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HECKE ACTIONS ON BRAUER GROUPS

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This paper draws a connection between a recent paper of DeMeyer [2] and an independent paper by Roggenkamp and Scott [11]. In [2] DeMeyer extends an earlier result of Janusz [5] by defining an action of a group G of automorphisms of a commutative ring S on its Brauer group $B(S)$. He then uses Amitzur cohomology to extend this action to the étale cohomology groups of the ring $H^n(S, G_m)$ with coefficients in the group of units G_m. In [11] Roggenkamp and Scott extend results of Perlis [10] by defining a contravariant additive functor from the Hecke category $\mathscr{H}_G$ to the category of abelian groups (the objects of $\mathscr{H}_G$ are the $\mathbb{Z}G$-modules $\mathbb{Z}G/H = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}$ for subgroups H of G and the morphisms of $\mathscr{H}_G$ are $\mathbb{Z}G$-module homomorphisms). Their functor sends $\mathbb{Z}G/H$ to $\mathrm{Pic}(S^H)$. In case G is finite they use Zariski derived functor cohomology to prove this. For the case when G is infinite they use 'generators and relations' for the category $(\mathscr{H}_G)^{\mathrm{op}}$ dual to $\mathscr{H}_G$. Recall that for an affine scheme $X = \operatorname{Spec} S$ the étale cohomology groups have the following interpretation for low n:

$$H^0(X, G_m) = \text{units of } S = S^*,$$

$$H^1(X, G_m) = \text{Picard group of } S = \operatorname{Pic} S,$$

$$\text{tors}\, H^2(X, G_m) = \text{Brauer group of } S = B(S).$$

In this paper we define a contravariant additive functor from $\mathscr{H}_G$ to the category of abelian groups which sends $\mathbb{Z}G/H$ to the Brauer group $B(S^H)$ when S is a Galois extension of R with (finite) group G. The maps on Azumaya algebras are defined explicitly. If 0 and 1 are the only idempotents of R, then this functor extends to $\mathscr{H}_{\mathrm{Gal}(S/R)}$ where S is the separable closure of R. We also show that there is a contravariant additive functor

$$\Phi^n : \mathscr{H}_G \to \text{abelian groups}$$

given by $\Phi^n(\mathbb{Z}G/H) = H^n(S^H, G_m)$. In this case we use Čech étale cohomology. Thus, when S/R is Galois we extend the results of Roggenkamp and Scott. Noting that $\mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \cong \mathbb{Z}G$ we also get DeMeyer's action of $\mathbb{Z}G$ on $B(S)$.

Throughout unadorned tensor products are over S. If H is a subgroup of G, then we write $H \leq G$. The set of all double cosets HgK will be denoted $H \backslash G/K$ for subgroups H, K of G. All other terminology will be as in [9]. The author wishes to thank F.R. DeMeyer for some helpful suggestions.

To define an additive functor on $\mathscr{H}_G$ one has to give its values for three types of maps: 'corestriction', 'restriction', and 'conjugation' and verify that these homomorphisms satisfy several relations [12]. Let S and R be commutative rings and assume that S is Galois over R with group G [4, p. 84]. For any pair of subgroups $H \leq K \leq G$ and element g in G we will define three homomorphisms

$$\mathrm{cor}^K : B(S^H) \to B(S^K) \qquad \text{(corestriction)},$$

$$\mathrm{res}_H : B(S^K) \to B(S^H) \qquad \text{(restriction)},$$

$$\mathrm{con}^g : B(S^H) \to B(S^{gHg^{-1}}) \qquad \text{(conjugation)}.$$

Since $S^K \leq S^H$, res is induced by the action $A \to A \otimes_{S^K} S^H$ for A an Azumaya S^K-algebra.

Since g induces an isomorphism

$$g|_{S^H} : S^H \to S^{gHg^{-1}}$$

we have an induced isomorphism on Brauer groups $B(S^H) \cong B(S^{gHg^{-1}})$. If A is an Azumaya S^H-algebra, then $\mathrm{con}^g(|A|) = |A \otimes_{S^H} S^{gHg^{-1}}|$. Now we define cor. This is a generalization of the norm map given in [8]. For any S-module M and g in G we define ${}_gM$ to be the S-module which is isomorphic to M as abelian groups and whose S-module action is given by $s * m = g^{-1}(s)m$. If M is an S-algebra, ${}_gM$ is equal to M as rings and equal to ${}_gM$ as an S-module. Let A be an S^H-module. Then for any x in G and y in H, y induces an isomorphism

$${}_xS \otimes_{S^H} A \cong {}_{y^{-1}x}S \otimes_{S^H} A$$

of S-modules. To see this just check that y induces an isomorphism of left S-modules ${}_xS \cong {}_{y^{-1}x}S$ and simultaneously of right S^H-modules. Then $y \otimes 1$ is an isomorphism. Now let $X = \{x_1, \dots, x_k\}$ be a full set of left coset representatives of H in K. Let A be an S^H-algebra and $B = S \otimes_{S^H} A$. Let $\boldsymbol{B} = \bigotimes_X {}_xB = {}_{x_1}B \otimes \cdots \otimes {}_{x_k}B$. If y is in K, then y induces a permutation of the set X. Also y induces a map of S-algebras ${}_xB \to {}_{y^{-1}x}B$. From the above, y induces an automorphism of $\boldsymbol{B}$. It is known, [1] or [8], that if A is an Azumaya S^H-algebra, then $\boldsymbol{B}$ is an Azumaya S-algebra, $\boldsymbol{B}^K$ is an Azumaya S^K-algebra and $\boldsymbol{B} = S \otimes_{S^K} \boldsymbol{B}^K$. To show that the correspondence $A \to \boldsymbol{B}^K$ induces cor we have to show that the image of $\mathrm{End}_{S^H}(P)$ for P an S^H-progenerator is of the form $\mathrm{End}_{S^K}(P')$ for P' an S^K-progenerator. Let $P' = P \otimes_{S^H} S$ and $\boldsymbol{P}' = \bigotimes_X {}_xP'$. Then P and $\boldsymbol{P}'$ are S-progenerators. By Galois descent [7], $\boldsymbol{P}' = S \otimes_{S^K} \boldsymbol{P}'^K$. Thus $\boldsymbol{P}'^K$ is an S^K-progenerator [7, Lemma 3.6]. Now let $A = \mathrm{End}_{S^H}(P)$. Then $B = \mathrm{End}_S(P')$ and $\boldsymbol{B} = \mathrm{End}_S(\boldsymbol{P}')$ [2, Lemma 1(b)]. Thus $\boldsymbol{B}^K = \mathrm{End}_{S^K}(\boldsymbol{P}'^K)$ and we see that $A \to \boldsymbol{B}^K$ induces the homomorphism cor.

Proposition 1. *Let H, K, L, D be subgroups of G, g, g' elements of G and y in $B(S^H)$. Then the homomorphism* cor, res, con *defined above satisfy the following axioms.*

(G.1) $\mathrm{cor}^H(y) = y, \quad \mathrm{cor}^L \mathrm{cor}^K(y) = \mathrm{cor}^L(y)$ *if* $H \le K \le L$.

(G.2) $\mathrm{res}_K(y) = y, \quad \mathrm{res}_D \mathrm{res}_H(y) = \mathrm{res}_D(y)$ *if* $D \le H \le K$.

(G.3) $\mathrm{con}^{g'} \mathrm{con}^g(y) = \mathrm{con}^{g'g}(y), \quad \mathrm{con}^h(y) = y$ *if h is in H.*

(G.4) $\mathrm{con}^g \mathrm{cor}^K(y) = \mathrm{cor}^{gKg^{-1}} \mathrm{con}^g(y),$

$\mathrm{con}^g \mathrm{res}_H(y) = \mathrm{res}_{gHg^{-1}} \mathrm{con}^g(y)$ *if* $H \le K$.

(G.5) (Mackey Formula) *If H and K are subgroups of L, then*

$$\mathrm{res}_K \mathrm{cor}^L(y) = \sum_{KgH} \mathrm{cor}^K \mathrm{res}_{ghg^{-1} \cap K} \mathrm{con}^g(y)$$

where g runs over a full set of representatives of the double cosets KgH in $K \backslash L/H$.

(c) $\mathrm{cor}^K \mathrm{res}_H(y) = [K:H]y$ *if* $H \le K$.

Proof. The above axioms follow from Galois descent and looking at the appropriate commutative diagrams. We will check the Mackey Formula and leave the rest to the reader. Let H and K be subgroups of L. Let KxH be a double coset in $K \backslash L/H$. Let A be an S^H-module and $B = S \otimes_{S^H} A$. Let X be a full set of left coset representatives for H in L. Then X is a disjoint union of the sets $X \cap KxH$ and we can choose X to begin with so that $Y_x = (X \cap KxH)x^{-1}$ is a full set of left coset representatives for $xHx^{-1} \cap K$ in K. Therefore we have

$$\bigotimes_X {}_x B = \bigotimes_{KxH} \left(\bigotimes_{y \text{ in } Y_x} {}_{yx} B \right) \tag{1}$$

where the tensor product $\otimes_{KxH}$ is taken over S^K and the other two are over S. The image of A under 'corestriction' to S^L and 'restriction' to S^K is $(\otimes_X B)^L \otimes_{S^L} S^K$. The image of A under 'conjugation' by x, 'restriction' to $S^{xHx^{-1} \cap K}$ and 'corestriction' to S^K is $(\otimes_{y \text{ in } Y_x\, yx} B)^L$. It remains to show that

$$\left(\bigotimes_X {}_x B \right)^H \otimes_{S^H} S^L \cong \bigotimes_{KxH} \left(\bigotimes_{Y_x} {}_{yx} B \right)^L. \tag{2}$$

Tensoring both sides of (2) with S over S^L over S^L yields (1). The result follows by descent [7, Théorème 5.1].

Corollary 2. *Let S and R be commutative rings with S a Galois extension of R with finite group G. There is a contravariant additive functor*

$\Phi : \mathscr{H}_G \to$ *abelian groups* *where* $\Phi(\mathbb{Z}G/H) = B(S^H)$.

Proof. This is an immediate consequence of Proposition 2 and [12].

When Spec R is connected we can extend the above results to the infinite group case. Let R be a ring with 0 and 1 the only idempotents of R, S the separable closure of R and G the Galois group of S over R (see [4]). To define a contravariant additive functor from $\mathscr{H}_G$ to the category of abelian groups one defines con and res as above and cor is defined for every pair $H \leq K$ such that $[K:H]$ is finite [12]. The three homomorphisms must satisfy axioms as in Proposition 1. Let $H \leq K \leq G$ with $[K:H]$ finite. We define the corestriction map from S^H to S^K in the following way. First note that S is the separable closure of S^K and S^H is finite over S^K. Also S^H is separable over S^K because $[K:H]$ finite implies $[\mathrm{Gal}(S/S^K):H]$ is finite which implies H is closed in $\mathrm{Gal}(S/S^K)$. Thus we can imbed S^H in its normal closure T over S^K in S. Thus T is Galois over S^K. We define cor^K as above using T in place of S. It is now a formality to check that the six axioms of Proposition 1 are satisfied.

Theorem 3. *Let S, R be commutative rings with S a Galois extension of R with finite group G. Then there is a contravariant additive functor*

$$\Phi^n : \mathscr{H}_G \to \text{abelian groups}$$

given by $\Phi^n(\mathbb{Z}G/H) = H^n(S^H, G_m)$ for each $n \geq 0$.

Proof. We will prove the theorem for Čech étale cohomology. The Čech groups agree with the derived functor groups for the sheaf of units G_m by [9, Theorem III 2.17]. The Čech cohomology groups for a scheme X are defined to be

$$\check{H}^n(X, G_m) = \varinjlim \check{H}^n(\mathscr{U}, G_m)$$

where the limit is taken over all étale covers $\mathscr{U} = (U_i \to X)$. If $\mathscr{U}$ is a cover for X and $\mathscr{V}$ is a refinement of $\mathscr{U}$, then there is a homomorphism induced on cohomology groups $\check{H}^n(\mathscr{U}, G_m) \to \check{H}^n(\mathscr{V}, G_m)$. First we show that every étale cover of Spec S has a refinement on which G acts. Let $G = \{x_1, \ldots, x_n\}$. Let U be an S-algebra and $\mathbf{U} = {}_{x_1}U \otimes \cdots \otimes {}_{x_n}U$. If $S \to U$ is étale, then $S \to \mathbf{U}$ is étale. We have seen above that G extends to a group of automorphisms of $\mathbf{U}$. If $\mathscr{U} = (\mathrm{Spec}\ U_i \to \mathrm{Spec}\ S)$ is an étale cover of Spec S, then $\boldsymbol{\mathscr{U}} = (\mathrm{Spec}\ \mathbf{U}_i \to \mathrm{Spec}\ S)$ is a refinement of $\mathscr{U}$ and G acts on $\boldsymbol{\mathscr{U}}$. Now assume that $S \to U$ is étale and that every automorphism x in G extends to an automorphism of U. Let H be a subgroup of G. Since S is Galois over S^H with group H it follows from descent theory that $U = U^H \otimes_{S^H} S$ and that U^H is étale over S^H. Let $\mathscr{V} = (\mathrm{Spec}\ V_i \to \mathrm{Spec}\ S^H)$ be an étale cover of S^H. Then $\mathscr{U} = (\mathrm{Spec}\ V_i \otimes S \to \mathrm{Spec}\ S)$ is an étale cover of S and $\boldsymbol{\mathscr{U}}$ is a refinement of $\mathscr{U}$. Let $\boldsymbol{\mathscr{V}} = \boldsymbol{\mathscr{U}}^H = (\mathrm{Spec}\ \mathbf{U}_i^H \to \mathrm{Spec}\ S^H)$. Then $\boldsymbol{\mathscr{V}}$ is a refinement of $\mathscr{V}$. Let $\mathbb{Z}G/H_2 \to \mathbb{Z}G/H_1$ be a morphism in $\mathscr{H}_G$. For any $\mathbb{Z}G$-module M we have $M^H \cong \mathrm{Hom}_G(\mathbb{Z}G/H, M)$. We have homomorphisms $(\mathbf{U}_i^{H_1})^* \to (\mathbf{U}_i^{H_2})^*$ and these maps induce a homomorphism $\check{H}^n(\boldsymbol{\mathscr{U}}^{H_1}, G_m) \to \check{H}^n(\boldsymbol{\mathscr{U}}^{H_2}, G_m)$. Taking the limit over all covers $\mathscr{V}$ of Spec S^{H_1}

induces $\check{H}^n(S^{H_1}, G_m) \to \check{H}^n(S^{H_2}, G_m)$. Since $(U^H)^* = (U^*)^H = \operatorname{Hom}_G(\mathbb{Z}G/H, U^*)$ we see that the functor is additive.

One can check that the maps defined in Theorem 3 on H^2 restrict to the maps of Corollary 2 on Brauer groups. The restriction and conjugation maps clearly agree. The corestriction map also agrees. Details for the corestriction maps are in [8, 6.2]. Most of the results of Roggenkamp and Scott extend to our context. We restate a few particularly useful propositions in terms of Brauer groups. If R is a commutative ring then the category $\mathscr{H}_{RG}$ is defined by taking permutation modules $RG/H = R \otimes \mathbb{Z}G/H$ as objects and RG-module homomorphisms as maps. It turns out that any contravariant additive functor

$$\Phi : \mathscr{H}_G \to \text{abelian groups}$$

induces a contravariant R-linear functor

$$\Phi_R : \mathscr{H}_{RG} \to R\text{-modules}$$

with $\Phi_R(RG/H) = R \otimes \Phi(\mathbb{Z}G/H)$. By $\hat{\mathscr{H}}_{RG}$ we denote the category of finite direct sums of objects in $\mathscr{H}_{RG}$.

Proposition 4 [11, 4.2.1]. *Let G be a group, R a commutative ring and suppose that we are given a contravariant additive functor*

$$\Phi : \mathscr{H}_G \to \textit{abelian groups}.$$

Extend Φ in the obvious way to the category $\hat{\mathscr{H}}_G$. Then for any objects A, B in $\hat{\mathscr{H}}_G$: if $R \otimes A \cong R \otimes B$, then $R \otimes \Phi(A) \cong R \otimes \Phi(B)$.

If M is a Z-module and p a prime then we denote by $Z_{(p)}$ the local ring at (p) and $M_{(p)} = M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Proposition 5 [11, 4.7.4]. *Let R be a commutative ring and G a finite group of automorphisms of R such that R is Galois over R^G with group G. If H is a subgroup of G and p a prime which does not divide $[G:H]$, then the map $B(R^G)_{(p)} \to B(R^H)_{(p)}$ is injective.*

Proposition 6 [11, 4.7.3]. *Let R and G be as above. Let $\{H_i\}$ be a family of subgroups of G with $\mathrm{GCD}_i\{[G:H_i]\} = 1$. Then the natural map $B(R^G) \to \bigoplus_i B(R^{H_i})$ is injective.*

For example, if R is a subring of $\mathbb{R}[x, y]$, which contains $\mathbb{R}$, then $S = R \otimes_{\mathbb{R}} \mathbb{C}$ is Galois over R with group $\mathbb{Z}/\langle 2 \rangle$. Proposition 5 implies that $B(S/R) = \operatorname{Ker}\{B(R) \to B(S)\}$ is a 2-group since the Brauer group is torsion. If moreover $\mathbb{R}[x, y]$ is a finitely generated R-module, then it is shown in [3] that $B(S) = 0$. In this case we see that $B(R)$ is a 2-group. If R is the subring of $\mathbb{R}[x, y]$ fixed by a group H of $\mathbb{R}$-automorphisms such that H is generated by elements of finite order then the following proposition implies that $B(R) = B(\mathbb{R})$.

The proof of the next proposition does not use the properties of the Hecke category.

Proposition 7. *Let K/k be a Galois extension of fields with finite group G. Let H be a group of k-automorphisms of $k[x_1, \dots, x_n]$ which is generated by elements of finite order. Let R be the subring of $k[x_1, \dots, x_n]$ fixed by H and $S = K \otimes_k R$. Then $B(S/R) \cong B(K/k)$.*

Proof. Since K/k is Galois with group G it follows that S/R is Galois with group G. We have the following spectral sequence [4] relating $H^2(G, S^*)$ and $B(S/R)$:

$$0 \to H^1(G, S^*) \to \operatorname{Pic} R \to (\operatorname{Pic} S)^G \to H^2(G, S^*) \to B(S/R) \to H^2(G, \operatorname{Pic} S).$$

Since H is a group of k-automorphisms, H extends to a group of K-automorphisms of $K[x_1, \dots, x_n]$ and $S = K[x_1, \dots, x_n]^H$. It is shown in [6] that $\operatorname{Pic} S = 0$. Therefore $H^2(G, S^*) \cong B(S/R)$. But $S^* = K^*$, so $B(S/R) \cong H^2(G, K^*) \cong B(K/k)$.

As a final example we show that if G is abelian the p-torsion subgroup of $B(S)$ can be calculated in terms of G, the p-torsion subgroup of $B(R)$ and the p-torsion subgroups of the Brauer groups of subrings of S with Galois group the direct product of a p-group with a cyclic group (see [11, 2.4]). Suppose G contains a subgroup H of order q^2 and exponent q for some $q \neq p$. Let $H_1, \dots, H_{q+1}$ denote the subgroups of H of order q. Then we have an isomorphism of $\mathbb{C}H$ modules

$$\mathbb{C}H \oplus \mathbb{C}^q \cong \mathbb{C}H/H_1 \oplus \cdots \oplus \mathbb{C}H/H_{q+1}.$$

Since p does not divide $|H|$, standard results in representation theory allow us to replace $\mathbb{C}$ in the displayed isomorphism with $\mathbb{Z}_p$, the ring of p-adic integers. Tensor both sides with $\mathbb{Z}_pG$ over $\mathbb{Z}_pH$. We obtain the following isomorphism $\hat{\mathscr{H}}_{\mathbb{Z}_pG}$:

$$\mathbb{Z}_pG/1 \oplus (\mathbb{Z}_pG/H)^q \cong \mathbb{Z}_pG/H_1 \oplus \cdots \oplus \mathbb{Z}_pG/H_{q+1}.$$

Proposition 4 now yields

$$B(S)_p \oplus B(R)_p^q \cong B(S^{H_1})_p \oplus \cdots \oplus B(S^{H_{q+1}})_p.$$

Since the Brauer group of a ring is a torsion group $\mathbb{Z}_p \otimes B(A) = B(A)_p$ is the subgroup of $B(A)$ consisting of elements of order a power of p for any ring A. If we assume that G is abelian, G contains a subgroup H as above unless G is a direct sum of a p-group with a cyclic group.

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