A solution to van Douwen’s problem on Bohr topologies

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Received 12 May 1998; received in revised form 5 June 2000
Communicated by A. Blass

Abstract

We show, in ZFC, that there are two groups of the same cardinality with nonhomeomorphic Bohr topologies. © 2001 Elsevier Science B.V. All rights reserved.

MSC: Primary 05D10; 20K45; secondary 22A05; 54H11

1. Introduction

The Bohr compactification of an abelian group $G$ is a compact group $bG$ that contains $G$ as a dense subgroup such that every homomorphism of $G$ into a compact group $K$ extends to a continuous homomorphism of the group $bG$ in $K$. The Bohr topology of $G$ is the topology induced by the Bohr compactification $bG$. This is precisely the initial topology on $G$ with respect to the family of all homomorphisms of $G$ into the circle group. The group $G$ equipped with the Bohr topology will be denoted by $G^b$ [4]. We answer negatively the following question of van Douwen [3].
Question. If $G$ and $H$ are discrete abelian groups of the same cardinality are then $G^\#$ and $H^\#$ homeomorphic as topological spaces?

The groups in our counterexample are uncountable (see Theorem). A negative answer to this question in the countable case, based on different ideas, was given independently and around the same time by Kunen [6].

For a natural $m>1$ denote by $\mathbb{Z}_m$ the cyclic group of order $m$, $\kappa$ will be a fixed cardinal number and $G_m$ will be the group of functions from $\kappa$ to $\mathbb{Z}_m$ with finite support (i.e., $G_m$ is the direct sum of $\kappa$ many copies of $\mathbb{Z}_m$).

Theorem. If $\kappa>2^{2^{\aleph_0}}$, then there is no homeomorphism between $G^\#_2$ and $G^\#_3$.

We show that every continuous map $G^\#_2 \to G^\#_3$ is constant on an infinite subset of $G_2$ hence, it cannot be a homeomorphism. Our proof is based on a combinatorial lemma (see [9,10]), that permits us to easily apply the elementary convergence properties of the groups $G^\#_2$ and $G^\#_3$ (Lemmas 1–3 and 6).

Of course, the theorem, as well as Kunen’s paper [6] leave many open questions. For example:

- is $G^\#_2$ homeomorphic to $\mathbb{Z}^\#$ when $\kappa$ is countable?
- are $\mathbb{Z}^\#$ and $\mathbb{Q}^\#$ homeomorphic?

It will be nice to study this phenomenon more deeply and classify, up to homeomorphism, all spaces $G^\#$ with $G$ discrete abelian group of a given cardinality (e.g., $G=\mathbb{Z}$ or $G=G_2$). It was proved recently by Comfort, Hernández and Trigos-Arrieta [1,2] that $\mathbb{Q}^\#$ and $\mathbb{Z}^\# \times (\mathbb{Q}/\mathbb{Z})^\#$ are homeomorphic.

2. Convergence in $G^\#_m$

We use the convention that a finite function $\sigma$ from some finite set $F$ of ordinals in $\kappa$ into $\{1\}$ (or into $\{1,2\}$) is to be identified with the function $f$ from $\kappa$ into $\{0,1\}$ (or $\{0,1,2\}$) defined by $f(\alpha)=0$ when $\alpha$ is not in $F$ and by $f(\alpha)=\sigma(\alpha)$ otherwise. We set $\text{supp } f = F$. The group $G_2$ is actually $[\kappa]^{<\omega}$ equipped with the operation symmetric difference. Most often we shall be dealing with the subset $[\kappa]^2$ consisting of all doubletons. Analogously, $[\kappa]^4$ will denote the set of all four-element subsets of $\kappa$.

Since the image of every homomorphism of $G_m$ to the circle group is contained in $\mathbb{Z}_m$, a typical subbasic open set $U_\zeta$ around $0$ in $G^\#_m$ is given by a function $\zeta : \kappa \to m$ and is defined by $U_\zeta = \{ f \in G_2 : f \zeta = 0 \}$ where the multiplication is the inner product as vectors. The characteristic function $\kappa \to m$ of a set $A \subseteq \kappa$, will be denoted by $\zeta_A$.

In the sequel, we consider doubletons $(\alpha, \beta)$ and four-element subsets $(\alpha, \beta, \gamma, \delta)$ of $\kappa$ which will be identified as above with elements of $G_2$. In such a case we always assume that $\alpha<\beta$ and $\alpha<\beta<\gamma<\delta$, respectively.

We begin with description of the nets of doubletons converging to 0 in $G^\#_2$. 
Lemma 1. For a net \( \{(z_d, \beta_d) : d \in D\} \) of doubletons of \( \kappa \) the following are equivalent:

- the net \( S = \{(z_d, \beta_d) : d \in D\} \) converges to 0 in the Bohr topology of \( G_2 \);
- for any \( A \subset \kappa \), there is \( d' \in D \) such that for all \( d > d' \), either \( \{z_d, \beta_d\} \subset A \) or \( \{z_d, \beta_d\} \cap A = \emptyset \).

Proof. It suffices to observe that a tail of the net \( S \) lies inside \( U \), iff a tail of the net consists of pairs whose ordinals are either both in \( A \) or both outside \( A \).

Lemma 2. Let \( S = \{(x, \beta, \gamma, \delta)\} \) be a net of four-element sets. If the corresponding nets \((x, \gamma)\) and \((\beta, \delta)\) converge in the Bohr topology to 0, then the net \( S \) converges in the Bohr topology to 0.

Proof. It suffices to apply Lemma 1 noting that the Bohr topology is a group topology, and \((x, \beta, \gamma, \delta) = (x, \beta) + (\gamma, \delta)\).

Lemma 3. For every partition \( Z' \cup Z'' \) of \( \omega \) into infinite disjoint subsets \( Z', Z'' \) there are nets \((z, \beta, \gamma, \delta)\) such that

1. \( z, \gamma \in Z' \), \( \beta, \delta \in Z'' \) and the corresponding nets \((x, \gamma)\) and \((\beta, \delta)\) converge to 0 in \( G_2^\# \).
2. \( z, \delta \in Z' \), \( \beta, \gamma \in Z'' \) and the corresponding nets \((x, \delta)\) and \((\beta, \gamma)\) converge to 0 in \( G_2^\# \).
3. \( z, \beta \in Z' \), \( \gamma, \delta \in Z'' \) and the corresponding nets \((x, \beta)\) and \((\gamma, \delta)\) converge to 0 in \( G_2^\# \).

Proof. Just use finite families \( \mathcal{F} \) of sets \( A \subset \omega \) ordered by inclusion as the index set \( D \). Then, for any particular \( \mathcal{F} \), it is possible to choose infinite \( W' \subset Z' \) and \( W'' \subset Z'' \) so that each of \( W' \) and \( W'' \) lie entirely inside or outside each \( A \in \mathcal{F} \). This can be easily proved by induction on \( |\mathcal{F}| \). Then for (1) choose \( z < \beta < \gamma < \delta \) for this particular \( \mathcal{F} \) so that \( z, \gamma \in W' \) and \( \beta, \delta \in W'' \). This is possible since \( W' \) and \( W'' \) are infinite. Now it is easy to check with Lemma 1 that the corresponding nets \((x, \gamma)\) and \((\beta, \delta)\) converge to 0 in \( G_2^\# \).

Define analogously \( z < \beta < \gamma < \delta \) in the other two cases.

Now we describe the nets converging to 0 in \( G_3^\# \) and their splitting in the following sense.

Definition 4. Suppose that \( \{n_d : d \in D\} \) is a net in \( G_3^\# \). We say that \( n_d \) splits into a sum of nets \( m_d^{(i)} \), \( i = 1, \ldots, n \), if:

1. \( n_d = \sum_{i=1}^n m_d^{(i)} \) for every \( d \);
2. \( \text{supp } m_d^{(i)} \cap \text{supp } m_d^{(j)} = \emptyset \) for \( i \neq j \) and every \( d, e \in D \).

Note that property (2) (all “cross-intersections” are empty) implies the weaker property “all supports \( \text{supp } m_d^{(i)} \) in \( n_d \) are pairwise disjoint for every \( d \in D \).
Definition 5. A family \( \{A_i\}_{i \in I} \) of subsets of some set \( X \) is called weakly disjoint, if for every \( i, j \in I \) the sets \( A_i \) and \( A_j \) are either disjoint or coincide.

Clearly, every family of singletons is weakly disjoint.

Lemma 6. (1) Let \( m_d \to 0 \) in \( G_3^p \). If \( \{\text{supp } m_d : d \in D\} \) is a weakly disjoint family, then \( m_d = 0 \) for some tail of the net.

(2) Assume \( n_d \to 0 \) and there is a splitting \( n_d = \sum_{i=1}^{n} m_{d}^{(i)} \). Then \( m_{d}^{(i)} \to 0 \) for every \( i \). Moreover, if \( \text{supp } m_{d}^{(i)} \cap \text{supp } m_{d}^{(i)} = \emptyset \) for some \( i \) and for every \( d \neq e \) in \( D \), then \( m_{d}^{(i)} = 0 \) on some tail of the net.

Proof. (1) Assume that \( \text{supp } m_d \neq \emptyset \) for cofinally many \( d \in D' \subseteq D \) and choose for those \( d \) the least point \( p_d \in \text{supp } m_d \). Since \( \{\text{supp } m_d : d \in D'\} \) form a weakly disjoint family, for \( d \neq d' \) either \( m_d \) and \( m_{d'} \) have disjoint supports, or \( p_d = p_{d'} \). Now for \( P = \{p_d : d \in D'\} \) and for every \( d \in D' \) \( \zeta_p(n_d) = n_d(p_d) \) is always one or two, so that the subnet \( \{n_d : d \in D'\} \) cannot converge to 0, a contradiction.

(2) Clearly, it suffices to consider the case of splitting in two nets. So we have to prove that if \( n_d \) splits in \( n_d = m_d + m_d' \), then \( n_d \to 0 \) implies both \( m_d \to 0 \) and \( m_d' \to 0 \). Set \( S = \bigcup_{d \in D} \text{supp } m_d \) and \( S' = \kappa \setminus S \). Let \( G_3 = L \oplus L' \) be the splitting of \( G_3 \) defined by \( L = \bigoplus_{i \in \kappa} \mathbb{Z}_3 \) and \( L_1 = \bigoplus_{i \in S} \mathbb{Z}_3 \). Then \( m_d \in L \) and \( m_d' \in L' \) for every \( d \in D \). For every character \( \chi : G_3 \to \mathbb{Z}_3 \) denote by \( \chi_1 \) the composition of \( \chi \) with the natural injection of \( L \hookrightarrow G_3 \) (i.e., \( \chi_1 = \chi | L \)). Since \( m_d \) vanishes on \( S' \) one has \( \chi(m_d) = \chi_1(m_d) \). Now \( \chi_1 \) vanishes on \( L' \), therefore \( \chi_1(m_d') = 0 \) and consequently, \( \chi(m_d) = \chi_1(n_d) \). The Bohr convergence of the net \( n_d \) yields \( \chi_1(n_d) \to 0 \). Hence \( \chi(m_d) \to 0 \) too. Therefore, \( m_d \to 0 \) in \( G_3^p \).

The final part of (2) follows from (1). \( \square \)

3. Proof of Theorem

For \( n \in \mathbb{N} \) we denote by \( [G_3]_n \) the set of finite functions in \( G_3 \) with support of size \( n \). In particular, we set for completeness \( [G_3]_0 = \{0\} \) and \( [S]_0 = \{0\} \) when we refer to these sets as subsets of the groups \( G_2 \) and \( G_3 \).

Definition 7. Let \( S \subseteq \kappa \) and let \( k, n, k_1, k_2 \in \mathbb{N} \).

(a) Two functions \( f_1 : [S]^{k_1} \to [G_3]_n \) and \( f_2 : [S]^{k_2} \to [G_3]_n \) are disjoint if \( \text{supp } f_1(A_1) \cap \text{supp } f_2(A_2) = \emptyset \) for every \( A_1 \in [S]^{k_1} \) and \( A_2 \in [S]^{k_2} \) (with \( A_1 \neq A_2 \) in case \( k_1 = k_2 \)).

(b) A function \( f : [S]^k \to [G_3]_n \) is standard, if \( \text{supp } f(A) \cap \text{supp } f(A') = \emptyset \) for \( A \neq A' \in [S]^k \).

Note that every constant function is standard (take \( k = 0 \) in (b)). The standard functions present a certain prototype of a “base” for continuous maps \( G_2^p \to G_3^p \) in some sense (see Section 3.1). As a first step we show in the next claim an “independence”
property of standard 1-variable functions with respect to convergence to 0 (if \( \tau_1(\alpha) + \tau_2(\beta) \to 0 \) for every net \((\alpha, \beta)\) converging to 0 in \( G_2^n \), then \( \tau_1, \tau_2 \) are linearly dependent on a cofinite subset of \( \kappa \)).

**Claim 8.** Let \( n, n_1, n_2 \in \mathbb{N} \) and let \( S \subseteq \kappa \) be an infinite set:

(i) If \( \tau : S \to [G_3]_{n_1} \) is standard and \( \tau(\alpha) \to 0 \) for some net \( \alpha \) in \( S \), then \( \tau \) vanishes.

(ii) If \( \tau_1 : S \to [G_3]_{n_1}, \tau_2 : S \to [G_3]_{n_2} \) are disjoint standard functions such that \( n_{\alpha, \beta} = \tau_1(\alpha) + \tau_2(\beta) \to 0 \) for every net \((\alpha, \beta)\) converging to 0 in \( G_2^n \), then there exists a cofinite subset \( S' \) of \( S \) where \( \tau_2 = -\tau_1 \).

**Proof.** (i) If \( n = 0 \) there is nothing to prove. Assume for a contradiction that \( n > 0 \). Hence \( \tau(\alpha) \neq 0 \) for every \( \alpha \in S \). Since \( \tau \) is standard, the family \( \tau(\alpha) \) must be weakly disjoint. Hence, Lemma 6 yields \( \tau(\alpha) = 0 \) on a tail of the net – a contradiction.

(ii) If \( n_1 = n_2 = 0 \), then \( \tau_1 = \tau_2 = 0 \), so take \( S' = S \). Assume \( n_1 > 0 \). Hence \( \tau_1(\alpha) \neq 0 \) for every \( \alpha \in S \). This yields \( n_2 > 0 \) too. Indeed, if \( \tau_2 = 0 \) then \( \tau_1(\alpha) \to 0 \) for a net \((\alpha, \beta)\) converging to 0 in \( G_2^n \), then \( \tau_2 = -\tau_1 \).

Let \( S' = \{ \gamma \in S : \tau_1(\gamma) \neq -\tau_2(\gamma) \} \). We show that \( S' \) is finite. Assume that \( S' \) is infinite and take a net \((\alpha, \beta)\) in \( S' \) converging to 0 in the Bohr topology. Let \( S_0 = \text{supp} \tau_1(\gamma) \cup \text{supp} \tau_2(\gamma) \). Then the family \( \{ S_\gamma : \gamma \in S \} \) is disjoint. Moreover, if \( \gamma \in S' \) then \( \tau_1(\gamma) = -\tau_2(\gamma) \), so that either \( \tau_1(\gamma) = \tau_2(\gamma) \) or \( \tau_1(\gamma) \) and \( \tau_2(\gamma) \) are linearly independent over \( \mathbb{Z}_3 \) since both they are non-zero. Therefore, for the subgroup \( H_\gamma = \langle \tau_1(\gamma), \tau_2(\gamma) \rangle \) of \( G_3 \) one can find a character \( \xi_\gamma : H_\gamma \to \mathbb{Z}_3 \) such that \( \xi_\gamma(\tau_1(\gamma)) = \xi_\gamma(\tau_2(\gamma)) = 1 \). For \( \gamma \not\in S' \) set \( \xi_\gamma = 0 \). Since the subgroups \( H_\gamma \) form an independent family, there exists a character \( \xi : \mathbb{H} = \bigoplus H_\gamma \to \mathbb{Z}_3 \) that extends all \( \xi_\gamma \). Since \( H \) is a direct summand of \( G_3 \), we can extend \( \xi \) to a character \( \xi' \) of \( G_3 \). Now \( \xi(n_{\alpha, \beta}) = \xi(\tau_1(\alpha) + \tau_2(\beta)) = 2 \to 0 \) – a contradiction.

3.1. The resolving set and the Combinatorial Lemma

The key point of our proof is that the standard functions are also sufficient to ensure that every function \([\kappa]^k\) to \([G_3]_{n_1}\) can be spanned (in appropriate sense) in a sum of pairwise disjoint standard functions. To give a more precise meaning of this phenomenon we need the following definition.

**Definition 9 (Resolving Set).** Let \( n > 0, \kappa \) be an infinite cardinal and let \( \pi \) be any map from \([\kappa]^{\ell}\) to \([G_3]_{n_1}\). A subset \( Z \subseteq \kappa \) is resolving for \( \pi \), if there are 16 standard pairwise disjoint standard functions \( \sigma_{i,j,k,l} \) defined for every \( i,j,k,l = 0,1 \) such that \( \sigma_{i,j,k,l} : [Z]^{i+j+k+l} \to [G_3]_{n_{i,j,k,l}} \) and

\[
\pi(\alpha, \beta, \gamma, \delta) = \sigma_{1111}(\alpha, \beta, \gamma, \delta) + \sigma_{1110}(\alpha, \beta, \gamma) + \sigma_{1101}(\alpha, \beta, \delta) + \sigma_{1011}(\alpha, \gamma, \delta) + \sigma_{0111}(\beta, \gamma, \delta) + \sigma_{1100}(\alpha, \beta) + \sigma_{1010}(\alpha, \gamma) + \sigma_{0110}(\beta, \gamma)
\]


\[
+ \sigma_{1001}(\alpha, \delta) + \sigma_{0101}(\beta, \delta) + \sigma_{0011}(\gamma, \delta) + \sigma_{1000}(\alpha) \\
+ \sigma_{0100}(\beta) + \sigma_{0010}(\gamma) + \sigma_{0001}(\delta) + \sigma_{0000}.
\]

for every \( \alpha < \beta < \gamma < \delta \) in \( Z \).

According to our convention, \( \sigma_{0000} \) is a constant function.

The following combinatorial lemma ensures the existence of infinite resolving set.

**Lemma 10** (Combinatorial Lemma). If \( \kappa \) is countable and \( \pi \) is any map from \( [\kappa]^4 \) to \( [G_3]^n \), for some \( n \in \mathbb{N} \), then there is an infinite resolving set \( Z \) of \( \pi \).

The Combinatorial Lemma will be proved below, we give first the following important consequence.

**Main Lemma.** Let \( \pi : G_2^2 \rightarrow [G_3]^n \) be a continuous map with \( \pi(0) = 0 \), sending \( [\kappa]^4 \) to \( [G_3]^n \) for some \( n \in \mathbb{N} \). If \( Z \) is an infinite resolving set of \( \pi \), then \( \pi \) vanishes on \( [Z]^4 \).

The proof of the Main Lemma is deferred to Section 3.3. Let us see now how the Theorem can be deduced from the Main Lemma.

**Proof of Theorem.** Recall that according to Erdős–Rado’s theorem ([5, Example 29.1]) every coloring of the set of 4-tuples of a set of size \( > \omega \) with countably many colors admits a homogeneous set \( H \) of size \( > \omega \), i.e., the induced coloring of \( [H]^4 \) is constant (in one color). Therefore, our hypothesis \( \kappa > 2^\omega \) permits to conclude that for every continuous map \( \pi : G_2 \rightarrow G_3 \) there exists an uncountable \( H \subseteq \kappa \) such that all supports of \( \pi \) on \( [H]^4 \) have the same size \( n \), i.e., \( \pi : [H]^4 \rightarrow [G_3]^n \). If \( n = 0 \), we are done. If \( n > 0 \) apply the Combinatorial Lemma to get an infinite resolving set \( Z \subseteq H \).

By the Main Lemma \( \pi \) vanishes on \( [Z]^4 \). \( \Box \)

3.2. **Proof of the Combinatorial Lemma 10**

Probably this lemma is known in set theory (see [7] for a simpler version). Nevertheless, we give here a complete proof for maps from \( [\kappa]^2 \) to \( G_3 \) (the case of maps \( [\kappa]^1 \rightarrow G_3 \) trivially follows from the Delta-lemma [5]). The proof for the version of maps from \( [\kappa]^4 \) to \( G_3 \) is essentially the same, requiring only a more careful and lengthy book-keeping.

So we have to show that for every map \( \pi : [\kappa]^2 \rightarrow [G_3]^n \) there exists an infinite subset \( Z \subseteq \kappa \) and four pairwise disjoint standard functions \( \sigma_{ij} \) (\( i, j = 0, 1 \)), such that \( \sigma_{00} \) is constant,

1. \( \sigma_{10} : S \rightarrow [G_3]^n \) and \( \sigma_{01} : S \rightarrow [G_3]^n \) are one-variable functions \((k_1, k_2 \in \mathbb{N})\),
2. \( \sigma_{11} : [S]^2 \rightarrow [G_3]^n \) \((k \in \mathbb{N})\), and
3. \( \pi(x, \beta) = \sigma_{00}(x, \beta) + \sigma_{10}(x) + \sigma_{01}(\beta) + \sigma_{11}(x, \beta) \) for all \( x < \beta \) in \( Z \).

If \( n = 0 \), then \( \pi \) vanishes on \( [\kappa]^2 \), so we can set \( \sigma_{00} = \pi |_{[\kappa]^2} = 0 \) and \( \sigma_{01} = \sigma_{10} = \sigma_{11} = 0 \).
Assume $n > 0$ and denote for brevity the composition $\sigma \circ \pi$ by $\rho : [\kappa]^2 \to [\kappa]^n$. Since the main properties of the functions $\sigma_{ij}$ ($i, j = 0, 1$) concern their supports, we shall be working mainly with $\rho$. For a non-empty subset $I = \{i_1, \ldots, i_s\}$ of the index set $\{1, 2, \ldots, n\}$ denote by $p_i : [\kappa]^n \to [\kappa]^i$ the natural projection sending the $n$-tuple $x_1 < \cdots < x_n$ to the $s$-tuple $x_{i_1} < \cdots < x_{i_s}$. For $I = \{i\}$ we write simply $p_i : [\kappa]^n \to [\kappa]$ for the projection on the $i$th component. Finally, set $\rho_i = p_i \circ \rho$ and $\rho_I = p_I \circ \rho$.

For two $n$-tuples $A, A' \in [\kappa]^n$ define the incidence matrix $e(A, A') = (e_{ij}(A, A'))$ of $A$ and $A'$ by setting

$$e_{ij}(A, A') = \begin{cases} 1 & \text{if } p_i(A) = p_j(A'), \\ 0 & \text{if } p_i(A) \neq p_j(A'). \end{cases}$$

(1) for every $i, j = 1, 2, \ldots, n$. This (symmetric) matrix describes the precise positions where the intersection of the $n$-tuples $A = \{a_1, \ldots, a_n\}$, $A' = \{a'_1, \ldots, a'_n\}$ occurs. Indeed, $A \cap A' \neq \emptyset$ iff there exists $0 < s \leq n$ and indices $1 \leq i_1 < \cdots < i_s \leq n$, $1 \leq j_1 < \cdots < j_s \leq n$ such that $a_{i_v} = a'_{j_v}$ for $v = 1, 2, \ldots, s$, i.e., $e_{i_vj_v}(A, A') = 1$ for all $v = 1, 2, \ldots, s$ and all others are 0.

Now take $x < y < u < v$ in $\kappa$ and consider the intersection between the $n$-tuples $\rho(x, y)$ and $\rho(u, v)$ making recourse to the above defined matrix. We consider different cases depending on the configuration of the two disjoint pairs (the arguments of $\rho$) in the quadruple $(x, y, u, v)$:

1. $\chi(x, y, u, v) = e(\rho(x, y), \rho(u, v))$ ("disjoint" pairs);
2. $\chi'(x, y, u, v) = e(\rho(x, u), \rho(y, v))$ ("overlapping" pairs);
3. $\chi''(x, y, u, v) = e(\rho(x, v), \rho(y, u))$ ("nested" pairs).

These 3 matrices determine a $2^{3n^2}$-coloring of $[\kappa]^4$. By Ramsey theorem there exists an infinite homogeneous set $S_1 \subseteq [\kappa]$, where these three matrices are constant (i.e., all values $\chi_{ij}$, $\chi'_{ij}$, $\chi''_{ij}$ depend only on the indices $i, j$, but not on the choice of the quadruple $(x, y, u, v)$).

Next, we consider all triples $x < y < z$ in $S_1$ and define analogously three matrices $\Xi = (\xi_{ij})$, $\Xi' = (\xi'_{ij})$ and $\Xi'' = (\xi''_{ij})$:

1. $\Xi(x, y, z) = e(\rho(x, y), \rho(y, z))$ (mixed pairs);
2. $\Xi'(x, y, z) = e(\rho(x, y), \rho(x, z))$ (pairs with the same first coordinate);
3. $\Xi''(x, y, z) = e(\rho(x, z), \rho(y, z))$ (pairs with the same second coordinate).

These 3 matrices determine a $2^{3n^2}$-coloring of $[S_1]^3$. By Ramsey theorem there exists an infinite homogeneous set $S \subseteq S_1$, where these three matrices are constant. In other words, all six matrices $X, X', X'', \Xi, \Xi'$ and $\Xi''$ are constant on $[S]^2$.

Now we show that the non-zero entries in the matrices $X, X', X'', \Xi, \Xi'$ and $\Xi''$ are placed on the diagonal, i.e., if some of the constants $\xi_{ij}$, $\xi'_{ij}$, $\xi''_{ij}$, $\xi_{ij}$, $\xi'_{ij}$, $\xi''_{ij}$ takes value 1, then necessarily $i = j$. Indeed, otherwise take three distinct pairs $(x, y)$, $(u, v)$, and $(u', v')$ such that pairwise they are all in the same required configuration (1)–(3) or (ii)–(iii) depending on the matrix. Then we have

$$\rho_i(x, y) = \rho_j(u, v) = \rho_1(u', v') = \rho_j(x, y),$$
a contradiction, since \( \rho_i(x, y) \) and \( \rho_j(x, y) \) are distinct coordinates of the same \( n \)-tuple \( \rho(x, y) \).

It should be noted here that the case \( \xi_{ij} = 1 \), with \( i \neq j \), cannot be excluded, as it may occur (see (d) and Example 11 below).

Now assume that \( \chi_{ii} = 1 \) for some \( i \). Then \( \rho_i \) is constant on \( S \). In fact, fix a pair \((x_0, y_0)\) and set \( z_0 = \rho_i(x_0, y_0) \). Obviously \( \rho_i(x, y) = \rho_i(x_0, y_0) = z_0 \) for all pairs \((x, y)\) that are “disjoint” with \((x_0, y_0)\). In the general case, take a third pair \((x', y')\) “disjoint” from both pairs \((x, y)\) and \((x_0, y_0)\). Then \( \rho_i(x, y) = \rho_i(x', y') = z_0 \) is constant. This implies that also the entries \( \chi_{ii}', \chi_{ii}'', \xi_{ii}, \xi_{ii}', \xi_{ii}''' \) in the remaining matrices have value 1.

Analogously, one proves that \( \rho_i \) is constant on \([S]^2\) whenever some of the constants \( \chi_{ii}', \chi_{ii}'', \xi_{ii}, \xi_{ii}', \xi_{ii}''' \) is 1. In such a case the \( ii \)-entries of the remaining 5 matrices have value 1.

This completely determines the matrices \( \chi, \chi' \) and \( \chi'' \).

Next assume that \( \chi_{ii} = \chi_{ii}' = \chi_{ii}'' = \xi_{ii} = 0 \) and \( \xi_{ii}' = 1 \). Then \( \xi_{ii}'' = 0 \). Indeed, if \( \xi_{ii}'' = 1 \), then \( \rho_i(x, z) = \rho_i(y, z) \) for all triples \( x < y < z \). On the other hand, \( \xi_{ii}' = 1 \) yields \( \rho_i(x, y) = \rho_i(x, z) \) for all triples \( x < y < z \). This implies that \( \rho_i(x, y) = \rho_i(y, z) \), so \( \xi_{ii} = 1 \) a contradiction. This proves also that the function \( x \mapsto \rho_i(x, y) \) is injective.

Analogously, \( \chi_{ii}' = \chi_{ii}'' = \xi_{ii} = 0 \) and \( \xi_{ii}' = 1 \) yield \( \xi_{ii}'' = 0 \) and injectivity of the function \( y \mapsto \rho_i(x, y) \).

This defines four sets of indices:

- \( I_{00} = \{ i \in \{1, 2, \ldots, n\} : \chi_{ii} = \chi_{ii}' = \chi_{ii}'' = \xi_{ii} = \xi_{ii}' = \xi_{ii}'' = 1 \} \);
- \( I_{10} = \{ i \in \{1, 2, \ldots, n\} : \chi_{ii} = \chi_{ii}' = \chi_{ii}'' = \xi_{ii} = 0, \xi_{ii}' = 1 \} \);
- \( I_{01} = \{ i \in \{1, 2, \ldots, n\} : \chi_{ii} = \chi_{ii}' = \chi_{ii}'' = \xi_{ii}' = 0, \xi_{ii} = 1 \} \);
- \( I_{11} = \{ i \in \{1, 2, \ldots, n\} : \chi_{ii} = \chi_{ii}' = \chi_{ii}'' = \xi_{ii}' = \xi_{ii} = 0 \} \).

Note, that \( I_{11} = \{1, 2, \ldots, n\} \setminus (I_{00} \cup I_{10} \cup I_{01}) \). Furthermore,

- (a) for \( i \in I_{10} \) the function \( x \mapsto \rho_i(x, y) \) is injective,
- (b) for \( i \in I_{01} \) the function \( y \mapsto \rho_i(x, y) \) is injective,
- (c) for \( i \in I_{11} \) the function \( (x, y) \mapsto \rho_i(x, y) \) is injective,
- (d) if \( \xi_{ii} = 1 \), then \( j \in I_{10} \) and \( i \in I_{01} \) (indeed, for every \( x < y < z \) and \( x' < y < z' \) we have \( \rho_i(x', y) = \rho_i(y, z) = \rho_i(x, y) \), so \( i \in I_{01} \), and \( \rho_j(y, z') = \rho_i(x, y) = \rho_i(y, z) \) so \( j \in I_{10} \).

Now take the maximal constant part \( \rho(x, y) \upharpoonright p_{I_{10}} \) of the support of \( \pi \) as “the root” of \( \pi(x, y) \) and define \( \sigma_{00}(x, y) \) for every \( x < y \) in \( S \) as the restriction of \( \pi(x, y) \) to the root. Clearly, the support of the function \( \sigma_{00} \) is constant, but the function itself need not be constant. Since there are only finitely many functions with that fixed finite domain, by a further application of Ramsey theorem we can get an infinite subset \( Z \) of \( S \) such that the function \( \sigma_{00} \) is constant on \([Z]^2\). Consequently, \( \sigma_{00} \) is a standard function.

Define the remaining \( \sigma_{ij} \) analogously. Let \( k_1 = |I_{10}| \). First note that the map \([Z]^2 \rightarrow [G_3]_{k_1}\) defined by \( (x, y) \mapsto \pi(x, y) \upharpoonright p_{I_{10}} \) for every \( x < y \) in \( Z \) depends only on the first coordinate \( x \). Denote by \( \sigma_{10}(x) \) the so defined function \( Z \rightarrow [G_3]_{k_1} \) of \( x \). Analogously define \( \sigma_{01}(y) \) as the restriction \( \pi(x, y) \upharpoonright p_{I_{10}} \) of this gives the second one-variable function \( \sigma_{01} : Z \rightarrow [G_3]_{k_1} \), where \( k_2 = |I_{01}| \). Finally, define \( \sigma_{11} : [Z]^2 \rightarrow [G_3]_{k} \), with \( k = |I_{11}| \) by
setting $\sigma_{11}(x, y) = \pi(x, y) \cup p_{11}$ for every $x < y$ in $Z$. Note that these four functions are pairwise disjoint and satisfy (1)–(3) by their definition.

It remains to verify that the functions $\sigma_{10}, \sigma_0$, and $\sigma_{11}$ are standard. In order to do it for $\sigma_{10}$ observe that if $\text{supp } \sigma_{10}(x) \cap \text{supp } \sigma_{10}(y) \neq \emptyset$ for some $x \neq y$, then there exists $i, j \in I_{10}$ and $t, z$ such that $\rho_i(x, t) = \rho_j(y, z)$. By (a) we may assume $i \neq j$. As $\Xi$ is the only matrix that may have non-zero entries out of the diagonal, we conclude that $\xi_{ij} = 1$ occurs, that leads to a contradiction by (d). Hence, $\sigma_{10}$ is standard. Analogously, one checks that $\sigma_0$ and $\sigma_{11}$ are standard applying (b) and (c), respectively, along with (d).

**Example 11.** The function $\mu : [\kappa]^2 \rightarrow [G_3]_2$ defined by $\mu(x, y) = (x, 1) - (\beta, 1)$ is not standard. Its presentation via the Combinatorial Lemma is $\mu(x, y) = \sigma(x) - \sigma(y)$, where $\sigma(x) = (x, 1)$ is standard. Now for every choice of the subset $Z \subseteq \kappa$ one obviously has the matrices

$$X = X' = X'' = 0, \quad \Xi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Xi' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Xi'' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$  

3.3. Proof of the main lemma

Since the functions $\sigma_{ijkl}$ have supports of uniform size, it is not restrictive to assume that $Z$ has type $\omega$. For the same reason it suffices to find just one zero value of the function $\sigma_{ijkl}$ in order to conclude it vanishes on $[Z]^{i+j+k+l}$.

First take a partition of $Z$ into a union of infinite disjoint sets $Z'$ and $Z''$ as in Lemma 3. Now find a net $(x, \beta, \gamma, \delta)$ such that $x, \beta, \gamma, \delta \in Z'$ and the corresponding nets $(x, \gamma)$ and $(\beta, \delta)$ converge to $0$ in $G_{\mu}^\kappa$, so that by Lemma 2, the net $(x, \beta, \gamma, \delta)$ Bohr-converges to $0$ in $G_{\mu}^\kappa$. This can be arranged by Lemma 3.

We have a splitting of $\pi$ as in Definition 4:

$$\pi(x, \beta, \gamma, \delta) = \sigma_{1111}(x, \beta, \gamma, \delta) + m(x, \beta, \gamma, \delta) + n(x, \beta, \gamma, \delta) + n'(x, \beta, \gamma, \delta) + k(x, \beta, \gamma, \delta)$$

$$+ k'(x, \beta, \gamma, \delta) + \sigma_{0000},$$

(2)

where

$$m(x, \beta, \gamma, \delta) = \sigma_{1110}(x, \beta, \gamma) + \sigma_{1101}(x, \beta, \gamma) + \sigma_{1011}(x, \gamma, \delta) + \sigma_{0111}(\beta, \gamma, \delta),$$

$$n(x, \beta, \gamma, \delta) = \sigma_{1100}(x, \beta) + \sigma_{0011}(\gamma, \delta) + \sigma_{1001}(x, \delta) + \sigma_{0110}(\beta, \gamma),$$

$$n'(x, \beta, \gamma, \delta) = \sigma_{1010}(x, \gamma) + \sigma_{0101}(\beta, \delta),$$

$$k(x, \beta, \gamma, \delta) = \sigma_{1000}(x) + \sigma_{0010}(\gamma) \quad \text{and} \quad k'(x, \beta, \gamma, \delta) = \sigma_{0001}(\beta) + \sigma_{0001}(\delta).$$

Now continuity of $\pi$ gives $\pi(x, \beta, \gamma, \delta) \rightarrow 0$, hence by Lemma 6 we have:

(a) $\sigma_{1111}(x, \beta, \gamma, \delta) \rightarrow 0$, consequently we conclude $\sigma_{1111}(x, \beta, \gamma, \delta) = 0$ on a tail of the net (since this is a net with pairwise disjoint supports, cf. (2) of Lemma 6).

(b) $\sigma_{0000} = 0$ as a constant net converging to 0.

(c) $m(x, \beta, \gamma, \delta) \rightarrow 0, n(x, \beta, \gamma, \delta) \rightarrow 0, n'(x, \beta, \gamma, \delta) \rightarrow 0, k(x, \beta, \gamma, \delta) \rightarrow 0$ and $k'(x, \beta, \gamma, \delta) \rightarrow 0$. 
With (a) and (b) we have
\[ \sigma_{1111} = \sigma_{0000} = 0. \]  
(3)

Now we consider step by step the consequences of the five limits 0 in (c).

**Step 1:** Let us see that \( m(x, \beta, \gamma, \delta) \to 0 \) implies that all four 3-variable functions vanish on \( Z \). Indeed, let us see first that \( m(x, \beta, \gamma, \delta) \) is splitting in the direct sum of its four components in the sense of Definition 4. Assume, for example, that for some \( x', \beta' < \gamma' < \delta' \) one has \( \text{supp} \sigma_{1110}(x, \beta, \gamma) \cap \text{supp} \sigma_{1101}(x', \beta', \delta') \neq \emptyset \). Then this yields \( \{x, \beta, \gamma\} = \{x', \beta', \delta'\} \). But the left hand side triple has two elements from \( Z' \), while the right hand side one has only one such element – a contradiction. Similar argument shows that the other cross-intersections cannot occur. This proves that \( m(x, \beta, \gamma, \delta) \) splits. Now Lemma 6 yields \( \sigma_{1110}(x, \beta, \gamma) \to 0, \sigma_{1101}(x, \beta, \delta) \to 0, \) etc. Let us note now that \( \sigma_{1110}(x, \beta, \gamma) \) is a weakly disjoint family when \( (x, \beta, \gamma, \delta) \) varies as described above. Now by Lemma 6 again we conclude that \( \sigma_{1110}(x, \beta, \gamma) \) vanishes on some tail of the net. By the “uniformity” properties of our functions, this gives \( \sigma_{1110} = 0 \) on \( Z \). Similar argument shows
\[ \sigma_{1110} = \sigma_{1011} = \sigma_{0110} = \sigma_{0110} = 0 \quad \text{on Z}. \]  
(4)

**Step 2:** Now, we see that \( n(x, \beta, \gamma, \delta) \to 0 \) and \( n'(x, \beta, \gamma, \delta) \to 0 \) imply that three of the 2-variable functions vanish on \( Z \).

Note that the net \( n(x, \beta, \gamma, \delta) \to 0 \) splits in \( n(x, \beta, \gamma, \delta) = s(x, \beta, \gamma, \delta) + \sigma_{0110}(\beta, \gamma) \), where \( s(x, \beta, \gamma, \delta) = \sigma_{1100}(x, \beta) + \sigma_{0011}(\gamma, \delta) + \sigma_{1001}(x, \delta) \). This follows from the fact that for \( (x', \beta', \gamma', \delta') \neq (x, \beta, \gamma, \delta) \) the support of \( s(x, \beta, \gamma, \delta) \) can meet the support of \( \sigma_{0110}(\beta', \gamma') \) only if \( (\beta', \gamma') \) coincides with some of the pairs \( (x, \beta), (\gamma, \delta), (x, \delta) \). But \( \beta' \in Z'' \) and \( \gamma' \in Z' \), while none of the other three pairs has this property (i.e., to have its first element in \( Z'' \) and the second one in \( Z' \)). Hence, another application of the lemma gives \( \sigma_{0110}(\beta, \gamma) \to 0 \). Since these supports form a weakly disjoint family, we conclude that then \( \sigma_{0110} \) must vanish on a tail of the net \( (\beta, \gamma) \) by (1) of Lemma 6. By uniformity of \( \sigma_{0110} \) this proves
\[ \sigma_{0110} = 0 \quad \text{on Z}. \]  
(5)

Now the net \( n'(x, \beta, \gamma, \delta) \to 0 \) splits, so that Lemma 6 gives \( \sigma_{1010}(x, \gamma) \to 0 \) and \( \sigma_{0101}(\beta, \delta) \to 0 \). As before, this gives
\[ \sigma_{1010} = \sigma_{0101} = 0 \quad \text{on Z}. \]  
(6)

**Step 3:** Now, we start dealing with the 1-variable functions. By Claim 8 the converging net \( k_{(x, \beta, \gamma, \delta)} = \sigma_{1000}(x) + \sigma_{0010}(\gamma) \to 0 \) gives
\[ \sigma_{0010}(\gamma) = - \sigma_{1000}(\gamma) \]  
(7)

for all \( \gamma \) from a cofinite subset \( Z_1' \) of \( Z' \). Analogously, the converging net \( k'_{(x, \beta, \gamma, \delta)} = \sigma_{0100}(\beta) + \sigma_{0001}(\delta) \to 0 \) gives
\[ \sigma_{0001}(\delta) = - \sigma_{0100}(\delta) \]  
(8)

for all \( \delta \) from a cofinite subset \( Z_2'' \) of \( Z'' \). Taking another net \( x < \beta < \gamma < \delta \) where the roles of \( Z' \) and \( Z'' \) are exchanged, we can find a cofinite subset \( Z_2' \) of \( Z' \) such that (8)
holds for all $\delta \in Z_2'$. Analogously, there exists a cofinite subset $Z_2''$ of $Z''$ such that (7) holds for every $\gamma \in Z_2''$. Now for the cofinite subset $Z_1 = (Z_1' \cap Z_1'') \cup (Z_1'' \cap Z_2'')$ of $Z$ clearly both (7) and (8) hold. From now on we shall work on $Z_1$ assuming for simplicity that $Z_1 = Z$, i.e., both (7) and (8) hold on $Z$.

Summing up (3)–(8), we see that we are left with

$$\pi(x, \beta, \gamma, \delta) = \sum_{i=0}^{9} \sigma_{i000}(x) + \sigma_{0100}(\beta) - \sigma_{0100}(\gamma) - \sigma_{0100}(\delta).$$

**Step 4**: In order to eliminate the remaining 2-variable functions, take a partition of $Z$ into a union of infinite disjoint sets $Z'$ and $Z''$ and find a net $(x, \beta, \gamma, \delta)$ such that $x, \delta \in Z'$, $\beta, \gamma \in Z''$ and the corresponding nets $(x, \delta)$ and $(\beta, \gamma)$ converge to 0 in $G_2''$, so that by an argument similar to that given in the proof of Lemma 2, the net $(x, \beta, \gamma, \delta)$ Bohr-converges to 0 in $G_2''$. Then by continuity, also $\pi(x, \beta, \gamma, \delta) \to 0$.

It is easy to check that one has a splitting

$$\pi(x, \beta, \gamma, \delta) = \sum_{i=0}^{9} \sigma_{i000}(x, \beta) + \sigma_{0100}(\gamma, \delta) + \sigma_{1001}(x, \delta) + [\sigma_{1000}(x) - \sigma_{0100}(\delta)].$$

By Lemma 6 $\sigma_{1000}(x, \beta) \to 0$, $\sigma_{0101}(\gamma, \delta) \to 0$ and $\sigma_{1001}(x, \delta) \to 0$. Since each one of these nets has weakly disjoint supports, all they vanish on a tail of the net, so we get

$$\sigma_{1000}(x, \beta) = \sigma_{0101}(\gamma, \delta) = \sigma_{1001}(x, \delta) = 0.$$  \hspace{1cm} (9)

A further application of Lemma 6 gives $\sigma_{1000}(x) - \sigma_{0100}(\delta) \to 0$ when the net $(x, \delta)$ converges to 0 in $[Z']^2$. Applying Claim 8 we can find a cofinite subset $Z_0$ of $Z'$ where

$$\sigma_{1000}(x) = \sigma_{0100}(x) \quad \text{for all } x \in Z_0.$$  \hspace{1cm} (10)

With (9) and (10) we have

$$\pi(x, \beta, \gamma, \delta) = [\sigma_{1000}(x) + \sigma_{1000}(\beta)] - [\sigma_{1000}(\gamma) + \sigma_{1000}(\delta)] \quad \text{for all } x, \beta, \gamma, \delta \in Z_0.$$  \hspace{1cm} (11)

**Step 5**: Now we are left with only one function $\sigma_{1000}$ of one variable. To finish the proof take a partition of $Z_0$ into a union of infinite disjoint sets $Z'$ and $Z''$ and find a net $(x, \beta, \gamma, \delta)$ such that $x, \beta \in Z'$, $\gamma, \delta \in Z''$ and the corresponding nets $(x, \beta)$ and $(\gamma, \delta)$ converge to 0 in $G_2''$, so that the net $(x, \beta, \gamma, \delta)$ Bohr-converges to 0 in $G_2''$ as before. By continuity, also $\pi(x, \beta, \gamma, \delta) \to 0$. It is easy to check that one has a splitting as indicated in (11). Again by Lemma 6 we conclude that $\sigma_{1000}(x) + \sigma_{1000}(\beta) \to 0$ when the net $(x, \beta)$ converges to 0 in $[Z']^2$. By Claim 8 $\sigma_{1000}(x) = -\sigma_{1000}(x)$ must hold on a cofinite subset of $Z'$. Thus $2\sigma_{1000}(x) = 0$ for cofinitely many $x \in Z'$. This yields that $\sigma_{1000}$ vanishes on a cofinite subset of $Z'$, hence on $Z$. Therefore, $\pi(x, \beta, \gamma, \delta)$ vanishes on $[Z]^4$ by (2)–(11).

**Acknowledgements**

We are indebted to the referee for her/his careful and patient reading of the original manuscript and valuable suggestions.
References