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Characterization of eccentric digraphs

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Abstract

The *eccentric digraph* $\mathrm{ED}(G)$ of a digraph G represents the binary relation, defined on the vertex set of G, of being 'eccentric'; that is, there is an arc from u to v in $\mathrm{ED}(G)$ if and only if v is at maximum distance from u in G. A digraph G is said to be *eccentric* if there exists a digraph H such that $G = \mathrm{ED}(H)$. This paper is devoted to the study of the following two questions: what digraphs are eccentric and when the relation of being eccentric is symmetric.

We present a characterization of eccentric digraphs, which in the undirected case says that a graph G is eccentric iff its complement graph $\overline{G}$ is either self-centered of radius two or it is the union of complete graphs. As a consequence, we obtain that all trees except those with diameter 3 are eccentric digraphs. We also determine when $\mathrm{ED}(G)$ is symmetric in the cases when G is a graph or a digraph that is not strongly connected.

Keywords: Eccentricity; Eccentric vertex; Distance; Eccentric digraph

1. Introduction and definitions

Interconnection networks are pervasive in today's society, including networks for the distribution of goods, communication networks, social networks and the Internet, to name just a few. The topology of an interconnection network is usually modelled by a graph, either directed or undirected, depending on the particular application. In all cases, there are some common fundamental characteristics of networks such as the number of nodes, number of connections at each node, total number of connections, clustering of nodes, etc. Many of the most important basic properties, underpinning the functionalities of a network, are related to the distance between the nodes in a network. Such properties includes the eccentricities of the nodes, the radius of the network and the diameter of the network (see [6]).

As a second level of abstraction, binary relations induced by distances in a graph can be also represented by a graph. Theoretical research in this direction includes the study of antipodal graphs (see [2,13]), antipodal digraphs (see [14]), eccentric graphs (see [1,12]) and eccentric digraphs (see [5,4]).

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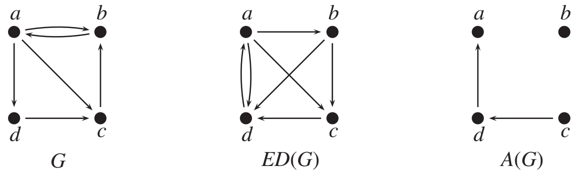

Fig. 1. A digraph G, its eccentric digraph $\mathrm{ED}(G)$ and its antipodal digraph $A(G)$.

The notion of the eccentric digraph of a graph was introduced by Buckley [5]. This construction was refined and extended by others, including Boland and Miller [4], to any digraph. This has led to the study of the behaviour of the iterated application of this operator (see [10]).

A *directed graph* or *digraph* $G=(V,E)$ consists of a finite nonempty set $V=V(G)$ of objects called *vertices* and a set $E=E(G)$ of ordered pairs of vertices called *arcs*; that is, $E(G)$ represents a binary relation defined on $V(G)$. For the purposes of this paper, a *graph* is a *symmetric digraph*; that is, a digraph G such that $(u,v)\in E(G)$ implies $(v,u)\in E(G)$. The *order* of G is the cardinality of $V(G)$ and is denoted by $|G|:=|V(G)|$. If (u,v) is an arc, it is said that u is *adjacent to* v and also that v is *adjacent from* u. The set of vertices which are adjacent from [to] a given vertex v is denoted by $N^+(v)$ $[N^-(v)]$ and its cardinality is the *out-degree* of v [*in-degree* of v]. A *walk* of length h from a vertex u to a vertex v ($u\to v$ walk) in G is a sequence of vertices $u=u_0,u_1,\ldots,u_{h-1},u_h=v$ such that each pair (u_{i-1},u_i) is an arc of G. A digraph G is *strongly connected* if there is a $u\to v$ walk for any pair of vertices u and v of G. The length of a shortest $u\to v$ walk is the *distance from u to v*, denoted by $\mathrm{dist}(u,v)$. If there is no $u\to v$ walk in G then we define $\mathrm{dist}(u,v)=\infty$. The *eccentricity* of a vertex u, denoted by $e(u)$, is the maximum distance from u to any vertex in G. If $\mathrm{dist}(u,v)=e(u)$ $(v\neq u)$ we say that v is an *eccentric vertex* of u. The *radius* of G, $\mathrm{rad}(G)$, is the minimum eccentricity of the vertices in G; the *diameter*, $\mathrm{diam}(G)$, is the maximum eccentricity of the vertices in G. Reader is referred to Chartrand and Lesniak [8] for additional graph concepts.

The *eccentric digraph* of a digraph G, denoted by $\mathrm{ED}(G)$, is the digraph on the same vertex set as G with an arc from vertex u to vertex v in $\mathrm{ED}(G)$ if and only if v is an eccentric vertex of u in G. The *antipodal digraph* of a digraph G, denoted by $A(G)$, has the same vertex set as G with an arc from vertex u to vertex v in $A(G)$ if and only if v is an antipodal vertex of u in G; that is $\mathrm{dist}(u,v)=\mathrm{diam}(G)$ (the notion of the antipodal digraph of a digraph was introduced by Johns and Sleno [14] as an extension of the definition of the antipodal graph of a graph given by Aravamudhan and Rajendran [2]). Note that $A(G)$ is always a subdigraph of $\mathrm{ED}(G)$, since $\mathrm{dist}(u,v)=\mathrm{diam}(G)$ implies $e(u)=\mathrm{diam}(G)$. Moreover, $A(G)=\mathrm{ED}(G)$ if and only if G is self-centered; that is all vertices of G have the same eccentricity ($\mathrm{rad}(G)=\mathrm{diam}(G)$). Fig. 1 shows an example of a digraph whose eccentric digraph and antipodal digraph are different.

The *eccentric graph* of a graph G, denoted by G_e, has the same set of vertices as G with two vertices u and v being adjacent in G_e if and only if either v is an eccentric vertex of u in G or u is an eccentric vertex of v in G; that is, $\mathrm{dist}_G(u,v)=\min\{e_G(u),e_G(v)\}$ (the notion of the eccentric graph of a graph was introduced by Akiyama et al. [1]). Note that G_e is the underlying graph of $\mathrm{ED}(G)$. We will prove that $G_e=\mathrm{ED}(G)$ if and only if G is self-centered (see Section 4).

Typical questions, related to the previous constructions, that have been investigated are listed below.

(1) Characterize those graphs and digraphs that are antipodal.
(2) Find hereditary properties of graphs and digraphs and their antipodal (respectively, eccentric).
(3) Investigate relationships between antipodal and eccentric graphs (respectively, digraphs). Find additional properties of graphs and digraphs that are self-antipodal (respectively, self-eccentric).

In this paper we present new results on eccentric digraphs. We give a characterization of eccentric digraphs (see Section 3) and completely answer the question of when is the eccentric digraph of a graph symmetric (see Section 4).

2. Eccentric and complement operators

In the undirected case, Buckley [5] proved that the eccentric digraph of a graph G is equal to its complement, $\mathrm{ED}(G)=\overline{G}$, if and only if G is either a self-centered graph of radius two or G is the union of $k\geqslant 2$ complete graphs.

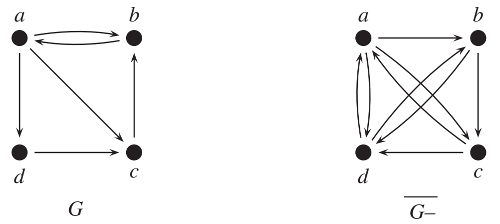

Fig. 2. A digraph G and the complement of its reduction $\overline{G^-}$.

We extend this result to the directed case by considering a modification of the complement operation. Unlike the usual complement, the new complement maintains all the adjacencies from vertices of eccentricity one as the eccentric digraph operation does. However, rather than introducing a new complement operation, we will produce the intended effect by using the usual complement operation on a modified version of the given graph.

More precisely, given a digraph G of order n, a "reduction" of G, denoted by G^-, is derived from G by removing all its arcs incident from vertices with out-degree $n-1$. We refer to the digraph $\overline{G^-}$ as the *complement of the reduction of* G. Note that $\mathrm{ED}(G)$ is a subdigraph of $\overline{G^-}$ and, moreover, their corresponding sets of vertices with out-degree $n-1$ are the same. Fig. 2 shows an example of a digraph and the complement of its reduction.

Next we characterize when the eccentric digraph of a digraph [graph] is equal to the complement of its reduction.

Proposition 2.1. *Let G be a digraph. Then $\mathrm{ED}(G) = \overline{G^-}$ if and only if for any vertex $u \in V(G)$ with eccentricity > 2 the following (local) transitive condition holds*:

$$(u, v), (v, w) \in E(G) \Rightarrow \ (u, w) \in E(G), \quad \forall v, w \in V(G) \text{ and } u \neq w. \tag{1}$$

Proof. Let G be a digraph of order $n > 1$ and let u be a vertex of G. From its definition, the eccentric vertices of u in G are precisely the out-neighbours of u in $\overline{G^-}$ if and only if the set of (non-null) distances from u in G is

$$\{\mathrm{dist}(u, v),\ v \in V(G)\backslash\{u\}\} = \{1\},\ \{\infty\},\ \{1, 2\} \text{ or } \{1, \infty\}.$$

So, if $e(u) > 2$, then any vertex $v \in V(G)\backslash\{u\}$ that is reachable from u ($\mathrm{dist}(u, v) < \infty$) must be at distance 1 from u, which is equivalent to saying that condition (1) holds. Conversely, if $e(u) > 2$ and u satisfies condition (1), then the set of distances from u in G is either $\{\infty\}$ or $\{1, \infty\}$. $\square$

Given two digraphs G and H, we define $G \to H$ to be the digraph $G \cup H$ with additional arcs from each vertex of G to each vertex of H.

Corollary 2.1. *Let G be a graph of order $n > 1$. Then $\mathrm{ED}(G) = \overline{G^-}$ if and only if G satisfies one of the following conditions*:

(1) $\mathrm{rad}(G) = 1$;
(2) *G is self-centered of radius* 2;
(3) *G is the union of $k \geqslant 2$ complete graphs.*

Proof. We identify G as a symmetric digraph and we distinguish different cases according to the radius r of G.

If $r = 1$ then each vertex of G has eccentricity $\leqslant 2$. So, from Proposition 2.1, $\mathrm{ED}(G) = \overline{G^-}$, which gives

$$\mathrm{ED}(G) = \begin{cases} K_n & \text{if } G = K_n, \\ K_{n-n'} \to \overline{H} & \text{if } G = K_{n-n'} + H, \end{cases}$$

where H is a graph with order n' and maximum degree $< n' - 1$.

If $1 < r < \infty$ then G is connected and has no vertices with degree $n-1$. From Proposition 2.1, $\mathrm{ED}(G) = \overline{G^-} = \overline{G}$ iff all vertices of G have eccentricity 2; that is, G is self-centered of radius 2.

Finally, if $r = \infty$ then $G = C_1 \cup \cdots \cup C_k$, with $k \geqslant 2$, where each C_i represents a connected component of G. In this case, G satisfies condition (1) iff all its connected components are transitive graphs; that is, each C_i is a complete graph. So, $G = K_{n_1} \cup \cdots \cup K_{n_k}$ and, consequently, $\mathrm{ED}(G) = \overline{G} = K_{n_1,\ldots,n_k}$ is a complete multipartite graph. □

3. Characterization of eccentric digraphs

We say that a digraph [graph] G is *eccentric* if there exists a digraph H such that $\mathrm{ED}(H) \cong G$, where $\cong$ denotes graph isomorphism. Since the eccentric digraph construction involves only graph invariants, as the distances are, we can substitute the isomorphism relation by the equality relation. Thus, if π is an isomorphism from G to $\mathrm{ED}(H)$, with $V(G) = V(H)$, then $\mathrm{ED}(H^\pi) = G$, where $V(H^\pi) = V(H)$ and $(u, v) \in E(H^\pi)$ iff $(\pi(u), \pi(v)) \in E(H)$. At this point, to avoid any confusion in terminology, we would like to mention that, in the undirected case, the term 'eccentric graph' has been used by Chartrand et al. [7] (see also [11]) to denote a graph G such that all its vertices are eccentric (that is, $\mathrm{ED}(G)$ has minimum in-degree $\geqslant 1$).

Given an eccentric digraph G, the list of digraphs H such that $\mathrm{ED}(H) = G$ can contain more than one digraph. Next we prove that the complement of the reduction of G is always in such a list, whenever G is eccentric. As a consequence, in order to determine if a digraph G is eccentric it will suffice to compute $\mathrm{ED}(\overline{G^-})$ and check that it is equal to G (see an example in Fig. 3). We point out that Aravamudhan and Rajendran [2] proved that an undirected graph G is antipodal if and only if G is the antipodal graph of its complement (see [13] for a shorter proof); Johns and Sleno [14] extended this result to the directed case.

Theorem 3.1. *A digraph G is eccentric if and only if* $\mathrm{ED}(\overline{G^-}) = G$.

Proof. Obviously if $\mathrm{ED}(\overline{G^-}) = G$ then G is eccentric.

Let us assume that G is an eccentric digraph with order $n > 1$; that is, there is a digraph H such that $\mathrm{ED}(H) = G$. Since any vertex of H has at least one eccentric vertex, G has no vertices with out-degree 0. This implies that the complement of the reduction of $\overline{G^-}$ is equal to G. Then, using Proposition 2.1, it turns out that the relation $\mathrm{ED}(\overline{G^-}) = G$ is equivalent to saying that $\overline{G^-}$ satisfies the transitivity condition (1) for each of its vertices with eccentricity > 2.

Now, let us suppose that condition (1) does not hold for at least one vertex u with eccentricity > 2 in $\overline{G^-}$. This implies that we can partition the vertices of the set $V \backslash \{u\}$, where $V = V(\overline{G^-}) = V(G)$, into at least three (non-empty) parts according to their distance from u in $\overline{G^-}$. Thus, we can take $V \backslash \{u\} = D_1 \cup D_2 \cup D_{>2}$, where D_1, D_2 and $D_{>2}$ consist of the vertices of $\overline{G^-}$ at distance 1, 2 and > 2 from u, respectively. From the definition of this 'distance partition' we can derive the following adjacency conditions in $\overline{G^-}$ (see Fig. 4):

(1) All vertices of D_1 must have out-degree $< n - 1$, otherwise the eccentricitity of u would be two;
(2) every vertex of D_2 is adjacent from at least one vertex in D_1;
(3) there is no arc from u to any of the vertices in $D_2 \cup D_{>2}$;
(4) there is no arc from any vertex in D_1 to any vertex in $D_{>2}$.

Let us see how these properties are reflected in G. First, we notice that all adjacencies from the vertices of the set $\{u\} \cup D_1$ are the same in $\overline{G}$ as in $\overline{G^-}$, since all these vertices have out-degree $< n - 1$ (see (1)).

Next, we obtain a partition of the set of out-neighbours of u in G, $N^+(u) = D_2 \cup D_{>2}$, such that

(2′) For any vertex $w \in D_2$ there is at least one vertex $v \in D_1$ such that (v, w) is not an arc in G;
(4′) every vertex in $D_{>2}$ is adjacent from all vertices in D_1.

We will show that a partition of $N^+(u)$ like this is not consistent with the assumption that G is the eccentric digraph of a digraph H. Since $G = \mathrm{ED}(H)$ is a subdigraph of $\overline{H^-}$ and taking into account that u cannot have out-degree 0 nor $n - 1$ in H (otherwise, u would have eccentricity 1 in G and, consequently, in $\overline{G^-}$), we obtain the following adjacency conditions in H:

(3″) The (non-empty) set of out-neighbours of u is contained in D_1;
(4″) there is no arc incident from any vertex in D_1 to any vertex in $D_{>2}$.

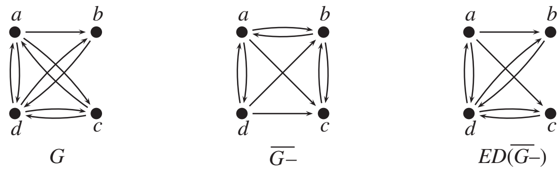

Fig. 3. A (non-eccentric) digraph G, the complement of its reduction $\overline{G^-}$, and the eccentric digraph $\mathrm{ED}(\overline{G^-})$.

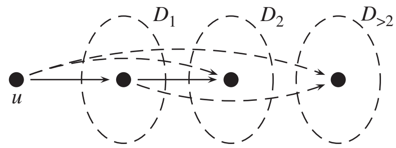

Fig. 4. Dashed lines indicated non-adjacencies.

So, if there is a walk in H from u to a vertex in $D_{>2}$ it must go through at least one vertex in D_2. But this would imply that

$$\min\{\mathrm{dist}_H(u,v),\ v \in D_2\} < \min\{\mathrm{dist}_H(u,v),\ v \in D_{>2}\},$$

which is impossible since all vertices in $N^+(u) = D_2 \cup D_{>2}$ are eccentric of u in H and, consequently, they are at the same distance from u. Therefore, no vertex of $D_{>2}$ is reachable from u in H, which implies that

$$\mathrm{dist}_H(u,v) = e_H(u) = \infty \quad \text{for any vertex } v \in D_2 \cup D_{>2}.$$

On the other hand, every vertex $v \in D_1$ must be reachable from u in H, since otherwise v would be an eccentric vertex of u in H and u is not adjacent to v in $G = \mathrm{ED}(H)$. Hence, there is no arc in H incident from a vertex in D_1 to a vertex $w \in D_2$, since it would imply that $\mathrm{dist}_H(u,w) < \infty$. This means that in H every vertex $w \in D_2$ is an eccentric vertex of all vertices in D_1, which contradicts the fact that there is at least one vertex in D_1 which is not adjacent to w in G (see (2′)), where $G = \mathrm{ED}(H)$. □

Note that an alternative proof can be obtained by applying ideas from a paper by Johns and Sleno (see [14]).

Restricting the previous characterization to the symmetric (undirected) case, we get the following

Theorem 3.2. *Let G be a graph of order $n > 1$. Then G is eccentric if and only if $\overline{G}$ is self-centered with radius two or $\overline{G}$ is the union of complete graphs.*

Proof. From the proof of Theorem 3.1, we know that G is eccentric iff G has minimum degree > 0 and $\mathrm{ED}(\overline{G^-})$ is equal to the complement of the reduction of $\overline{G^-}$. So, using Corollary 2.1, we have that G is eccentric iff $\overline{G^-}$ satisfies one of the following conditions:

(1) $\mathrm{rad}(\overline{G^-}) = 1$ and G has no vertex of degree 0;
(2) $\overline{G^-}$ is self-centered of radius 2;
(3) $\overline{G^-}$ is the union of $k \geqslant 2$ complete graphs.

First, let us consider the case that $\overline{G^-}$ is a digraph of radius 1. This means that $\overline{G^-}$ is either the complete graph K_n or the graph $K_{n-n'} + H$, where H is a graph of order $n' > 1$ and maximum degree $< n' - 1$. With the extra condition that G has no vertices of degree 0, we deduce that either $G = K_n$, or G is the complement of the reduction of $K_{n-n'} + H$ which gives $G = K_{n-n'} \to \overline{H}$. Since G is a symmetric digraph, we conclude that $G = K_n$. In such a case $\overline{G}$ is the union of n copies of the complete graph K_1.

If the radius of $\overline{G^-}$ is greater than one, then $\overline{G^-} = \overline{G}$. Thus, we can reformulate conditions (2) and (3) by saying that either $\overline{G}$ is self-centered of radius 2 or $\overline{G} = K_{n_1} \cup \cdots \cup K_{n_k}$, with $k \geqslant 2$. □

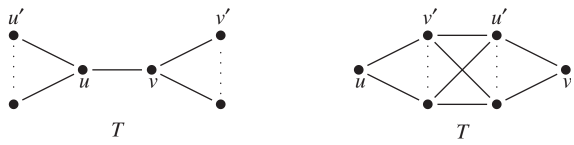

Fig. 5. Tree of diameter 3 and its complement.

As an application, we determine the eccentric character of some classes of graphs.

Corollary 3.1.

(i) *Every non-connected graph with minimum degree* >0 *is eccentric.*
(ii) *The eccentric graphs of radius* 1 *are the complete multipartite graphs with at least one partite set of cardinality* 1.
(iii) *Every connected graph with radius* $\geqslant 3$ *or diameter* $\geqslant 4$ *is eccentric.*

Proof. (i) If G is a non-connected graph with minimum degree >0 then $G=C_1\cup\cdots\cup C_k$, with $k\geqslant 2$, where each C_i represents a connected component of G (with order >1). Therefore, $\overline{G}$ is a self-centered graph with radius 2. Hence, G is eccentric ($\mathrm{ED}(\overline{G})=\overline{\overline{G}}=G$).

(ii) If G is a graph of radius 1 such that G is eccentric, which means that $\overline{G}$ is the union of complete graphs (at least one of them of order one), then $G=\overline{\overline{G}}$ is a complete multipartite graph with at least one partite set of cardinality one.

(iii) We will show that if G is a graph with minimum degree $\delta(G)>0$ such that its complement graph $\overline{G}$ is not self-centered of radius 2, then the radius of G is $\leqslant 2$ and the diameter of G is $\leqslant 3$. As a consequence, any connected graph G with radius $\geqslant 3$ or diameter $\geqslant 4$ will be eccentric, since $\mathrm{ED}(\overline{G})=\overline{\overline{G}}=G$.

Let us suppose that $\overline{G}$ neither has a vertex with eccentricity one ($\delta(G)>0$) nor is self-centered of radius two. This means that there is at least one pair of nonadjacent vertices u and v in $\overline{G}$ such that u and v do not share any common out-neighbour in $\overline{G}$; that is, $e_{\overline{G}}(u)\geqslant \mathrm{dist}_{\overline{G}}(u,v)>2$. So, in its complement graph, which is G, any vertex in $V\backslash\{u,v\}$ is adjacent with either u or v and, moreover, u and v are also adjacent. Therefore, the eccentricity of u in G is $\leqslant 2$ and the maximum distance between any pair of vertices w and z in G is $\leqslant 3$, since there is at least one $w-z$ walk of length 3 using u and v as step vertices. Hence, $\mathrm{rad}(G)\leqslant 2$ and $\mathrm{diam}(G)\leqslant 3$. □

Corollary 3.2. *A tree is eccentric if and only if its diameter is not equal to* 3.

Proof. Let T be a tree of order $n>1$. We distinguish between different cases according to the diameter of T.

If $\mathrm{diam}(T)\leqslant 2$ then T is a star, $T=K_{1,n-1}$. Since its complement is the union of complete graphs, T is eccentric.

If $\mathrm{diam}(T)=3$ then T has two central and adjacent vertices u and v. Any other vertex w of T must have degree 1 (w is a leaf) and be adjacent either with u or v (but not with both). Moreover, there are at least two leaves u' and v' in T adjacent to u and v, respectively. So, if we take the graph complement of T, we can see that the distance between u and v in $\overline{T}$ is equal to three, since u, v', u', v is a shortest $u-v$ walk in $\overline{T}$ (see Fig. 5).

Therefore, $\overline{T}$ is neither self-centered of radius two nor the union of complete graphs. Hence, from Theorem 3.2, we conclude that T is not eccentric.

In any other case, since $\mathrm{diam}(T)\geqslant 4$ and using Corollary 3.1 (part (iii)), we deduce that T is eccentric. □

To conclude this section we point out that Boland et al. [3] proved that if a digraph G is not eccentric then there exists an eccentric digraph H such that H contains G as an induced subdigraph and $|H|=|G|+1$.

4. Symmetric eccentric digraphs

In this section we study symmetric eccentric digraphs. In the undirected case, the condition of being self-centered guarantees that the corresponding eccentric digraph is in fact a graph since the distance is symmetric. We will see that this condition is also necessary.

Fig. 6. The directed cycle is self-centered but its eccentric digraph is not symmetric.

Proposition 4.1. *Let G be a graph. Then the eccentric digraph* $\mathrm{ED}(G)$ *is symmetric if and only if G is self-centered.*

Proof. Let us assume that G is a connected non-trivial graph such that $\mathrm{ED}(G)$ is symmetric. Note that if G is disconnected (or $G = K_1$) then G is trivially self-centered.

Now let us consider a central vertex u of G and one of its adjacent vertices v. Taking into account the triangular inequality and the symmetry of the distance in graphs, we have

$$|\mathrm{dist}(u, w) - \mathrm{dist}(v, w)| \leqslant \mathrm{dist}(u, v) = 1, \quad \forall w \in V(G). \tag{2}$$

Since $e(u) = \mathrm{rad}(G)$, it follows that $\mathrm{rad}(G) \leqslant e(v) \leqslant \mathrm{rad}(G) + 1$.

Let us suppose that v is not a central vertex. Then $e(v) = \mathrm{rad}(G) + 1$ and there is a vertex w such that $\mathrm{dist}(v, w) = \mathrm{rad}(G) + 1$. From (2), $\mathrm{dist}(u, w) = \mathrm{rad}(G)$, which means that w is an eccentric vertex of u and v. However, since $\mathrm{ED}(G)$ is a symmetric digraph,

$$e(w) = \mathrm{dist}(w, u) = \mathrm{dist}(w, v),$$

which contradicts that $\mathrm{dist}(u, w) \neq \mathrm{dist}(v, w)$.

Hence all vertices adjacent to a central vertex of G are also central and, since G is connected, we conclude that G is self-centered.

The previous argument cannot be extended to the non-symmetric (directed) case, since then the distance is not a metric. Thus, the condition of being self-centered is no longer necessary, in general, to have a symmetric eccentric digraph, as can be seen by taking $G = K_{n'} \to N_{n-n'}$ ($\mathrm{ED}(G) = K_n$), where N_n denotes a null graph of order n. Such a condition is not sufficient either, as we can see in Fig. 6.

Evidently, the characterization of digraphs G such that $\mathrm{ED}(G)$ is symmetric seems to be more complicated. We can provide a complete characterization in the non-strongly connected case.

Proposition 4.2. *Let G be a non-strongly connected digraph. Then* $\mathrm{ED}(G)$ *is a symmetric digraph if and only if*

$$G = C_1 \cup \cdots \cup C_k \quad (k \geqslant 2) \quad \textit{or} \quad G = K_{n'} \to (C_1 \cup \cdots \cup C_k) \quad (k \geqslant 1),$$

where $C_1, \ldots, C_k$ are strongly connected digraphs.

Proof. If $G = C_1 \cup \cdots \cup C_k$ [$G = K_{n'} \to (C_1 \cup \cdots \cup C_k)$] then $\mathrm{ED}(G) = K_{n_1,\ldots,n_k}$ [$\mathrm{ED}(G) = K_{1,\overset{n'}{\ldots},1,n_1,\ldots,n_k}$, respectively], where n_i is the order of C_i. So, in both cases, $\mathrm{ED}(G)$ is a symmetric digraph.

Next, let us assume that G is a non-strongly connected digraph such that $\mathrm{ED}(G)$ is a symmetric digraph. Let u and v be two vertices of G belonging to different strongly connected components of G, say C_u and C_v, respectively.

Suppose that $e(u) = e(v) = \infty$. If there are vertices $u' \in V(C_u)$ and $v' \in V(C_v)$ such that $(u', v') \in E(G)$, then $\mathrm{dist}(v', u') = \infty$, as both u' and v' belong to distinct strongly connected components. Hence, $(v', u') \in E(\mathrm{ED}(G))$ and from the symmetry we have $(u', v') \in E(\mathrm{ED}(G))$, which implies $(u', v') \notin E(G)$, a contradiction.

Now, let us suppose that $e(u) < \infty$ and $e(v) = \infty$. For any pair of vertices $u' \in V(C_u)$ and $v' \in V(C_v)$, we have $\mathrm{dist}(u', v') = e(u')$, since $\mathrm{dist}(v', u') = e(v') = \infty$ and $\mathrm{ED}(G)$ is symmetric. Then, if a vertex $u' \in V(C_u)$ is adjacent to a vertex in C_v, we have $e(u') = 1$. Since there must be at least one arc from a vertex of C_u to a vertex of C_v, the existence of such a vertex $u' \in V(C_u)$ with $e(u') = 1$ is guaranteed. As a consequence, all vertices adjacent to u' must also have eccentricity 1, property that can be extended, by connection, to any other vertex of C_u. Hence, $G = K_{|C_u|} \to (C_1 \cup \cdots \cup C_k)$, where each C_i is a strongly connected digraph. □

While in the strongly connected case we do not have a complete characterization, we are able to provide the following partial results.

Proposition 4.3. *Let G be a strongly connected digraph such that* $\mathrm{ED}(G)$ *is symmetric. Then the following conditions hold*:

(i) $\mathrm{rad}(G) > 1$, *unless G is a complete digraph.*
(ii) *If* $\mathrm{diam}(G) = 2$ *then G is a self-centered graph.*

Proof. Let G be a connected digraph such that $\mathrm{ED}(G)$ is symmetric. In the case that $\mathrm{rad}(G) = 1$ and $\mathrm{diam}(G) > 1$, we can consider the vertex subset $V_1 = \{v \in V(G) \mid e(v) = 1\} \neq V(G)$. Since G is strongly connected, there is a vertex $u \in \overline{V_1}$ such that u is adjacent to a vertex v of V_1. Then, since u is an eccentric vertex of v and $\mathrm{ED}(G)$ is symmetric, $e(u) = \mathrm{dist}(u, v) = 1$, which is impossible. The second part of the proposition is a straightforward consequence of the first one. □

Next, we restrict our search for eccentric symmetric digraphs to the case of stable digraphs. We recall that a digraph G with girth g is *stable* if for any pair of vertices $u, v \in V(G)$ it holds that

$$\mathrm{dist}(u, v) + \mathrm{dist}(v, u) = g, \quad \text{if } 0 < \mathrm{dist}(u, v) < g. \tag{3}$$

In particular, notice that a stable digraph with girth 2 is a graph.

Proposition 4.4. *Let G be a connected and stable digraph with girth $g \geqslant 3$. Then*, $\mathrm{ED}(G)$ *is symmetric if and only if G is self-centered with radius g.*

Proof. The diameter D of a connected and stable digraph G with girth $g \geqslant 3$ is either $D = g - 1$ or $D = g$ (see [9]). As a consequence, if we consider an arc (u, v) of G then from the stability condition (3) we have $\mathrm{dist}(v, u) = g - 1 \leqslant e(v) \leqslant D$. Thus, if $e(v) = g - 1$ then u is an eccentric vertex of v. Assuming that $\mathrm{ED}(G)$ is symmetric, we have $e(u) = \mathrm{dist}(u, v) = 1$, which means that G has radius 1. From Proposition 4.3, this can only happen when G is a complete digraph, which is not our case since $g \geqslant 3$. So $e(v) = g$ for every vertex v of G, i.e., G is self-centered with radius g.

Conversely, let us assume that G is a self-centered digraph with radius g and let us consider a pair of vertices u and v of G such that $(u, v) \in E(\mathrm{ED}(G))$, i.e., $\mathrm{dist}(u, v) = e(u) = g$. Since G is stable, $\mathrm{dist}(v, u) = g = e(v)$, which means that $(v, u) \in E(\mathrm{ED}(G))$. Hence, $\mathrm{ED}(G)$ is symmetric. □

Since Damerell [9] showed that any distance-regular digraph is stable, we can take distance-regular digraphs with equal girth and diameter, known as *long digraphs*, as examples of connected digraphs G such that $\mathrm{ED}(G)$ is symmetric. Thus, for instance, from the directed cycle Z_{k+1} and the complete digraph with loops K_n^* we construct their *conjunction digraph*, $Z_{k+1} \wedge K_n^*$, which has as a vertex set the cartesian product $V(Z_{k+1}) \times V(K_n^*)$ and with an arc from (u, v) to (u', v') iff $(u, u') \in E(Z_{k+1})$. It turns out that $Z_{k+1} \wedge K_n^*$ is a long digraph (with girth and diameter equals to $k + 1$) and, consequently, its eccentric digraph is symmetric. In fact, $\mathrm{ED}(Z_{k+1} \wedge K_n^*)$ is the union of $k + 1$ copies of K_n. We conclude this section by showing, for arbitrarily large diameter D, the existence of strongly connected digraphs with radius r, $1 < r < D$, whose eccentric digraph is symmetric.

Proposition 4.5. *Given two positive integers a and b, where $2 \leqslant a < b$, there exists a strongly connected digraph G, with* $\mathrm{rad}(G) = a$ *and* $\mathrm{diam}(G) = b$, *such that* $\mathrm{ED}(G)$ *is a symmetric digraph.*

Proof. Let us consider the dragon graph $D_{m,2n}$ obtained by identifying a vertex v_0 of the cycle $C_{2n} : v_0 v_1 \ldots v_{2n-2} v_{2n-1} v_0$ with a pendant vertex u_0 of the path $P_m : u_0 u_1 \ldots u_{m-2} u_{m-1}$, where $m, n \geqslant 2$. That is,

$$V(D_{m,2n}) = \{v_0, v_1, \ldots, v_{2n-1}\} \cup \{u_1, u_2, \ldots, u_{m-1}\},$$
$$E(D_{m,2n}) = \{v_0 v_1, v_1 v_2, \ldots, v_{2n-1} v_0\} \cup \{v_0 u_1, u_1 u_2, \ldots, u_{m-2} u_{m-1}\}.$$

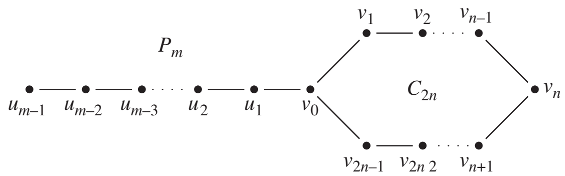

Fig. 7. The dragon graph $D_{m,2n}$.

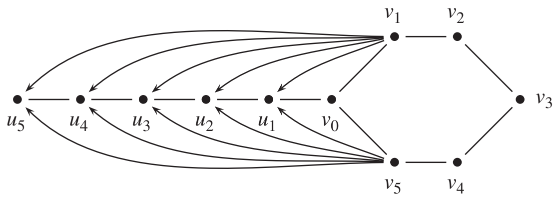

Fig. 8. The digraph $G_{6,6}$.

(See Fig. 7). This is a particular case of a more general construction given by Mao and Liu [15], which involves the concatenation of a self-centered graph with a path.

Now we add some arcs to the graph $D_{m,2n}$ in order to get a digraph whose radius and diameter still depend on both parameters m, n and whose eccentric digraph is symmetric. Thus, we define the digraph $G_{m,2n}$ as follows:

$$V(G_{m,2n}) = V(D_{m,2n}),$$
$$E(G_{m,2n}) = E(D_{m,2n}) \cup \{(v_1, u_i), (v_{2n-1}, u_i) \mid 1 \leqslant i \leqslant m-1\} \cup E_0,$$

where $E_0 = \{(v_0, u_i) \mid 2 \leqslant i \leqslant m-1\}$, if $n = 2$ and $m \geqslant 3$, and $E_0 = \emptyset$, otherwise. (See, for instance, Fig. 8).

The following relations about distances and eccentricities in the digraph $G_{m,2n}$ can be easily derived from its construction:

$$e(v_i) = n = \mathrm{dist}(v_i, v_{i+n(\mathrm{mod}\ 2n)}) > \mathrm{dist}(v_i, u_j), \quad \text{if } i \neq n,$$
$$e(v_n) = n = \mathrm{dist}(v_n, v_0) = \mathrm{dist}(v_n, u_j),$$
$$e(u_j) = n + j = \mathrm{dist}(u_j, v_n), \quad j = 1, \ldots, m-1.$$

As a result, $\mathrm{rad}(G_{m,2n}) = n$ and $\mathrm{diam}(G_{m,2n}) = m + n - 1$. Moreover, taking into account that each vertex v_i has a unique eccentric vertex $(v_{i+n\ (\mathrm{mod}\ 2n)})$, apart from v_n, which is also mutually eccentric with all vertices of the set $\{u_1, \ldots, u_{m-1}\}$, we have that $\mathrm{ED}(G_{m,2n})$ is the union of $K_{1,m}$ and $n-1$ copies of K_2. So, $\mathrm{ED}(G_{m,2n})$ is a symmetric digraph.

Hence, given any two positive integers a and b, with $2 \leqslant a < b$, we can take $m = a$ and $n = b - a + 1$ and construct the digraph $G_{m,2n}$, which satisfies the conditions given in the Proposition. $\square$

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