatrix} \]
and let \( C_2 = [(1) \oplus C_1] (U \oplus [1]) \). Then
\[ A' = C_2^* A C_2 = \begin{bmatrix} a & 0 & \sqrt{k^2 - d^2} \\ 0 & a & d \\ 0 & 0 & b' \end{bmatrix}. \]
Because \( \text{arg}(ab') = \text{arg}(bc) \) and \( b, c \in \text{int} F'(A'[2, 3]) \), by [4,11], there is a nonsingular matrix \( C_3 \in M_2 \) such that
\[ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_3^* A' C_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_3 \end{bmatrix} = \begin{bmatrix} a & x' & y' \\ 0 & b & z' \\ 0 & 0 & c \end{bmatrix} \]
for some complex numbers \( x', y' \) and \( z' \). Since \( B[2, 3] \) is congruent to \( Q' \), if \( 0 \in \text{int} F(A[2, 3]) \), then \( 0 \in \text{int} F(B[2, 3]) \). Moreover, \( b, c \in \text{int} F'(B[2, 3]) \), which implies \( z' \neq 0 \). Clearly, \([x' \ y'] \neq 0\). \( \square \)
3. The case $0 \in \text{int } F(A)$

The following result follows from the work of [1] and is stated in [4].

**Theorem 5.** Let $A \in M_{n,n}$, $n \geq 3$, be a nonsingular matrix such that $0 \in \text{int } F(A)$. Then $A$ is congruent to a matrix of the form
\[
\begin{bmatrix}
1 & * \\
0 & B
\end{bmatrix},
\]
in which $B \in M_{n-1}$ and $0 \in \text{int } F(B)$.

**Lemma 6.** Let $A \in M_{n,n}$, $n \geq 3$, be a nonsingular matrix such that $0 \in \text{int } F(A)$. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C} \setminus \{0\}$ be such that $\arg(\lambda_1 \cdots \lambda_n) = \arg(\det(A))$. Then $A$ is congruent to an upper triangular matrix of the form
\[
\begin{bmatrix}
B_{11} & B_{12} \\
0 & B_{22}
\end{bmatrix},
\]
with $B_{11} \in M_{n-2}$ such that $\sigma(B_{11}) = [\lambda_1, \ldots, \lambda_{n-2}]$, $B_{22} \in M_2$ such that $\sigma(B_{22}) = [\lambda_{n-1}, \lambda_n]$ and $0 \in \text{int } F(B_{22})$.

**Proof.** Since $0 \in \text{int } F(\lambda_1^{-1}A)$, by Theorem 5, $\lambda_1^{-1}A$ is congruent to a matrix of the form
\[
\begin{bmatrix}
1 & * \\
0 & B
\end{bmatrix},
\]
with $B \in M_{n-1}$ such that $0 \in \text{int } F(B)$. Then $A$ is congruent to
\[
A' = \begin{bmatrix}
\lambda_1 & * \\
0 & B'
\end{bmatrix},
\]
with $B' \in M_{n-1}$ such that $0 \in \text{int } F(B')$. Since $\arg(\det(A)) = \arg(\det(A'))$, then $\arg(\det(B')) = \arg(\lambda_2 \cdots \lambda_n)$. The remainder of the proof is by induction on $n$. If $n = 3$, by [4] and using Schur’s triangularization theorem, $B'$ is congruent to an upper triangular matrix with spectrum $[\lambda_2, \lambda_3]$ and the proof of the base case is complete. Now suppose that $n > 3$. According to the induction hypothesis, there is a nonsingular $C \in M_{n-1}$ such that
\[
C^* B' C = \begin{bmatrix}
B_{11}' & \ast \\
0 & B_{22}'
\end{bmatrix},
\]
with $B_{11}' \in M_{n-3}$ upper triangular such that $\sigma(B_{11}') = [\lambda_2, \ldots, \lambda_{n-2}]$, $B_{22}'$ upper triangular such that $\sigma(B_{22}') = [\lambda_{n-1}, \lambda_n]$ and $0 \in \text{int } F(B_{22}')$. Then $(\{1\} \oplus C^*) A' (\{1\} \oplus C)$ has the desired form. \(\square\)

**Theorem 7.** Let $A \in M_{n,n}$, $n \geq 3$, be a nonsingular matrix such that $0 \in \text{int } F(A)$. Then $C E(A)$ is multi-unitary.
Proof. According to Lemma 6, \( A \) is congruent to an upper triangular matrix \( B \) of the form (2), with \( \sigma(B) = \sigma(A) \) and \( 0 \in \text{int} \, F(B_{22}) \). Note that \( B \in CE(A) \) and, since \( 0 \in \text{int} \, F(B_{22}) \), the upper triangular matrix \( B_{22} \) is not diagonal. We assume that \( B_{12} \neq 0 \), for, if it is 0, we replace \( B \) by \( S^*BS \), which \( S = I_{n-3} \oplus C \) and \( C \) is obtained by Lemma 4. Bearing in mind Lemma 3, there is \( R \in \text{Aut}(B_{22}) \) such that \( ||B_{12}R||_F = ||R^*B_{12}^*||_F \) is arbitrarily large. Therefore, the Frobenius norm of

\[
\begin{pmatrix}
I_{n-2} \oplus R^* & B_{12} \\
0 & B_{22}
\end{pmatrix} \in CE(B)
\]

ranges over an infinite set as \( R \) ranges over \( \text{Aut}(B_{22}) \). Thus, \( CE(A) (= CE(B)) \) is multi-unitary. \( \square \)

As noted, the situation is different for \( n = 2 \). In general, it is also different for \( CS(A) \) in place of \( CE(A) \), it can occur that a nonsingular \( A \in M_n \), with \( 0 \in \text{int} \, F(A) \), has \( CS(A) \) mono-unitary. See Theorem 14. However, when \( A \) has distinct eigenvalues, \( CS(A) = CE(A) \) so that we have immediately the following.

Corollary 8. Let \( A \in M_n, n \geq 3 \), be a nonsingular matrix with distinct eigenvalues such that \( 0 \in \text{int} \, F(A) \). Then \( CS(A) \) is multi-unitary.

Corollary 8 may be compared with the main result of [5], in which \( A \) is required to be normal but may have repeated eigenvalues.

Note that if \( A \) is a singular matrix such that \( 0 \in \text{int} \, F(A) \) then \( CE(A) \) is multi-unitary unless \( A \) is unitarily similar to \( 0 \oplus B \), with \( B \in M_2 \) nonsingular. In fact, if \( A \) is unitarily similar to \( 0 \oplus B \), with \( B \in M_k \) nonsingular, then it is easily seen that \( 0 \in \text{int} \, F(B) \), and, by Theorem 7, for \( k \geq 3 \), \( CE(B) \) is multi-unitary, which implies \( CE(A) \) multi-unitary.

4. Exceptional matrices

We say that a nonsingular \( A \in M_n \) is rotationally rank one if there is a \( \theta \in \mathbb{R} \) such that \( \text{rank}(e^{i\theta}A + e^{-i\theta}A^*) = 1 \). The property of being rotationally rank one is invariant under congruence. Because \( \text{rank}(e^{i\theta}A + e^{-i\theta}A^*) = 1 \) implies the Hermitian part of \( e^{i\theta}A \) semidefinite, if \( A \) is rotationally rank one, then \( 0 \notin \text{int} \, F(A) \).

Lemma 9. Let \( A \in M_n \) be a rotationally rank one matrix. Then \( CE(A) \), and thus \( CS(A) \), is mono-unitary.

Proof. First, suppose that \( \text{rank}(A + A^*) = 1 \). Let \( B \in CE(A) \) and \( \sigma(A) (= \sigma(B)) = \{\lambda_1, \ldots, \lambda_n\} \). Without loss of generality, suppose that \( \text{Re}(\lambda_i) \neq 0 \) for \( i \leq k \) and \( \text{Re}(\lambda_i) = 0 \) for \( i > k \). By Schur’s unitary triangularization theorem, \( A \) and \( B \) are unitarily similar to upper triangular matrices of the forms, respectively,
for some $a_{ij}, b_{ij} \in \mathbb{C}$. With additional unitary similarities via diagonal unitary matrices, we can assume $a_{1j} \geq 0$ and $b_{1j} \geq 0$, for $j = 2, \ldots, n$. Because $\text{rank}(A' + A'^*) = \text{rank}(B' + B'^*) = 1$, all 2-by-2 minors of both $A' + A'^*$ and $B' + B'^*$ are 0. This implies $a_{ij} = b_{ij} = 0$ for $i = 1, \ldots, k$, $j = k + 1, \ldots, n$. Since $A' + A'^*$ is nonzero, $k \geq 1$. Then, it also follows $a_{ij} = b_{ij} = 0$ for $i = 1, \ldots, k$, $j = k + 1, \ldots, n$.

By a possible previous multiplication of $A$ by $-1$, we can assume that $\text{Re}(\lambda_1) > 0$.

A calculation now shows that $\text{Re}(\lambda_i) \text{Re}(\lambda_j) > 0$ and $a_{ij} = b_{ij} = 2\sqrt{\text{Re}(\lambda_i)\text{Re}(\lambda_j)}$, for $i, j = 1, \ldots, k$ with $j > i$. Thus, $A' = B'$ and $B$ is unitarily similar to $A$. Because $B$ is an arbitrary element in $CE(A)$, it follows that $CE(A)$ is mono-unitary.

If rank$(G + G^*) = 1$, with $G = e^{i\theta}A$ for some $\theta \in \mathbb{R}$, then, by the first part of the proof, $CE(G)$ is mono-unitary. Because $A = e^{-i\theta}G$, also $CE(A)$ is mono-unitary.

We call a nonsingular $A \in M_n$ exceptional if $0 \notin \text{int}F(A)$ and $A$ is unitarily similar to $\text{diag}(k_1, \ldots, k_r) \oplus B$, for some $k_1, \ldots, k_r \in \mathbb{C}$ and $B \in M_{n-r}$ such that

(a) $k_j \notin \text{int}F'(B)$, $j = 1, \ldots, r$, and
(b) $B$ is rotationally rank one.

It can occur that either $B$ does not appear or $r = 0$. In either case, we still call $A$ exceptional. So a nonsingular normal matrix $A$ such that $0 \notin \text{int}F(A)$ is exceptional.

In particular, a nonsingular Hermitian matrix is exceptional.

Also, for example, the matrices

\[
A = \begin{bmatrix}
i & 0 & 0 & 0 \\
0 & 1 & 2 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
i & 0 & 0 & 0 \\
0 & e^{-\frac{\pi}{2}i} & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

are exceptional.

Note that, in contrast to the rotationally rank one property, the property of being exceptional is not generally congruentially invariant.

It was shown in [5] that if $A$ is normal and $0 \notin \text{int}F(A)$ then $CS(A)$ is mono-unitary. The nonsingular case of this statement is a special case of the next theorem.

**Theorem 10.** Let $A \in M_n$ be an exceptional matrix. Then $CE(A)$, and thus $CS(A)$, is mono-unitary.
**Proof.** The proof is by induction on \( r \), the number of diagonal entries in the diagonal part of a representation of a unitary similarity of \( A \), as in the definition of exceptional. In case \( r = n \) the result reduces to the normal case, which is covered in [5]. If \( A \) is rotationally rank one, the conclusion follows from Lemma 9. This also provides the base case, \( r = 0 \), of the induction. Next, suppose that \( A \) is neither rotationally rank one nor normal. Then \( A \) is unitarily similar to a matrix of the form \([k] \oplus B\), in which \( 0 \neq k \in \partial F'(A) \) and \( B \) is exceptional. Let \( A' \in CE(A) \). Since \( F'(A') \) and \( F'(A) \) coincide, \( k \in \partial F'(A') \). Since \( k \) is also an eigenvalue of \( A' \), \( A' \) is unitarily similar to a matrix of the form

\[
A'' = \begin{bmatrix} k & 0 \\ 0 & B' \end{bmatrix},
\]

with \( B' \in M_{n-1} \) (otherwise a neighborhood of \( k \) would lie in \( F(A') \), which is not the case). By Lemma 1, \( B' \) is congruent to \( B \). Since \( B' \) has the same eigenvalues as \( B, B' \in CE(B) \), as well. According to the induction hypothesis, \( CE(B) \) is mono-unitary. Thus, \( B' \) is unitarily similar to \( B \), which implies that \( A'' \), and thus \( A' \), is unitarily similar to \( A \). Since \( A' \) is any element of \( CE(A) \), we conclude that \( CE(A) \) is mono-unitary. □

5. The 3-by-3 case

Here we determine the number of unitary similarity classes in the C-E and C-S classes of each \( A \in M_3 \).

If \( A \in M_3 \) is nonsingular and not exceptional, we show in the next subsection that \( CE(A) \) is multi-unitary. (This is quite different from the 2-by-2 case, in which all nonsingular matrices are mono-unitary, exceptional or not.) We then return, in the second subsection, to C-S equivalence. We exploit the C-E result to show that the only additional exceptions to multi-unitary are derogatory matrices: if \( A \in M_3 \) is nonsingular, then \( CS(A) \) is multi-unitary if and only if \( A \) is nonderogatory and not exceptional.

If \( A \in M_3 \) is singular then \( CE(A) \) (and \( CS(A) \)) is multi-unitary unless \( A \) is either zero or unitarily similar to \( 0 \oplus B \), with \( B \in M_k \) nonsingular. Note that in this case \( k \leq 2 \), so that \( CE(B) \) is mono-unitary.

5.1. The C-E class of 3-by-3 matrices

**Lemma 11.** Let \( A \in M_2 \) be a nonsingular, nonrotationally Hermitian matrix such that \( 0 \notin \text{int} F(A) \). If \( x \in \mathbb{C}^2 \) is invariant for \( A \), there is \( \theta \in \mathbb{R} \) such that

\[
\text{rank}[e^{i\theta}A + e^{-i\theta}A^*x] = 1.
\]
Proof. Case 1: Suppose \( 0 \in \partial F(A) \). Since \( A \) is not rotationally Hermitian, by [6,8] there is a nonsingular \( C \in M_2 \) such that
\[
A' = C^*AC = e^{i\gamma} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}
\]
for some \( \gamma \in \mathbb{R} \). Because \( x \) is invariant for \( A \), \( C^*x \) is invariant for \( A' \). According to Lemma 2,
\[
C^*x = \begin{bmatrix} y \\ y \end{bmatrix}
\]
for some \( y \in \mathbb{C} \). A simple calculation shows that \( \text{rank}\{B + B^*[C^*x]\} = 1 \), with \( B = e^{-i\gamma}A' \). Thus,
\[
\text{rank}\{(C^*)^{-1}[B + B^*[C^*x]](C^{-1} \oplus [1])\} = 1,
\]
which implies the claim.

Case 2: Suppose \( 0 \notin F(A) \). In this case \( A \) is unitoid. Then, there is a nonsingular \( C \in M_2 \) such that
\[
A' = C^*AC = \begin{bmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{bmatrix}
\]
for some \( \gamma_1, \gamma_2 \in \mathbb{R} \). Note that, because \( A \) is not rotationally Hermitian, \( e^{i\gamma_1} \) and \( e^{i\gamma_2} \) do not lie on the same line through the origin. According to the work in [7], Aut\((A')\) is the group of the 2-by-2 unitary diagonal matrices. Since \( x \) is invariant for \( A \), \( C^*x \) is invariant for \( A' \). Thus \( C^*x \) has a zero entry. Without loss of generality, suppose that
\[
C^*x = \begin{bmatrix} 0 \\ y \end{bmatrix}
\]
for some \( y \in \mathbb{C} \). Then \( \text{rank}\{B + B^*[C^*x]\} = 1 \), with \( B = e^{i(\frac{\pi}{2} - \gamma_1)}A' \), which implies the claim. \( \Box \)

Note that, in particular, Lemma 11 implies that any 2-by-2 nonsingular nonrotationally Hermitian matrix \( A \) such that \( 0 \notin \text{int} F(A) \) is rotationally rank one.

Lemma 12. Let
\[
A = \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix} \in M_3
\]
be nonsingular, with \( x, z \neq 0 \). Suppose that \( 0 \notin \text{int} F(A) \). If \([y \ z]^T\) is invariant for \( A[1, 2] \) and \([x \ y]^*\) is invariant for \( A[2, 3] \), then \( A \) is rotationally rank one.

Proof. According to Lemma 11, there are \( \theta_1, \theta_2 \in \mathbb{R} \) such that
\[
\text{rank}\left[ A_1 + A_1^* \begin{bmatrix} y \\ z \end{bmatrix} \right] = 1 \quad \text{and} \quad \text{rank}\left[ A_2^* + A_2^* \begin{bmatrix} x \\ y \end{bmatrix} \right] = 1
\] (3)
with $A_1 = e^{i\theta_1} A[1, 2]$ and $A_2 = e^{i\theta_2} A[2, 3]$. By a possible previous multiplication of $A$ by $e^{-i\theta_1}$, assume, without loss of generality, that $\theta_1 = 0$. A straightforward calculation shows that (3) implies that $\text{Re}(b) \neq 0$, $y = \frac{z}{2\text{Re}(b)}$, $\theta_2$ is an integer multiple of $\pi$ and, therefore, rank$(A + A^*) = 1$. \[ \Box \]

We may now classify the C-E equivalence classes for nonsingular elements of $M_3$. The result lies in stark contrast to the 2-by-2 case, in which mono-unitary is generic, and shows that “exceptional” is justified.

**Theorem 13.** Let $A \in M_3$ be a nonsingular matrix. Then $CE(A)$ is multi-unitary if and only if $A$ is not exceptional.

**Proof.** According to Theorem 10, if $CE(A)$ is multi-unitary then $A$ is not exceptional. Now we will prove the converse. Suppose that $A$ is not exceptional. If $0 \in \text{int} F(A)$, the claim follows from Theorem 7. Next suppose that $0 \notin \text{int} F(A)$. By a possible unitary similarity, suppose, without loss of generality, that $A = \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix}$.

Note that since $A$ is not exceptional, $A$ has at least one nonzero off-diagonal entry. If $A$ has exactly one nonzero off-diagonal entry, with an auxiliary unitary similarity via a permutation matrix, we can assume $x = y = 0$. Since $A$ is not exceptional, $a \in \text{int} F'(A[2, 3])$. Note that, because $0 \notin \text{int} F(A[2, 3])$, $A[2, 3]$ is rotationally rank one. By Lemma 4 there is an upper triangular matrix $B \in CE(A)$ with at least two nonzero off-diagonal entries. Because $CE(B) = CE(A)$, it is enough to show that $CE(B)$ is multi-unitary. Thus, assume, without loss of generality, that $A$ has at least two nonzero off-diagonal entries. In particular, assume $x \neq 0$ and either $y \neq 0$ or $z \neq 0$. If $x = 0$ it will become clear that the proof can be dealt with similarly, by focus upon the (2,3) principal submatrix of $A$ instead of the (1,2). If $[y \ z]^T$ is not invariant for $A[1, 2]$, according to Lemma 3, the Euclidean norm of $R^* [y \ z]^T$ ranges over an infinite set as $R$ ranges over $\text{Aut}(A[1, 2])$. Since, for each $R \in Aut(A[1, 2])$,

$$ (R^* \oplus [1])A(R \oplus [1]) = \begin{bmatrix} a & x \\ 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y & c \\ z & 0 \end{bmatrix} \in CE(A), $$

then there are infinitely many Frobenius norms for matrices in $CE(A)$ and, therefore, $CE(A)$ is multi-unitary. If $[y \ z]^T$ is invariant for $A[1, 2]$, then, because the nonscalar automorphisms of $A[1, 2]$ (which exist; see the proofs of Lemma 2 and Lemma 3(a)), are nontriangular, both $y \neq 0$ and $z \neq 0$. In this case $[x \ y]^*$ is not invariant for $A[2, 3]$, otherwise, by Lemma 12, $A$ would be rotationally rank one. The proof follows as in the previous case, by focussing upon $A[2, 3]$ in place of $A[1, 2]$. \[ \Box \]
5.2. The C-S class of 3-by-3 matrices

Since there are few possible Jordan structures for a 3-by-3 matrix with given eigenvalues, Theorem 13 goes a long way toward classifying C-S classes. Now, derogatory matrices may provide exceptions as well to the generic status of multi-unitary matrices. The general situation is summarized in the following.

Theorem 14. Let $A \in \mathbb{M}_3$ be a nonsingular matrix. Then $CS(A)$ is multi-unitary if and only if $A$ is neither derogatory nor exceptional.

Proof. ($\Rightarrow$) If $A$ is exceptional, it follows from Theorem 10 that $CE(A)$, and thus $CS(A)$, is mono-unitary. If $A$ is derogatory then either the minimum polynomial of $A$ has degree 1 or degree 2. In the first case $A$ is scalar and the result is trivial. In the second case the result follows from the work in [8], where it is shown that any nonsingular $A$ with minimum polynomial of degree 2 is such that $CS(A)$ is mono-unitary.

($\Leftarrow$) Now suppose that $A$ is nonderogatory and not exceptional. If $A$ has distinct eigenvalues then $CS(A)$ and $CE(A)$ coincide and it follows from Theorem 13 that $CS(A)$ is multi-unitary. If $A$ has at least two eigenvalues equal, then, by the first part of the proof, each (in fact, there is at most one) C-S class of derogatory matrices in $CE(A)$ is mono-unitary. According to Theorem 13, $CE(A)$ is multi-unitary. Thus, there are infinitely many unitary similarity classes among nonderogatory matrices in $CE(A)$. Since all these matrices are similar to $A$, it follows that $CS(A)$ is multi-unitary.

6. A density result for $n \geq 3$

We are convinced that for $n > 3$ the situation regarding classification of C-E and C-S equivalence classes is rather similar to that for $n = 3$. It is not straightforward to prove this with the machinery we have developed so far or with other conventional techniques (e.g. Pearcy’s Theorem [9, p. 76]), though it is surely likely that the norm again can be changed by an automorphism of a principal submatrix in block-triangular form. Here, we confirm the latter intuition and are able to give a density result for multi-unitary matrices, for $n > 2$ (exactly in contrast to the situation for $n = 2$).

Lemma 15. If $A \in \mathbb{M}_n$ is such that $0 \notin F(A)$ then $Aut(A)$ is connected.

Proof. According to the work in [7], $Aut(A)$ is uniformly similar to a direct sum of unitary groups. Since any complex unitary group is connected, it follows that $Aut(A)$ is connected.
Lemma 16. Let $R \in M_n$ be a nonsingular nonunitary matrix and $x \in \mathbb{C}^n$. Then in any neighborhood of $x$ there is $y \in \mathbb{C}^n$ such that \( \|Ry\| = \|y\| \).

Proof. If \( \|Rx\| \neq \|x\| \) the claim holds with $y = x$. Now suppose that \( \|Rx\| = \|x\| \) or, equivalently, \( x^*R^*Rx = x^*x \). Let \( \lambda_1, \ldots, \lambda_n > 0 \) be the eigenvalues of the positive definite matrix \( R^*R \). Because $R$ is not unitary, there is $i$ such that $\lambda_i \neq 1$. Without loss of generality, suppose that $\lambda_1 = 1$. Let $U \in M_n$ be a unitary matrix such that $U^*R^*RU = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and let $a_1, \ldots, a_n$ be complex numbers such that $x = U[a_1, \ldots, a_n]^T$. It follows from the hypothesis that

$$
\sum_{i=1}^n |a_i|^2 = \sum_{i=1}^n \lambda_i |a_i|^2.
$$

(4)

If $a_1 \neq 0$, let $y = U[(1 + \varepsilon)a_1, a_2, \ldots, a_n]^T$, $\varepsilon > 0$. Then

$$
\|y\|^2 = \sum_{i=2}^n |a_i|^2 + (1 + \varepsilon)^2 |a_1|^2
$$

and

$$
\|Ry\|^2 = \sum_{i=2}^n \lambda_i |a_i|^2 + (1 + \varepsilon)^2 \lambda_1 |a_1|^2.
$$

Clearly, (4) implies \( \|Ry\| \neq \|y\| \). If $a_1 = 0$, a similar calculation shows that for $y = U[\varepsilon, a_2, \ldots, a_n]^T$, with $\varepsilon > 0$, \( \|Ry\| \neq \|y\| \). Since in any case $\varepsilon$ can be taken arbitrarily small, the proof is complete. $\square$

Theorem 17. The matrices in $M_n$, $n \geq 3$, whose C-S equivalence classes are multi-unitary, are dense in $M_n$.

Proof. We will show that in any neighborhood of $A \in M_n$, $n \geq 3$, there is a $B$ such that $CS(B)$ is multi-unitary.

Case 1: Suppose $0 \in \text{int } F(A)$. Clearly, there is a nonsingular $B \in M_n$ arbitrarily close to $A$ such that the eigenvalues of $B$ are pairwise distinct and $0 \in \text{int } F(B)$. Since, in this event, $CS(B)$ coincides with $CE(B)$, according to Theorem 7, $CS(B)$ is multi-unitary.

Case 2: Suppose $0 \notin F(A)$. By a unitary similarity, suppose, without loss of generality, that

$$
A = \begin{bmatrix} A_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix},
$$

with $A_{11} \in M_{n-1}$. Note that $0 \notin F(A_{11})$. A nonnormal $B_{11} \in M_{n-1}$, with $0 \notin F(B_{11})$ and eigenvalues distinct from $a_{22}$, may be chosen arbitrarily close to $A_{11}$. Clearly, $B_{11}^{-1}B_{11}^* \in \text{Aut}(B_{11})$. Also, $B_{11}^{-1}B_{11}^*$ is not unitary [3]. By Lemma 16, there is a $b_{12}$ arbitrarily close to $a_{12}$ such that $\|(B_{11}^{-1}B_{11}^*)^*b_{12}\| \neq \|b_{12}\|$. According to
Lemma 15, there is a continuous path $Q_\lambda$, $\lambda \in [0, 1]$, of matrices in Aut$(B_{11})$ such that $Q_0 = B_{11}^{-1}B_{11}^*$ and $Q_1 = I_{n-1}$. Let

$$B = \begin{bmatrix} B_{11} & b_{12} \\ 0 & a_{22} \end{bmatrix}.$$ 

As $a_{22} \notin \sigma(B_{11})$, for any $\lambda \in [0, 1]$, $(Q_\lambda \oplus [1])^*B(Q_\lambda \oplus [1]) \in CS(B)$. Since

$$\| (Q_0 \oplus [1])^*B(Q_0 \oplus [1]) \|_F \neq \| (Q_1 \oplus [1])^*B(Q_1 \oplus [1]) \|_F = \| B \|_F,$$

by continuity, the Frobenius norm of $(Q_\lambda \oplus [1])^*B(Q_\lambda \oplus [1])$ runs over a continuum as $\lambda$ runs over $[0,1]$. Therefore, $CS(B)$ is multi-unitary. Note that any perturbation can be taken sufficiently small so that $B$ is arbitrarily close to $A$.

Case 3: Suppose $0 \notin \partial F(A)$. Let $\theta$ be an angle in the direction opposite to a normal to $F(A)$ at 0. Then, for $\epsilon > 0$, $0 \notin F(A + \epsilon e^{i\theta}I_n)$. According to case 2, in any neighborhood of $A + \epsilon e^{i\theta}I_n$ there is a matrix $B$ such that $CS(B)$ is multi-unitary. Since $\epsilon$ can be taken arbitrarily small, then $B$ can be arbitrarily close to $A$. □

References