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Linear Algebra and its Applications 318 (2000) 173–179

LINEAR ALGEBRA AND ITS APPLICATIONS

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A note on affine toric varieties

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Received 25 September 1999; accepted 30 April 2000

Submitted by J. Dias da Silva

Abstract

Let k be an arbitrary field and Γ a toric set in the affine space $\mathbb{A}_k^n$ given parametrically by monomials. Using linear algebra we give necessary and sufficient conditions for Γ to be an affine toric variety, and show some applications. © 2000 Elsevier Science Inc. All rights reserved.

AMS classification: Primary 14M25; 15A36; Secondary 13F20

1. Introduction

Let k be any field and $D = (d_{ij})$ a fixed $m \times n$ matrix with non-negative integer entries d_{ij} and with non-zero columns. Let $k[x_1, \dots, x_n]$ and $k[t_1, \dots, t_m]$ be two polynomial rings over k, and ϕ the graded homomorphism of k-algebras,

$$\phi : R = k[x_1, \dots, x_n] \to k[t_1, \dots, t_m], \quad \text{induced by } \phi(x_i) = t^{d_i},$$

where $d_i = (d_{1i}, \dots, d_{mi})$ is the ith column of D and $t^{d_i} = t_1^{d_{1i}} \cdots t_m^{d_{mi}}$. Then the polynomial rings are graded by assigning $\deg(t_i) = 1$ and $\deg(x_j) = \deg(t^{d_j})$ for all i, j. The kernel of ϕ, denoted by P, is called the *toric ideal* associated with D. If $\alpha = (\alpha_i) \in \mathbb{N}^n$, we set $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ for the corresponding monomial in R.

Note that the map ϕ is closely related to the homomorphism $\psi : \mathbb{Z}^n \to \mathbb{Z}^m$, determined by the matrix D in the standard bases of $\mathbb{Z}^n$ and $\mathbb{Z}^m$. Indeed, one can easily

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[1] Partially supported by CONACyT grant 27931E and SNI, Mexico.

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verify that *a binomial* $g = x^\alpha - x^\beta$ *belongs to* $P = \ker(\phi)$ *if and only if* $\alpha - \beta$ *belongs to* $\ker(\psi)$; see [2] for a detailed study of the relation between ϕ and ψ.

The *affine space* of dimension n over k, denoted by $\mathbb{A}_k^n$, is the Cartesian product $k^n = k \times \cdots \times k$ of n-copies of k. Given a subset $I \subset R$ its *zero set* or *variety*, denoted by $V(I)$, is the set of $a \in \mathbb{A}_k^n$ such that $f(a) = 0$ for all $f \in I$.

The *toric set* Γ determined by D is the subset of the affine space $\mathbb{A}_k^n$ given parametrically by $x_i = t_1^{d_{1i}} \cdots t_m^{d_{mi}}$ for all i, that is, one has

$$\Gamma = \left\{ \left(t_1^{d_{11}} \cdots t_m^{d_{m1}}, \ldots, t_1^{d_{1n}} \cdots t_m^{d_{mn}} \right) \in \mathbb{A}_k^n \;\middle|\; t_1, \ldots, t_m \in k \right\}.$$

We say that Γ is an *affine toric variety* if Γ is the zero set of the toric ideal P associated with D.

Toric ideals and their varieties occur naturally in algebra and geometry [1,9], some of their properties have been linked to polyhedral geometry [8] and graph theory [5,7,10]. Of particular interest for this note is the fact that toric ideals are generated by binomials [4]; here by a binomial we mean a difference of two monomials.

Our aim is to use linear algebra to characterize when a toric set Γ is an affine toric variety in terms of:

(a) the existence of solutions in k of equations of the form $z^{\lambda_i} = c$, where $c \in k$ and λ_i is an invariant factor of the matrix D;

(b) the vanishing condition "$V(P, x_i) \subset \Gamma$ for all i", that in some cases can be checked recursively.

Some applications will be presented to illustrate the usefulness of our characterization.

To prove the main result (see Theorem 2.3) we make use of the fact that any integral matrix is equivalent to a diagonal matrix which is in Smith normal form [6, Theorem II.9], together with a description of a certain generating set of a system of linear diophantine equations (see Proposition 2.2).

2. Affine toric varieties

First we fix some notation. Let Γ be a toric set defined by an $m \times n$ matrix $D = (d_{ij})$. Then there are unimodular integral matrices $U = (u_{ij})$ and $Q = (q_{ij})$ of orders m and n, respectively, such that

$$L = UDQ = \operatorname{diag}(\lambda_1, \ldots, \lambda_s, 0, \ldots, 0),$$

where s is the rank of D and $\lambda_1, \ldots, \lambda_s$ are the invariant factors of D, that is, λ_i divides λ_{i+1} and $\lambda_i > 0$ for all i. For the use in the following, set $U^{-1} = (f_{ij})$ and $Q^{-1} = (b_{ij})$. In the sequel e_i will denote the ith unit vector in $\mathbb{Z}^n$.

For convenience we state the following version of well-known descriptions for the solution set of a homogeneous system of linear diophantine equations, see [6, Chapter 2].

Lemma 2.1. *If $\psi : \mathbb{Z}^n \to \mathbb{Z}^m$ is the homomorphism determined by D, then* $\ker(\psi) = \mathbb{Z}q_{s+1} \oplus \cdots \oplus \mathbb{Z}q_n$, *where q_i corresponds to the* ith *column of Q.*

Proof. Let $x \in \mathbb{Z}^n$ and make the change of variables $y = Q^{-1}x$. As $L = UDQ$ it follows that $Dx = 0$ if and only if $Ly = 0$. Set $y = (y_1, \dots, y_n)$.

First note $q_i \in \ker(\psi)$ for $i \geqslant s+1$, because $LQ^{-1}q_i = Le_i$, where e_i is the ith unit vector in $\mathbb{Z}^n$. On the other hand, if x is in $\ker(\psi)$, then $\lambda_i y_i = 0$ for $i = 1, \dots, s$. Thus, $x = Qy = \sum_{i=s+1}^{n} y_i q_i$. To complete the proof observe that the columns of Q are a basis for $\mathbb{Z}^n$. $\square$

Proposition 2.2. *Let $\psi : \mathbb{Z}^n \to \mathbb{Z}^m$ be the linear map determined by D and $v_j = \sum_{i=1}^{s} b_{ij} q_i$, where q_i corresponds to the* ith *column of Q. If $e_1, \dots, e_n$ is the standard basis of $\mathbb{Z}^n$, then $\{v_i - e_i\}_{i=1}^{n}$ is a generating set for* $\ker(\psi)$.

Proof. Set $Q^{-1} = (b_{ij})$. Note $e_j = \sum_{i=1}^{n} b_{ij} q_i$ for all j, because $QQ^{-1} = I$. Hence, one can write

$$v_j = \sum_{i=1}^{s} b_{ij} q_i = e_j - \sum_{i=s+1}^{n} b_{ij} q_i \quad (j = 1, 2, \dots, n),$$

and using Lemma 2.1 we obtain $v_j - e_j \in \ker(\psi)$ for $j = 1, \dots, n$. Set $\delta_{ik} = 1$ if $i = k$ and $\delta_{ik} = 0$ otherwise. From the equality above

$$\begin{aligned}
\sum_{j=1}^{n} q_{jk}(e_j - v_j) &= \sum_{j=1}^{n} q_{jk} \left(\sum_{i=s+1}^{n} b_{ij} q_i \right) \\
&= \sum_{i=s+1}^{n} q_i \left(\sum_{j=1}^{n} q_{jk} b_{ij} \right) \\
&= \sum_{i=s+1}^{n} q_i \delta_{ik} \\
&= q_k
\end{aligned}$$

for $k \geqslant s+1$. Hence, q_k is in the subgroup of $\mathbb{Z}^n$ generated by $\{v_i - e_i\}_{i=1}^{n}$ for $k \geqslant s+1$, as required. $\square$

For the use in the following, note that every vector $v \in \mathbb{Z}^n$ can be written uniquely as $v = v_+ - v_-$, where v_+ and v_- are vectors with non-negative entries and have disjoint support.

In the sequel we use the notation introduced above. Our main result is:

Theorem 2.3. *Let k be a field, Γ the toric set determined by the matrix D and P its toric ideal. Then $\Gamma = V(P)$ if and only if the following two conditions are satisfied*:

(a) *If* $(a_i) \in V(P)$ *and* $a_i \neq 0\ \forall i$, *then* $a_1^{q_{1i}} \cdots a_n^{q_{ni}}$ *has a* λ_i*-root in* k *for* $i = 1, \ldots, s$.
(b) $V(P, x_i) \subset \Gamma$ *for* $i = 1, \ldots, n$.

Proof. ($\Leftarrow$): One invariably has $\Gamma \subset V(P)$. To prove the other contention take a point $a = (a_1, \ldots, a_n)$ in $V(P)$, by condition (b) one may assume that $a_i \neq 0$ for all i. Thus, using (a) there are $t'_1, \ldots, t'_s$ in k such that

$$\left(t'_i\right)^{\lambda_i} = a_1^{q_{1i}} \cdots a_n^{q_{ni}} = a^{q_i} \quad (i = 1, \ldots, s). \tag{1}$$

For convenience of notation we extend the definition of t'_i by putting $t'_i = 1$ for $i = s+1, \ldots, m$ and $t' = (t'_1, \ldots, t'_m)$. Set

$$t_j = \left(t'_1\right)^{u_{1j}} \cdots \left(t'_m\right)^{u_{mj}} \quad (j = 1, \ldots, m), \tag{2}$$

where $U = (u_{ij})$. We claim that $t^{d_k} = t_1^{d_{1k}} \cdots t_m^{d_{mk}} = a_k$ for $k = 1, \ldots, n$. Setting $U^{-1} = (f_{ij})$ and comparing columns in the equality $U^{-1}L = DQ$ one has

$$\lambda_i f_i = \sum_{j=1}^{n} q_{ji} d_j \quad (i = 1, 2, \ldots, s), \tag{3}$$

where $f_i = (f_{1i}, \ldots, f_{mi})$ and $d_j = (d_{1j}, \ldots, d_{mj})$ denote the ith and jth columns of U^{-1} and D, respectively. Next we compare columns in the equality $D = (U^{-1}L)$ Q^{-1} to get

$$d_k = \sum_{j=1}^{s} \lambda_j b_{jk} f_j \quad (k = 1, 2, \ldots, n), \tag{4}$$

where $Q^{-1} = (b_{ij})$. Using $UU^{-1} = I$ and Eq. (2) we rapidly conclude that

$$t^{f_k} = t'_k \quad (k = 1, \ldots, m). \tag{5}$$

From Proposition 2.2 we derive $Dv_j = De_j = d_j$ for $j = 1, \ldots, n$, where

$$v_j = \sum_{i=1}^{s} b_{ij} q_i = \left(\sum_{\ell=1}^{s} q_{1\ell} b_{\ell j}, \ldots, \sum_{\ell=1}^{s} q_{n\ell} b_{\ell j}\right) \quad (j = 1, \ldots, n). \tag{6}$$

Hence, $D(v_j)_+ = D(e_j + (v_j)_-)$, that is, $x^{(v_j)_+} - x^{e_j + (v_j)_-}$ belongs to the toric ideal P. Using that $a \in V(P)$ yields $a^{(v_j)_+} = a^{e_j + (v_j)_-}$, and thus

$$a^{v_j} = a^{e_j} = a_j \quad (j = 1, \ldots, n). \tag{7}$$

Therefore, putting altogether

$$\begin{aligned} t^{d_k} &\overset{(4)}{=} t^{\sum_{j=1}^{s} \lambda_j b_{jk} f_j} = \left(t^{f_1}\right)^{\lambda_1 b_{1k}} \cdots \left(t^{f_s}\right)^{\lambda_s b_{sk}} \\ &\overset{(5)}{=} \left(t'_1\right)^{\lambda_1 b_{1k}} \cdots \left(t'_s\right)^{\lambda_s b_{sk}} \end{aligned}$$

$$\overset{(1)}{=} \left(a^{q_1}\right)^{b_{1k}} \cdots \left(a^{q_s}\right)^{b_{sk}} = a^{q_1 b_{1k}+\cdots+q_s b_{sk}}$$
$$\overset{(6)}{=} a^{v_k}$$
$$\overset{(7)}{=} a_k$$

for $k = 1, \ldots, n$. Thus, $a \in \Gamma$, as required.

($\Rightarrow$): It is clear that (b) holds because $V(P, x_i) \subset V(P)$. To prove (a) take (a_i) in $V(P)$ with $a_i \neq 0$ for all i. Then by definition of Γ there are $t_1, \ldots, t_m$ in k such that $a_j = t^{d_j}$ for $j = 1, \ldots, n$. Therefore, by Eq. (3), one has

$$t^{\lambda_i f_i} = t^{q_{1i} d_1} \cdots t^{q_{ni} d_n} = a_1^{q_{1i}} \cdots a_n^{q_{ni}}.$$

Thus, $(t^{f_i})^{\lambda_i} = a_1^{q_{1i}} \cdots a_n^{q_{ni}}$, as required. □

Corollary 2.4. *If k is algebraically closed, then $V(P) \subset \Gamma \cup V(x_1 \cdots x_n)$.*

Proof. Let $a = (a_i) \in V(P)$ such that $a_i \neq 0$ for all i. Since k is algebraically closed condition (a) above holds. Therefore, one may proceed as in the first part of the proof of Theorem 2.3 to get $a \in \Gamma$. □

Corollary 2.5. *If k is algebraically closed, then $\Gamma = V(P)$ if and only if $V(P, x_i) \subset \Gamma$ for all i.*

Proof. If k is algebraically closed, then (a) is satisfied. Thus, Γ is a toric variety if and only if $V(P, x_i) \subset \Gamma$ for all i. □

Remark 2.6. The last two corollaries are valid if we assume condition (a), instead of assuming k algebraically closed.

As a more concrete application we now show that *Veronese toric sets* are affine toric varieties.

Proposition 2.7. *Let d be a positive integer and*

$$A = \left\{(a_1, \ldots, a_m) \in \mathbb{N}^m \;\middle|\; a_1 + \cdots + a_m = d\right\}.$$

If k is an algebraically closed field and D the matrix whose columns are the vectors in A, then the toric set Γ determined by D is an affine toric variety.

Proof. Let

$$\mathscr{B} = \left\{t^a \;\middle|\; a \in A\right\} = \{f_1, \ldots, f_m, f_{m+1}, \ldots, f_s\},$$

where

$$s = \binom{d+m-1}{m-1}.$$

One can order the f_i so that $f_i = t_i^d$ for $i = 1, \ldots, m$ and $|\mathrm{supp}(f_i)| \geqslant 2$ for $i > m$, where $\mathrm{supp}(t^a) = \{t_i \mid a_i > 0\}$.

Fix an integer $1 \leqslant i \leqslant s$, it suffices to prove $V(P, x_i) \subset \Gamma$, where P is the toric ideal associated with D. We use induction on m. Take $a \in V(P, x_i)$. If $i > m$ and $\phi(x_i) = f_i = t_1^{r_1} \cdots t_m^{r_m}$, note that the binomial $x_i^d - x_1^{r_1} \cdots x_m^{r_m}$ belongs to P, and hence $a \in V(P, x_j)$ for some $1 \leqslant j \leqslant m$. Therefore, one may harmlessly assume $1 \leqslant i \leqslant m$ and $\phi(x_i) = t_i^d$; for simplicity of notation we assume $i = 1$. Observe that for every $f_j = t_1^{r_1} \cdots t_m^{r_m}$ with $r_1 > 0$ one has $a_j = 0$; indeed since $x_j^d - x_1^{r_1} \cdots x_m^{r_m}$ belongs to P and $a \in V(P, x_1)$ one has $a_j = 0$. Let D' be the submatrix of D obtained by removing the first row and all the columns with non-zero first entry (from top to bottom), and P' the toric ideal of D'. The vector $a' = (a_i \mid t_1 \notin \mathrm{supp}(f_i))$ is in $V(P')$, because $P' \subset P$. Since $V(P') \subset \Gamma' \cup V(x_2 \cdots x_s)$, where Γ' is the toric set associated with D', by induction one readily obtain $a \in \Gamma$. $\square$

Next we present another consequence that can be used to prove that monomial curves over arbitrary fields are affine toric varieties.

Corollary 2.8. *If the columns of D generate $\mathbb{Z}^m$ as $\mathbb{Z}$-module, then $\Gamma = V(P)$ if and only if $V(P, x_i) \subset \Gamma$ for all i.*

Proof. Since $\mathbb{Z}d_1 + \cdots + \mathbb{Z}d_n = \mathbb{Z}^m$, one has $\lambda_i = 1$ for all i, and thus condition (a) holds. Therefore, Γ is an affine toric variety if and only if (b) holds. $\square$

A toric set Γ in the affine space $\mathbb{A}_k^n$ is called a *monomial curve* if its corresponding matrix D has only one row, namely, $D = (d_1, \ldots, d_n)$, and $d_1, \ldots, d_n$ are relatively prime positive integers.

Proposition 2.9 [2]. *Let k be an arbitrary field and Γ a monomial curve. Then $\Gamma = V(P)$.*

Proof. As $\mathbb{Z} = \mathbb{Z}d_1 + \cdots + \mathbb{Z}d_n$, by Corollary 2.8, it suffices to show $V(P, x_i) \subset \Gamma$. Let $a \in V(P, x_i)$. Since all the binomials $x_i^{d_j} - x_j^{d_i}$ vanish on a, one obtains $a = 0$ and $a \in \Gamma$. $\square$

Remark 2.10. If k is algebraically closed, from Corollary 2.5, it follows that the conclusion of Proposition 2.9 remains valid even without the assumption $\gcd(d_1, \ldots, d_n) = 1$.

Remark 2.11. If Γ is a toric set over an infinite field k and $\Gamma = V(I)$ for some $I \subset R$, then Γ is equal to $V(P)$, see [3, Chapter 1]. Thus, if k is infinite and Γ is a variety, then Γ must be an affine toric variety.

In the light of this remark, a natural question is whether Γ can be a variety but not a toric variety; to clarify consider:

Example 2.12. Let $k = \mathbb{Z}_3$ and $D = (2, 4)$. Then

$$\Gamma = \{(0, 0)\} \cup \{(1, 1)\} = V\big(x_1 - x_2, x_2^2 - x_2\big).$$

On the other hand $P = (x_1 - x_2^2)$ and $(1, 2) \in V(P)$. Thus, $\Gamma \neq V(P)$.

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