Construction of a Lyapunov functional for 1D-viscous compressible barotropic fluid equations admitting vacua

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Abstract

The Navier–Stokes equations for a compressible barotropic fluid in 1D with zero velocity boundary conditions are considered. We study the case of large initial data in $H^1$ as well as the mass force such that the stationary density is uniquely determined but admits vacua. Missing uniform lower bound for the density is compensated by a careful modification of the construction procedure for a Lyapunov functional known for the case of solutions which are globally away from zero [I. Straškraba, A.A. Zlotnik, On a decay rate for 1D-viscous compressible barotropic fluid equations, J. Evol. Equ. 2 (2002) 69–96]. An immediate consequence of this construction is a decay rate estimate for this highly singular problem. The results are proved in the Eulerian coordinates for a large class of increasing state functions including $p(\rho) = a\rho^\gamma$ with any $\gamma > 0$ ($a > 0$ a constant).

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Résumé

Pour un fluide compressible barotropique, nous considérons les équations de Navier–Stokes dans le cas unidimensionnel associées à des conditions aux limites homogènes de Dirichlet. Les données initiales et de larges forces externes sont telles que la densité à l’équilibre soit déterminée de façon unique, et puisse s’annuler sur un ensemble de mesure nulle. Perdre toute borne inférieure pour la densité nous conduit à modifier soigneusement la procédure connue de construction d’une fonctionnelle de Lyapounov, comme conséquence nous obtenons une estimation nouvelle du taux de convergence. Les résultats sont établis en utilisant des coordonnées eulériennes pour une large classe de fonctions d’état décrivant la pression $p(\rho) = a\rho^\gamma$, $\gamma > 0$, $a > 0$.

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0. Introduction

The purpose of this study is the construction of a Lyapunov functional for 1D Navier–Stokes equations of a viscous compressible barotropic fluid under the influence of a large mass force in the case when the stationary density admits points of zero density, more precisely when \( \text{meas}\{x; \, \rho_\infty(x) = 0\} = 0 \), \( \rho_\infty \) being a solution to the steady state problem. We assume standard initial-boundary value problem with the Dirichlet boundary condition for the velocity as in (1.1)–(1.3) below. An immediate product of our construction is a result on a decay rate of evolutionary solution to the stationary one as time tends to infinity (see Theorem 1.1).

There are many results about the global behavior of solutions to Eqs. (1.1), (1.2) below under different boundary conditions and other data and we refer e.g. to [3,7,8,10–12] and the references therein, see also results and comments in a recent monograph [5, Chapter 8]. For optimal global existence result related to the considered problem (1.1)–(1.3) we refer to [9]: The unique solution has strictly positive density at any finite time provided the initial density has the same property.

So, in this work we continue the research results of which are summarized in [10], where a Lyapunov functional has been constructed for the case of positive stationary density given by Eqs. (1.15), (1.16) below. Let us note that the explicit necessary and sufficient conditions for such a positivity are known (see Proposition 1.2). Since for large class of external forces the stationary densities can contain points of vacua while being uniquely determined, we believe that Lyapunov analysis is important also for this case. To our knowledge, the only result in this direction and generality is in [13], where an analogous problem with a free boundary has been tackled.

Here, we are working with Dirichlet boundary condition (naturally no boundary condition for the density can be prescribed). A global lower bound for the density in terms of the stationary density would simplify considerably the problem. Unfortunately we do not have it in our case. Thus we have to find an alternative argument. This argument is given by a careful use of a comparison quasistationary density approximating the original one. Two crucial a priori estimates play decisive role in the construction. An appropriate form of the energy equality and an estimate utilizing the monotonicity of the state function and the analysis of approximative relation between the quasistationary density and the original density \( \rho \).

Despite of the singularity of the problem, a large class of mass forces and state functions is admitted. First, we give a survey of already known results which play an important role in the following arguments. Then we present the construction of a special differential equality including the velocity, the density, stationary density and quasistationary density. The terms including quasistationary density are carefully analyzed with the aim to exclude it from the differential equality and modify it to a differential inequality including a suitable Lyapunov functional. Resolving the Lyapunov differential inequality we obtain a decay rate for the convergence of the evolutionary solution to the stationary one. The result presented here has been announced in already published note [6]. Let us remark that this is purely 1D-result and there is hardly hope to extend it to higher space dimension.
1. Basic known facts and the main result

We consider the following system of equations describing 1D-flow of a viscous compressible barotropic fluid

\[ \rho_t + (\rho u)_x = 0, \]  
\[ (\rho u)_t + (\rho u^2)_x - (\mu u_x - p(\rho))_x = \rho f, \quad (1.1) \]

in the domain \( Q_\infty = (0, \ell) \times (0, \infty) \) with the boundary and initial conditions

\[ u|_{x=0,\ell} = 0, \quad \rho|_{t=0} = \rho^0(x), \quad u|_{t=0} = u^0(x) \quad \text{in} \ (0, \ell). \quad (1.3) \]

Suppose that

\[ f(x, t) = f_\infty(x) + g(x, t) \quad \text{with} \quad f_\infty \in W^{1,\infty}(0, \ell) \quad \text{and} \quad g \in L^{\infty,2}(Q_\infty). \quad (1.4) \]

Here \( Q_T = (0, \ell) \times (0, T) \). Throughout the paper we use the anisotropic Lebesgue space \( L^{q,s}(Q) \) equipped with the norm \( \| w \|_{L^{q,s}(Q)} := \| \| w \|_{L^q(0,\ell)} \|_{L^s(0,\infty)} \). Let the initial functions satisfy

\[ \rho^0, u^0 \in H^1(0, \ell), \quad 0 < \rho^0 \leq \rho_0^0, \quad u^0|_{x=0,\ell} = 0. \quad (1.5) \]

Our main requirements on the state function \( p(\cdot) \) are as follows.

\[ p(\cdot) \text{ is continuous, increasing function on } [0, \infty), \quad p(0) = 0, \quad p(\infty) = \infty; \quad (1.6) \]
\[ p'(r) \in L^\infty_{\text{loc}}(0, \infty), \quad p'(r) > 0, \quad r > 0; \quad (1.7) \]
\[ p(r) \sim r^\gamma \quad \text{as} \ r \to 0^+ \quad \text{with a} \ \gamma > 0; \quad (1.8) \]
\[ r p'(r) \leq \text{const} \quad \text{as} \ r \to 0^+. \quad (1.9) \]

Finally, the viscosity coefficient \( \mu \) is assumed to be positive and constant.

We shall study the asymptotic behavior of the \textit{strong generalized solution} to problem (1.1)–(1.3) having the following properties: \( \rho \in C(Q_T), \rho_x, \rho_t \in L^2(Q_T), \rho > 0 \) and \( u \in H^1(Q_T) \cap L^2(0, T; H^1(0, \ell)), u_{xx} \in L^2(Q_T) \) for any \( T > 0 \). Define also

\[ P(r) := r \int_1^r \frac{p(s) - p(1)}{s^2} ds, \quad (1.10) \]
\[ \Pi(r, s) = \int_s^r \frac{p(\sigma) - p(s)}{\sigma^2} d\sigma, \quad r, s \geq 0, \]

and \( F := I f_\infty \). We use the notation \( Ih := \int_0^x h(y) \, dy \) for any function \( h \in L^1(0, \ell) \).

First of all, we remind the mass and energy conservation laws:

\[ \int_0^\ell \rho(x, t) \, dx = \int_0^\ell \rho^0(x) \, dx =: m, \quad (1.11) \]
\[ \frac{d}{dt} \left( \int_0^\ell \frac{1}{2} \rho u^2 + P(\rho) - \rho F \right) \, dx + \mu \int_0^\ell (u_x)^2 \, dx = \int_0^\ell \rho g u \, dx \quad (1.12) \]
or
\[
\frac{d}{dt} \int_0^\ell \frac{1}{2} \rho u^2 \, dx + \mu \int_0^\ell (u_x)^2 \, dx = \int_0^\ell (\rho f\nabla u + \rho g u - p(\rho) x u) \, dx.
\] (1.13)

Denote the initial total energy by
\[
E_0 := \int_0^\ell \left( \frac{\rho_0 u_0^2}{2} + P(\rho_0) - \rho_0 F \right) \, dx.
\] (1.14)

In the whole paper we will assume that the stationary problem which is given by
\[
\rho(x) = 0 \quad \text{on } (0, \ell),
\]
\[
\int_0^\ell \rho(x) \, dx = m, \quad \rho(x) \geq 0,
\] (1.15)

has a unique solution \(\rho_\infty \in L^\infty(0, \ell)\).

Our main result is contained in the following theorem.

**Theorem 1.1** (Main result). Let conditions (1.4)–(1.9) be satisfied and the stationary problem (1.15), (1.16) have a unique solution \(\rho_\infty \in L^\infty(0, \ell)\) such that
\[
\text{meas} \{ x; \rho_\infty(x) = 0 \} = 0.
\]

Then for any \(t_0 \geq 0\) there are positive constants \(K := K(t_0, \ell, m, \mu, E_0, \| f\nabla \|_{W^{1,\infty}(0, \ell)}\) and \(\alpha := \alpha(t_0, \ell, m, \mu, E_0, \| f\nabla \|_{W^{1,\infty}(0, \ell)}\) such that
\[
\int_0^\ell \left( \rho u^2 + \rho \Pi(\rho, \rho_\infty) + |\rho - \rho_\infty|^{\beta} + (p(\rho) - p(\bar{\rho}))^2 \right)(x, t) \, dx
\]
\[
\leq K \left\{ e^{-\alpha(t-t_0)} \left[ 1 + \int_{t_0}^t e^{\alpha s} \| g(s) \|_2^2 \, ds \right] + \int_{t}^{\infty} \| g(s) \|_2^2 \, ds \right\}^{1/2}, \quad t \geq t_0,
\] (1.17)

where \(\bar{\rho}\) is given by (1.26) below, and \(\beta \geq 2\) if \(\gamma < 2\) or \(\beta \geq \gamma\) if \(\gamma \geq 2\) is arbitrary but fixed.

Theorem 1.1 will be proved in Section 2 after the following preliminaries.

First, a well-known consequence of energy equation (1.12) is

**Proposition 1.1.** (See [10].) Suppose in addition to (1.6), (1.7) that the conditions
\[
0 < \rho^0 \leq N, \quad \| u^0 \|_{L^2(0, \ell)} \leq N, \quad \| f\nabla \|_{L^\infty(0, \ell)} \leq N, \quad \| g \|_{L^{2,1}_x(Q)} \leq N
\] (1.18)
\[
\| g \|_{L^{\infty}_x(Q)} \leq N
\] (1.19)

and \(\| P(\rho^0) \|_{L^1(0, \ell)} \leq N\) are satisfied. Then there exists a positive constant \(K(N)\) such that

\[
\| \sqrt{\rho} u \|_{L^{2,\infty}(Q)} + \| P(\rho) \|_{L^{1,\infty}(Q)} + \| u_x \|_{L^2(Q)} \leq K(N),
\] (1.20)
(ii) $\rho(x, t) \leq K(N)$ \hfill (1.21)

and

(iii) $\frac{1}{2} \int_0^{\ell} (\rho u^2)(x, t) \, dx \to 0 \quad \text{as } t \to \infty$. \hfill (1.22)

It was already mentioned that there is a necessary and sufficient condition for the solution $\rho_\infty \in C([0, \ell])$ of (1.15), (1.16) such that $p(\rho_\infty) \in L^\infty(0, \ell)$ to be positive (i.e., $\rho_\infty > 0$). Denoting

$$F_{\min} := \min_{[0, \ell]} F(x), \quad F_{\max} := \max_{[0, \ell]} F(x), \quad C_p := \int_0^{1} \frac{p(r)}{r^2} \, dr \leq \infty,$$

this condition reads:

**Proposition 1.2.** (See [10].) Let (1.6)–(1.7) be satisfied and $f_\infty \in L^\infty(0, \ell)$. Then the positive solution $\rho_\infty$ to the problem (1.15), (1.16) exists if and only if

$$C_p < \infty \quad \text{or} \quad F_{\max} - F_{\min} < \Psi(\infty),$$

$$\frac{1}{m} \int_0^{\ell} \frac{1}{\Psi^{-1}(F(x) - F_{\min})} \, dx < 1,$$

where $\Psi(r) := \frac{p(r)}{r} + \int_0^{r} \frac{p(s)}{s^2} \, ds$ for $r > 0$ and $\Psi(0) = 0$, with $\Psi^{-1}$ being the inverse of $\Psi$. Moreover, for $C_p < \infty$, the function $\Psi$ is continuous and increasing on $[0, \infty)$.

In addition, the positive solution is unique.

**Proposition 1.3.** (See [10].) Let conditions (1.4)–(1.7) be satisfied and $p(\cdot), \, f_\infty \in BV([0, \ell])$ and $m > 0$ be such that there is a unique solution of (1.15), (1.16). Then, for $t \to \infty$, we have

$$\|p(\rho(t)) - p(\bar{\rho}(t))\|_{L^q(0, \ell)} + \|\rho(t) - \rho_\infty\|_{L^q(0, \ell)} \to 0 \quad \forall q \in [1, \infty),$$

$$\|p(\bar{\rho}(t)) - p(\rho_\infty)\|_{C([0, \ell])} \to 0,$$

where $\bar{\rho} = \bar{\rho}(x, t)$ is such that

$$p(\bar{\rho}(x, t)) = \frac{1}{\ell} \int_0^{\ell} p(\rho(\xi, t)) \, d\xi + \frac{1}{\ell} \int_0^{\ell} \rho(\eta, t) f_\infty(\eta) \, d\eta \, d\xi - \int_x^{\ell} \rho(\xi, t) f_\infty(\xi) \, d\xi.$$

We will use $p(\bar{\rho})$ in the form

$$p(\bar{\rho}(x, t)) = \frac{1}{\ell} \int_0^{\ell} p(\rho(\xi, t)) \, d\xi + \frac{1}{\ell} \int_0^{\ell} I^*(\rho f_\infty)(\xi, t) \, d\xi - I^*(\rho f_\infty)(x, t).$$
where \( I^* h(x) := \int_x^\ell h(\eta) d\eta \). Notice that \( \rho \) satisfies
\[
p(\rho)_x = \rho f_\infty, \quad x \in (0, \ell), \quad t > 0, \quad \int_0^\ell p(\rho) \, dx = \int_0^\ell p(\rho) \, dx, \quad t > 0.
\]
Let us note that such “quasistationary density” \( \rho \) was for the first time used for stabilization in [4], where the case of 2 and 3 space variables has been treated.

Next proposition shows that there are fairly general explicit conditions for uniqueness of the solution to Eqs. (1.15), (1.16). We refer in this respect to [1] and the references therein.

**Proposition 1.4.** (See [2, 1].) Let in addition to (1.7) we have \( p \in C([0, \infty)) \cap C^1(0, \infty) \) and \( F = 1f \) be locally Lipschitz continuous on \((0, \ell)\).

If \( \int_0^1 \frac{dp(s)}{s} < \infty \), assume in addition, that the upper level sets \{\( x \in (0, \ell); \quad F(x) > k \}\} are connected in \((0, \ell)\) for any constant \( k \in \mathbb{R} \).

Then, given \( m > 0 \), there is at most one function \( \rho_\infty \in L^\infty_{\text{loc}}(0, \ell) \) satisfying (1.15), (1.16) in the sense of distributions.

Moreover, if such a function exists, it is given by the formula
\[
\rho_\infty = \Psi^{-1}(\left( F(x) - k\ell \right)^+)\]
for a certain constant \( k\ell \). (Here \([z]^+ := \max\{z, 0\}\).)

We will also need the following elementary lemma.

**Lemma 1.1.** Let \( r_0 > 0 \) and \( s_0 > 0 \) be arbitrary fixed numbers and assume \( p(r) \sim r^\gamma \) as \( r \to 0^+ \) with a constant \( \gamma > 0 \). Let \( \beta \geq 2 \) if \( \gamma < 2 \) and \( \beta \geq \gamma \) if \( \gamma \geq 2 \). Then there is a constant \( k = k(\beta) \) such that
\[
k(\beta) |r - s|^\beta \leq r \Pi(r, s) \quad \text{for all } r \in (0, r_0], \quad s \in [0, s_0].
\]

**Proof.** First, let \( s > 0 \) be fixed. Then by the l’Hospital rule
\[
\lim_{r \to s} \frac{|r - s|^\beta}{(r \Pi(r, s))} = \frac{\beta s}{p'(s)} \lim_{r \to s} |s - r|^{\beta - 2} = \begin{cases} \infty & \text{if } \beta < 2, \\ \frac{\beta s}{p'(s)} & \text{if } \beta = 2, \\ 0 & \text{if } \beta > 2. \end{cases}
\]
Let now \( s = 0 \) and use the assumption \( p(r) \sim r^\gamma \) near zero. Then
\[
\lim_{r \to 0^+} \frac{r^\beta}{r \Pi(r, 0)} = \lim_{r \to 0^+} (\beta - 1) \frac{r^\beta}{p(r)} = \begin{cases} \infty & \text{if } \beta < \gamma, \\ \beta - 1 & \text{if } \beta = \gamma, \\ 0 & \text{if } \beta > \gamma. \end{cases}
\]
The result immediately follows. \( \square \)

**2. Construction of a Lyapunov functional**

Let us subtract the differential equation in (1.27) from Eq. (1.2). We obtain the relation
\[
(\rho u)_t + (\rho u^2)_x - \mu u_{xx} + p(\rho)_x - p(\bar{\rho})_x = \rho g.
\]
(2.1)
Multiply (2.1) by \(-\varepsilon I(p(\rho) - p(\bar{\rho}))\) and integrate over \((0, \ell)\):

\[
-\varepsilon \frac{d}{dt} \int_{0}^{\ell} \rho u I(p(\rho) - p(\bar{\rho})) \, dx + \varepsilon \int_{0}^{\ell} \rho u I(p(\rho) - p(\bar{\rho})) \, dx \\
+ \varepsilon \int_{0}^{\ell} (\rho u^2 - \mu u_x)(p(\rho) - p(\bar{\rho})) \, dx + \varepsilon \int_{0}^{\ell} (p(\rho) - p(\bar{\rho}))^2 \, dx
\]

\[
= \varepsilon \int_{0}^{\ell} \rho g I(p(\bar{\rho}) - p(\rho)) \, dx.
\]

Adding (1.13) multiplied by a positive parameter \(\eta > 0\) and (2.2) we find

\[
\frac{d}{dt} \int_{0}^{\ell} \left( \eta \rho u^2/2 - \varepsilon \rho u I(p(\rho) - p(\bar{\rho})) \right) \, dx + \eta \int_{0}^{\ell} (p(\bar{\rho}) - p(\rho)) u_x \, dx \\
+ \varepsilon \int_{0}^{\ell} \rho u I(p(\rho) - p(\bar{\rho})) \, dx + \varepsilon \int_{0}^{\ell} (\rho u^2 - \eta \mu u_x)(p(\rho) - p(\bar{\rho})) \, dx \\
+ \varepsilon \int_{0}^{\ell} (p(\rho) - p(\bar{\rho}))^2 \, dx + \eta \mu \int_{0}^{\ell} u_x^2 \, dx
\]

\[
= \eta \int_{0}^{\ell} \rho u g \, dx + \varepsilon \int_{0}^{\ell} \rho g I(p(\bar{\rho}) - p(\rho)) \, dx.
\]

Next, we also have

\[
\frac{1}{2} \frac{d}{dt} \int_{0}^{\ell} (p(\rho) - p(\bar{\rho}))^2 = \int_{0}^{\ell} (p(\rho) - p(\bar{\rho}))(p(\rho) - p(\bar{\rho})) \, dx,
\]

where (by the equation of continuity and (1.26))

\[
p(\rho)_t - p(\bar{\rho})_t = -((p(\rho)u)_x + (p(\rho) - \rho p'(\rho))u_x
\]

\[
+ \frac{1}{\ell} \int_{0}^{\ell} (\rho p'(\rho) - p(\rho))u_x \, dx + \int_{0}^{\ell} \frac{1}{\ell} I^s((\rho u)_x f_\infty) \, dx - I^s((\rho u)_x f_\infty).
\]

Further we have (notice that \(\int_{0}^{\ell} (\int_{0}^{\ell} h d\xi)(p(\rho) - p(\bar{\rho})) \, dx = 0\) since \(\int_{0}^{\ell} p(\rho) \, dx = \int_{0}^{\ell} p(\bar{\rho}) \, dx\))

\[
- \int_{0}^{\ell} (p(\rho)u)_x (p(\rho) - p(\bar{\rho})) \, dx = -\frac{1}{2} \int_{0}^{\ell} p(\rho)^2 u_x \, dx - \int_{0}^{\ell} p(\rho) u p(\bar{\rho})_x \, dx
\]
\[
\begin{align*}
&= \frac{1}{2} \int_0^\ell (p(\bar{\rho})^2 - p(\rho)^2) u_x \, dx - \frac{1}{2} \int_0^\ell p(\bar{\rho})^2 u_x \, dx \\
&\quad + \int_0^\ell (p(\bar{\rho}) - p(\rho)) u p(\bar{\rho})_x \, dx - \int_0^\ell p(\bar{\rho}) p(\bar{\rho})_x u \, dx \\
&= \int_0^\ell (p(\bar{\rho}) - p(\rho)) u \rho f_\infty \, dx + \frac{1}{2} \int_0^\ell (p(\bar{\rho})^2 - p(\rho)^2) u_x \, dx,
\end{align*}
\]

and

\[I^*(\rho u_x f_\infty) = -\rho u f_\infty - \int_x^\ell \rho u f'_\infty \, dx. \tag{2.7}\]

Summarizing (2.4)–(2.7) we get

\[
\frac{1}{2} \frac{d}{dt} \int_0^\ell (p(\rho) - p(\bar{\rho}))^2 \, dx
\]

\[
= \frac{1}{2} \int_0^\ell (p(\bar{\rho})^2 - p(\rho)^2) u_x \, dx + \int_0^\ell (p(\rho) - p(\bar{\rho}))(p(\rho) - \rho p'(\rho)) u_x \, dx \tag{2.8}
\]

\[
+ \int_0^\ell (p(\rho) - p(\bar{\rho})) I^*(\rho u f'_\infty) \, dx.
\]

Multiply equality (2.8) by a parameter \(\delta > 0\) and add to (2.3):

\[
\frac{d}{dt} \int_0^\ell \left( \frac{\eta \rho u^2}{2} + \frac{\delta}{2} (p(\bar{\rho}) - p(\rho))^2 + \varepsilon \rho u I(p(\bar{\rho}) - p(\rho)) \right) \, dx + \eta \int_0^\ell (p(\bar{\rho}) - p(\rho)) u_x \, dx
\]

\[
+ \varepsilon \int_0^\ell \rho u I(p(\rho)_t - p(\bar{\rho})_t) \, dx + \varepsilon \int_0^\ell (\rho u^2 - \eta \mu u_x)(p(\rho) - p(\bar{\rho})) \, dx
\]

\[
+ \varepsilon \int_0^\ell (p(\rho) - p(\bar{\rho}))^2 \, dx + \eta \mu \int_0^\ell u_x^2 \, dx \tag{2.9}
\]

\[
+ \delta \left[ \frac{1}{2} \int_0^\ell (p(\bar{\rho})^2 - p(\rho)^2) u_x \, dx + \int_0^\ell (p(\rho) - p(\bar{\rho}))(p(\rho) - \rho p'(\rho)) u_x \, dx
\]

\[
+ \int_0^\ell (p(\rho) - p(\bar{\rho})) I^*(\rho u f'_\infty) \, dx \right].
\]
\[ = \eta \int_0^\ell \rho u g \, dx + \varepsilon \int_0^\ell \rho \varphi \left( p(\bar{\rho}) - p(\rho) \right) \, dx. \]

Our intention now is to compare the integral under \( \frac{d}{dt} \) with the remaining terms in equality (2.3).

**Lemma 2.2.** The following inequality holds true:

\[ V_{\varepsilon, \delta}(t) := \int_0^\ell \left( \frac{\eta \rho u^2}{2} + \frac{\delta}{2} \left( p(\rho) - p(\bar{\rho}) \right)^2 + \varepsilon \rho \varphi \left( p(\rho) - p(\bar{\rho}) \right) \right) \, dx \]

\[ \geq \left( \frac{\eta}{2} - \varepsilon m \beta \right) \int_0^\ell \rho u^2 \, dx + \left( \frac{\delta}{2} - \varepsilon \ell m \beta^{-1} \right) \int_0^\ell \left( p(\rho) - p(\bar{\rho}) \right)^2 \, dx. \] (2.10)

**Proof.** Indeed, we have

\[ \int_0^\ell \rho \varphi \left( p(\rho) - p(\bar{\rho}) \right) \, dx \leq \| \rho \|_1 \left( \int_0^\ell \rho u^2 \, dx \right)^{1/2} \left\| p(\bar{\rho}) - p(\rho) \right\|_1 \]

\[ \leq m \left( \beta \int_0^\ell \rho u^2 \, dx + \frac{\ell}{\beta} \left\| p(\bar{\rho}) - p(\rho) \right\|_2^2 \right) \] (2.11)

with any positive constant \( \beta \). Now estimate (2.10) immediately follows. □

**Lemma 2.3.** The following inequality holds true:

\[ \varepsilon \left| \int_0^\ell \rho \varphi \left( p(\rho) - p(\bar{\rho}) \right) \, dx \right| \leq \varepsilon c(\ell, m, \mu, E_0, \| f_\infty \|_{W^{1,\infty}_0(0, \ell)}) \| u_x \|_2^2. \] (2.12)

**Proof.** First, by the renormalized equation of continuity,

\[ \int_0^x p(\rho) \, d\xi = -p(\rho)u + \int_0^x (p(\rho) - \rho p(\rho'))u_x \, d\xi. \] (2.13)

Secondly, by (1.26) we have

\[ I p(\bar{\rho}) = \frac{x}{\ell} \int_0^\ell p(\rho) \, d\xi + \frac{x}{\ell} \int_0^\ell \rho \varphi f_\infty \, d\eta \, d\xi - \int_0^x \int_0^\xi \rho \varphi f_\infty \, d\eta \, d\xi. \] (2.14)

Then, again by the equation of continuity and by the help of several integrations by parts we finally obtain
\[ IP(\rho)_t = \frac{x}{\ell} \int_0^\ell (p(\rho) - \rho p' (\rho)) u_x \, d\xi + \frac{x}{\ell} \int_0^\ell \rho u (\xi f_\infty) \xi \, d\xi \]

\[ - x \rho u f_\infty + \int_0^x (\rho u) (\xi f_\infty) \xi \, d\xi. \]  

\[ (2.15) \]

Thus

\[ \int_0^\ell \rho u (p(\rho)_t - p(\bar{\rho})_t) \, dx \]

\[ = - \int_0^\ell \rho u^2 p(\rho) \, dx + \int_0^\ell \rho u \int_0^x (p(\rho) - \rho p' (\rho)) u_x \, d\xi \, dx \]

\[ - \int_0^\ell \int_0^x \rho u (p(\rho) - \rho p' (\rho)) u_x \, d\xi \, dx - \int_0^\ell \rho u \int_0^x \rho u (\xi f_\infty) \xi \, d\xi \, dx \]

\[ - \int_0^\ell x \rho u \int_0^\ell \rho u f_\infty \, d\xi \, dx + \int_0^\ell \rho u \int_0^\xi \rho u (\xi f_\infty) \xi \, d\xi \, dx. \]  

\[ (2.16) \]

Further we have

\[ \left| \int_0^\ell \rho p(\rho) u^2 \, dx \right| \leq \sup_{x, t} \rho p(\rho) \| u \|_2^2 \leq c \| u_x \|_2^2, \]

since by Proposition 1.1, \( \rho \) is globally bounded. Notice that \( p(\bar{\rho}) \) is globally bounded:

\[ |p(\bar{\rho})(x, t)| \leq \frac{1}{\ell} \int_0^\ell p(\rho) \, dx + 2m \| f_\infty \|_\infty \leq c(\ell, m, E_0, \| f_\infty \|_{W^1_\infty (0, \ell)}), \quad \forall x, t. \]  

\[ (2.17) \]

Similarly we have

\[ \left| \int_0^\ell \int_0^x (p(\rho) - \rho p' (\rho)) u_x \, d\xi \, dx \right| \leq m \sqrt{\ell} \| u_x \|_2 \left| \int_0^\ell (p(\rho) - \rho p' (\rho)) u_x \, dx \right| \leq c \| u_x \|_2^2. \]  

\[ (2.18) \]

where, for brevity, we do not mark the arguments of \( c(\cdot) \). Similarly we have

\[ \left| \int_0^\ell \int_0^\ell \rho u (p(\rho) - \rho p' (\rho)) u_x \, d\xi \, dx \right| \leq c \| u_x \|_2^2, \]

\[ \left| \int_0^\ell \int_0^\ell \rho u (\xi f_\infty) \xi \, d\xi \, dx \right| \leq \sqrt{\ell} m^2 \| u_x \|_2 \| (\xi f_\infty) \xi \|_\infty \leq c \| u_x \|_2^2, \]
\[
\int_0^{\ell} x \rho u f_{\infty} \, d\bar{\xi} \, dx \leq \ell^2 m^2 \| f_{\infty} \|_{\infty} \| u_x \|_2^2 \leq c \| u_x \|_2^2.
\] (2.19)

\[
\int_0^{\ell} \rho u (\xi f_{\infty}) \, d\bar{\xi} \, dx \leq m^2 \ell \| (\xi f_{\infty}) \|_{\infty} \| u_x \|_2^2 \leq c \| u_x \|_2^2.
\] (2.20)

The inequality (2.12) immediately follows. □

According to Lemma 2.3 the term on the right-hand side of (2.12) can be made subordinate to the term \( \mu \int_0^{\ell} u_x^2 \, dx \) when taking \( \varepsilon \) small enough, in particular if

\[
\varepsilon c(\ell, m, \mu, E_0, \| f_{\infty} \|_{W^{1,\infty}_{\infty}(0,\ell)}) < \mu.
\] (2.21)

Proceeding in (2.9) to the estimate of the term \( \int_0^{\ell} (\rho u^2 - \mu u_x)(p(\rho) - p(\bar{\rho})) \, dx \) we first observe that

\[
\int_0^{\ell} \rho u^2 (p(\rho) - p(\bar{\rho})) \, dx \leq \varepsilon(t) \| u_x \|_2^2,
\] (2.22)

where \( \varepsilon(t) : = \ell \int_0^{\ell} (p(\rho) - p(\bar{\rho}))(\rho - \bar{\rho}) \, dx + \sup_{x,t} \rho \| p(\rho) - p(\bar{\rho}) \|_1 \to 0 \) as \( t \to \infty \) by Proposition 1.3. Next we observe in (2.9) that the term \( \eta \int_0^{\ell} (p(\bar{\rho}) - p(\rho))u_x \, dx \) can be compounded with the term \( -\varepsilon \eta \mu \int_0^{\ell} (p(\rho) - p(\bar{\rho}))u_x \, dx \) to obtain \( (\eta + \varepsilon \eta \mu) \int_0^{\ell} (p(\rho) - p(\bar{\rho}))u_x \, dx \) which we estimate as

\[
(\eta + \varepsilon \eta \mu) \int_0^{\ell} (p(\rho) - p(\bar{\rho}))u_x \, dx \leq (\eta + \varepsilon \eta \mu) \left( \lambda_1 \| u_x \|_2^2 + \lambda_1^{-1} \| p(\rho) - p(\bar{\rho}) \|_2^2 \right).
\] (2.23)

Quite analogously is estimated the last inconvenient term on the left-hand side of (2.9):

\[
\delta \left| \int_0^{\ell} (p(\rho) - p(\bar{\rho}))u_x \, dx + \int_0^{\ell} (p(\rho) - p(\bar{\rho}))(p(\rho) - \rho p'(\rho))u_x \, dx \right|
\]

\[
\leq \delta c(\ell, m, \mu, E_0, \| f_{\infty} \|_{W^{1,\infty}_{\infty}(0,\ell)}) \left( \lambda_2 \| u_x \|_2^2 + \lambda_2^{-1} \| p(\rho) - p(\bar{\rho}) \|_2^2 \right).
\] (2.24)

Finally, \( \int_0^{\ell} \rho u g \, dx \leq \sqrt{\ell} \left( \sup_{x,t} \rho \right) \| u_x \|_2 \| g \|_2 \),

\[
\leq \lambda_3 \| u_x \|_2^2 + c(\ell, m, \mu, E_0, \| f_{\infty} \|_{W^{1,\infty}_{\infty}(0,\ell)}) \lambda_3^{-1} \| g \|_2^2.
\] (2.25)
and
\[
\varepsilon \left| \int_0^\ell \rho g I (p(\rho) - p(\overline{\rho})) \, dx \right| \leq \varepsilon \sqrt{\ell} \left( \sup_{x,t} \rho \right) \| g \|_2 \| p(\rho) - p(\overline{\rho}) \|_2 \]
\[
\leq \varepsilon (\lambda_4 \| p(\rho) - p(\overline{\rho}) \|^2_2 + c(\ell, m, \mu, E_0, \| f_\infty \|_{W^{1,\infty}(0, \ell)}) \lambda_4^{-1} \| g \|^2_2).
\]

Using estimates (2.12), (2.21), (2.22), (2.23), (2.24) and (2.25) in (2.9) we obtain
\[
\frac{d}{dt} \int_0^\ell \left( \frac{\eta \mu u^2}{2} + \frac{\delta}{2} (p(\rho) - p(\overline{\rho}))^2 + \varepsilon \rho I (p(\rho) - p(\overline{\rho})) \right) \, dx
\]
\[
+ \left( \eta \mu - \varepsilon \xi - \xi(t) - (\eta + \varepsilon \eta \mu) \lambda_1 - c \delta \lambda_2 - \eta \lambda_3 \right) \| u \|_2^2
\]
\[
+ \left( \varepsilon - (\eta + \varepsilon \eta \mu) \lambda_1^{-1} - c \delta \lambda_2^{-1} - \varepsilon \lambda_4 \right) \| p(\rho) - p(\overline{\rho}) \|^2_2
\]
\[
\leq c(\lambda_3^{-1} \eta + \varepsilon \lambda_4^{-1}) \| g \|^2_2.
\]

To get a decay of the functional \( V_{\varepsilon, \delta}(t) \) defined by (2.10) we need (observe that \( \xi(t) \to 0 \) as \( t \to \infty \))
\[
\eta \mu > c \varepsilon + \lambda_1 (\eta + \varepsilon \eta \mu) + c \delta \lambda_2 + \eta \lambda_3,
\]
\[
\varepsilon > \lambda_1^{-1} (\eta + \varepsilon \eta \mu) + c \lambda_2^{-1} \delta + \varepsilon \lambda_4.
\]  

(2.27)

Since the parameters \( \lambda_3, \lambda_4 \) can be chosen independently, so that, for example, sufficiently small, it suffices, instead of (2.27), to consider conditions
\[
\eta \mu > c \varepsilon + \lambda_1 (\eta + \varepsilon \eta \mu) + c \delta \lambda_2,
\]
\[
\varepsilon > \lambda_1^{-1} (\eta + \varepsilon \eta \mu) + c \lambda_2^{-1} \delta.
\]  

(2.28)

From (2.10) we get additional conditions for positivity of \( V_{\varepsilon, \delta}(t) \), namely
\[
\varepsilon \beta \sqrt{\varepsilon} m^2 < \frac{n}{2}, \quad \varepsilon \beta^{-1} \sqrt{\varepsilon} m^2 < \frac{\delta}{2}.
\]  

(2.29)

The choice of \( \beta \) which obeys (2.29) is possible if and only if
\[
4 \ell m \varepsilon^2 < \eta \delta.
\]  

(2.30)

Next, the choice of \( \lambda_1 \) satisfying (2.28) is possible if
\[
\varepsilon \lambda_2 > c \delta \quad \text{and} \quad \frac{(\eta + \varepsilon \eta \mu)^2}{\varepsilon - c \delta \lambda_2^{-1}} < \eta \mu - c \varepsilon - c \delta \lambda_2.
\]

Now choose \( \lambda_2 = 2 \varepsilon \delta \varepsilon^{-1} \). Then we have to require
\[
2(\eta + \varepsilon \eta \mu)^2 < \varepsilon (\eta \mu - c \varepsilon - 2 c^2 \delta^2 \varepsilon^{-1}).
\]  

(2.31)

Choose also \( \delta = \varepsilon^{3/4} \). By (2.30) we have the constraint \( 4 \ell m \varepsilon^{5/4} < \eta \).

Then we solve \( 2(\eta + \varepsilon \eta \mu)^2 < \varepsilon (\eta \mu - c \varepsilon - 2 c^2 \delta \varepsilon^{-1}) \). Choose \( \varepsilon \) so small that \( \eta \mu - c \varepsilon - 2 c^2 \delta \varepsilon^{-1} > \frac{n \mu}{2} \). Then it suffices to require
\[
4 \eta (1 + \varepsilon \mu)^2 < \varepsilon \mu.
\]  

(2.32)
Since \( \varepsilon \) may be chosen of order \( \eta^{4/5} \), for sufficiently small \( \eta \) the last inequality can be satisfied. Then, choosing \( \varepsilon \) so small that (2.30) and (2.32) hold, and other parameters as above, we can achieve that in (2.26) the coefficients at \( \|u_x\|_2^2 \) and \( \|p(\rho) - p(\bar{\rho})\|_2^2 \) are positive. Then (2.26) implies

\[
\frac{d}{dt} \int_0^\ell \left( \frac{\eta u^2}{2} + \frac{\delta}{2} (p(\rho) - p(\bar{\rho}))^2 + \varepsilon \rho u I(p(\rho) - p(\bar{\rho})) \right) (x, t) \, dx \\
+ a\left(\|u_x\|_2^2 + \|p(\rho) - p(\bar{\rho})\|_2^2\right) \leq k\|g\|_2^2, \quad t \geq t_0,
\]

with some positive constants \( a, k \) and \( t_0 \).

Further, we have

\[
V_{\varepsilon, \delta}(t) \equiv \int_0^\ell \left( \frac{\eta u^2}{2} + \frac{\delta}{2} (p(\rho) - p(\bar{\rho}))^2 + \varepsilon \rho u I(p(\rho) - p(\bar{\rho})) \right) \, dx \\
\leq \frac{\eta m \ell}{2} \|u_x\|_2^2 + \frac{\delta}{2} \|p(\rho) - p(\bar{\rho})\|_2^2 + \varepsilon m \ell \|u_x\|_2 \|p(\rho) - p(\bar{\rho})\|_2 \\
\leq \frac{1}{2} (\eta m \ell + \delta + \varepsilon m \ell) \left( \|u_x\|_2^2 + \|p(\rho) - p(\bar{\rho})\|_2^2 \right) .
\]

Putting

\[
\alpha := \frac{2a}{\delta + m \ell (\eta + \varepsilon)}
\]

we get from (2.33)

\[
\frac{dV_{\varepsilon, \delta}}{dt}(t) + \alpha V_{\varepsilon, \delta}(t) \leq k\|g\|_2^2, \quad t \geq t_0.
\]

By integration of (2.36) over the interval \((t_0, t)\) we arrive at the inequality

\[
V_{\varepsilon, \delta}(t) \leq ke^{-\alpha(t-t_0)} \left( V_{\varepsilon, \delta}(t_0) + \int_{t_0}^t e^{\alpha s} \|g(s)\|_2^2 \, ds \right), \quad t \geq t_0,
\]

with some constant \( k \geq 1 \). Note, that \( \alpha, k, \varepsilon \) and \( \delta \) are locally bounded functions of \( \ell, m, \mu, E_0 \) and \( \|f_{\infty}\|_{W^1_{\infty}(0, \ell)} \) and \( t_0 \geq 0 \), previously sufficiently large, can be chosen arbitrary, since, due to the regularity of the solution, (2.37) holds on any finite interval \([0, T_0]\) (the constant \( k \) may eventually change).

Now we need the following technical lemma.

**Lemma 2.4.** Let the set \( \{ x \in (0, \ell); \rho_{\infty}(x) = 0 \} \) be of measure zero and

\[
\limsup_{r \to 0^+} \int_0^r \frac{dp(s)}{s} \, ds < \infty.
\]
Then
\[
\frac{d}{dt} \int_0^\ell \rho \Pi(\rho, \rho_\infty) \, dx = \int_0^\ell \left( p(\bar{\rho}) - p(\rho) \right) u_x \, dx.
\] (2.38)

**Proof.** Let \( \rho_n = \rho_\infty + \frac{1}{n} \). Then by (1.10) and using (1.1)–(1.3) we have
\[
\frac{d}{dt} \int_0^\ell \rho \Pi(\rho, \rho_n) \, dx = \int_0^\ell \left( \Pi(\rho, \rho_n) + \rho \frac{p(\rho) - p(\rho_n)}{\rho^2} \right) \rho_t \, dx
\]
\[
= \int_0^\ell \rho u \left( \Pi(\rho, \rho_n) + \frac{p(\rho) - p(\rho_n)}{\rho} \right) \, dx.
\] (2.39)
Further,
\[
\rho \left( \Pi(\rho, \rho_n) + \frac{p(\rho) - p(\rho_n)}{\rho} \right)_x
\]
\[
= \rho \left( \frac{p(\rho) - p(\rho_n)}{\rho^2} \rho_x - \int_{\rho_n}^\rho \frac{p(\rho_n)_x}{\sigma^2} \, d\sigma + \frac{p(\rho)_x - p(\rho_n)_x}{\rho} - \frac{p(\rho) - p(\rho_n)}{\rho^2} \rho_x \right)
\]
\[
= \rho p(\rho_n)_x \left( \frac{1}{\rho} - \frac{1}{\rho_n} \right) + p(\rho)_x - p(\rho_n)_x = p(\rho)_x - \frac{\rho}{\rho_n} p(\rho_n)_x = p(\rho)_x - \rho \pi(\rho_n)_x,
\] where \( \pi(r) = \int_0^r \frac{\rho'(s)}{s} \, ds \). Since \( \{ \rho_\infty > 0 \} \) is an open set, we can write it in the form
\( \bigcup_{j \in S} (a_j, b_j) \), where \( S \subset N \) is countable. Notice that \( \rho_\infty(a_j), \rho_\infty(b_k) = 0 \) as soon as \( a_j, b_k \in (0, \ell) \). Let \( \varphi \in C^\infty(0, \ell), \varphi(0) = \varphi(\ell) = 0 \). Then, sending \( n \to +\infty \), we get
\[
\int_0^\ell \pi(\rho_n)_x \rho \varphi \, dx = \int_0^\ell \pi(\rho_n)(\rho \varphi)_x \, dx \to \int_0^\ell \pi(\rho_\infty)(\rho \varphi)_x \, dx
\]
\[
= - \sum_{j \in S_{a_j}} b_j \int_0^{b_j} \pi(\rho_\infty)(\rho \varphi)_x \, dx = \sum_{j \in S_{b_j}} b_j \int_{a_j}^{b_j} \pi(\rho_\infty)_x \rho \varphi \, dx
\]
\[
= \sum_{j \in S_{a_j}} b_j \int_{a_j}^{b_j} \frac{p(\rho_\infty)_x}{\rho_\infty} \rho \varphi \, dx = \int_{\rho_\infty > 0} \rho f \varphi \, dx = \int_0^\ell \rho f \varphi \, dx.
\]
The result immediately follows. \( \Box \)

By (2.38) and (2.37) we have
\[
\left| \frac{d}{dt} \int_0^\ell \rho \Pi(\rho, \rho_\infty) \, dx \right| \leq \| u_x \|_2 \| p(\bar{\rho}) - p(\rho) \|_2
\] (2.41)
\begin{align*}
&\leq \frac{2}{\delta} \|u_x\|_2 e^{-\frac{\alpha}{2}(t-t_0)} \left( V_{\epsilon,\delta}(t_0) + k \int_{t_0}^t e^{\alpha \sigma} \|g(\sigma)\|_2^2 d\sigma \right)^{1/2}, \quad t \geq t_0.
\end{align*}

Since by (1.24),
\begin{equation}
\lim_{t \to \infty} \int_0^\ell \rho(x,t) \Pi(\rho(x,t), \rho_\infty(x)) \, dx = \int_0^\ell \rho_\infty(x) \Pi(\rho_\infty(x), \rho_\infty(x)) \, dx = 0, \quad (2.42)
\end{equation}
we find
\begin{align*}
\int_0^\ell \rho(t) \Pi(\rho(t), \rho_\infty) \, dx
&- \int_0^\ell \rho(s) \Pi(\rho(s), \rho_\infty) \, dx \\
&= - \int_t^s \frac{d}{d\tau} \int_0^\ell \rho(\tau) \Pi(\rho(\tau), \rho_\infty) \, dx \, d\tau
\leq \int_t^s \frac{d}{d\tau} \int_0^\ell \rho(\tau) \Pi(\rho(\tau), \rho_\infty) \, dx \, d\tau
\leq \frac{2\sqrt{k}}{\delta} \left( \int_t^s \|u_x(\tau)\|_2^2 \, d\tau \right)^{1/2} \left[ \int_t^s e^{-\alpha(\tau-t_0)} \left( V_{\epsilon,\delta}(t_0) + \int_{t_0}^\tau e^{\alpha \sigma} \|g(\sigma)\|_2^2 d\sigma \right) \, d\tau \right]^{1/2}
\end{align*}

\begin{align*}
&= \frac{2\sqrt{k}}{\delta} \left( \int_t^s \|u_x(\tau)\|_2^2 \, d\tau \right)^{1/2} \left\{ - \frac{1}{\alpha} \left[ e^{-\alpha(\tau-t_0)} \int_{t_0}^\tau e^{\alpha \sigma} \|g(\sigma)\|_2^2 d\sigma \right]_{\tau=t_0}^t + \frac{V_{\epsilon,\delta}(t_0)}{\alpha} \left( e^{-\alpha(\tau-t_0)} - e^{-\alpha(s-t_0)} \right) + \frac{1}{\alpha} \int_t^s e^{-\alpha(\tau-t_0)} e^{\alpha \tau} \|g(\tau)\|_2^2 d\tau \right\}^{1/2}
\end{align*}

\begin{align*}
&\leq \frac{2\sqrt{k}}{\delta} \left( \int_t^\infty \|u_x(\tau)\|_2^2 \, d\tau \right)^{1/2} \left\{ \frac{1}{\alpha} e^{-\alpha(t-t_0)} \int_{t_0}^t e^{\alpha \sigma} \|g(\sigma)\|_2^2 d\sigma + \frac{V_{\epsilon,\delta}(t_0)}{\alpha} e^{-\alpha(t-t_0)} + \frac{1}{\alpha} \int_t^\infty e^{\alpha \sigma} \|g(\sigma)\|_2^2 d\sigma \right\}^{1/2}.
\end{align*}

Sending \( s \rightarrow \infty \) and using (2.42) we obtain
\begin{align*}
\int_0^\ell \rho(t) \Pi(\rho(t), \rho_\infty) \, dx
&\leq \kappa \left( \int_t^\infty \|u_x(s)\|_2^2 ds \right)^{1/2} \left[ e^{-\alpha(t-t_0)} \left( 1 + \int_{t_0}^t e^{\alpha \sigma} \|g(\sigma)\|_2^2 d\sigma \right) + \int_t^\infty \|g(\sigma)\|_2^2 d\sigma \right]^{1/2},
\end{align*}
with a constant \( \kappa = \kappa(t_0, \ell, m, \mu, E_0, \|f_\infty\|_{W^{1,\infty}(0,\ell)}) \). By (2.10) and (2.37) we also have
\[
\int_0^\ell \left( \rho u^2 + \left( p(\rho) - p(\bar{\rho}) \right)^2 \right) (x, t) \, dx \\
\leq a_0 V_{\varepsilon, \delta}(t) \\
\leq a_0 e^{-\alpha(t-t_0)} \left( V_{\varepsilon, \delta}(t_0) + k \int_{t_0}^t e^{\alpha s} \| g(s) \|_2^2 \, ds \right) \\
\leq a_1 e^{-\alpha(t-t_0)} \left[ \int_0^\ell \left( \rho u^2 + \left( p(\rho) - p(\bar{\rho}) \right)^2 \right) (x, t_0) \, dx + \int_{t_0}^t e^{\alpha s} \| g(s) \|_2^2 \, ds \right]
\]
with constants \( a_j = a_j (\ell, m, \mu, E_0, \| f_\infty \|_{W^{1, \infty}(0, \ell)}, \) \( j = 0, 1 \). This together with (2.44) yields

\[
\int_0^\ell \left( \rho u^2 + \rho \Pi(\rho, \rho_\infty) + \left( p(\rho) - p(\bar{\rho}) \right)^2 \right) (x, t) \, dx \\
\leq K \left\{ e^{-\alpha(t-t_0)} \left[ 1 + \int_{t_0}^t e^{\alpha s} \| g(s) \|_2^2 \, ds \right] + \int_t^\infty \| g(s) \|_2^2 \, ds \right\}^{1/2}
\]
for \( t_0 \) and \( K \) such that

\[
\kappa V_{\varepsilon, \delta}(t)^{1/2} + a_1 V_{\varepsilon, \delta}(t) \leq KV_{\varepsilon, \delta}(t)^{1/2} \quad \text{for} \; t \geq t_0,
\]
where \( K = K(t_0, \ell, m, \mu, E_0, \| f_\infty \|_{W^{1, \infty}(0, \ell)}). \) The estimate (2.46) in combination with Lemma 1.1 yields the desired estimate (1.17).

**Remark 1.** Let us note that since \( \lim_{t \to \infty} \int_0^t e^{-\alpha(t-s)} G(s) \, ds = 0 \) for all \( G \in L^q(R^+) \) with \( \alpha > 0 \) and \( 1 \leq q < \infty \), the right-hand side of (2.46) tends to zero as \( t \to \infty \). If, moreover, \( \| e^{bt} g(x, t) \|_{L^2(Q)} \leq N \) with some \( b \in (0, \alpha] \) (for example, if \( g \equiv 0 \)), then the decay rate is exponential, i.e.,

\[
\int_0^\ell \left( \rho u^2 + \rho \Pi(\rho, \rho_\infty) + |\rho - \rho_\infty|^\beta + \left\| p(\rho) - p(\bar{\rho}) \right\|_2^2 \right) \, dx \leq k(N)e^{-bt}, \quad t \geq 0.
\]

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