Uniform properties and hyperspace topologies for $\aleph$-uniformities

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Received 1 October 1989
Revised 3 April 1990

Abstract

Di Concilio, A., Uniform properties and hyperspace topologies for $\aleph$-uniformities, Topology and its Applications 44 (1992) 115-123.

Let $X$ be a completely regular space and $\aleph$ an infinite cardinal number. The $\aleph$-uniformity of $X$ generated by all open normal coverings of $X$ with cardinality $\leq \aleph$, is the weakest one with the following property: any continuous function from $X$ to any metric space of weight $\leq \aleph$ is uniformly continuous. Any continuous function from a uniform space $X$ to any metric space of weight $\leq \aleph$ is uniformly continuous iff any locally finite covering of cozero-sets of cardinality $\leq \aleph$ is uniform. With $\aleph$-collectionwise normality, any continuous function from $X$ to any metric space of weight $\leq \aleph$ and uniform dimension $\leq 1$ is uniformly continuous iff any discrete family of subsets of $X$ with cardinality $\leq \aleph$ is uniformly discrete. The uniform hypertopologies induced via the Hausdorff uniformity on the hyperspace $2^X$ of $X$ from the $\aleph'$-uniformity, generated by the family of all continuous functions from $X$ to any metric space of density $\leq \aleph$ and uniform dimension $\leq 1$ and from the $\aleph$-uniformity agree. Further, both agree with a Vietoris-type topology iff $X$ is normal.

Keywords: Hausdorff uniformity, Vietoris topology, Vietoris-type topology, hyperspace, $\aleph$-uniformity, uniform dimension.


Introduction

In metric setting if any real-valued continuous and bounded function is uniformly continuous, then any continuous function is uniformly continuous. In uniform setting a surprising lot of interesting forms of uc-ness (some continuity is uniform), involving leader uniformities and remarkable classes of metric spaces, arise. Let $(X, \mathcal{U})$ be a uniform space. In [7] Nagata, for the first, characterized a form of uc-ness like a uniform separation property proving that any real-valued continuous

* Research supported by Fondi di Ricerca M.U.R.S.T.
and bounded function on $X$ is uniformly continuous iff any two disjoint zero-sets of $X$ can be uniformly separated. Then, in [6] Michael related uniform normality of $X$ (any two disjoint closed subsets of $X$ can be uniformly separated) to the Vietoris topology on the hyperspace $2^X$ of $X$ (see Section 3) proving that, with normality, the uniform hypertopology induced via the Hausdorff uniformity from the Čech uniformity of $X$ coincides with the Vietoris topology. In [1] Atsuji proved that, with normality, any real-valued continuous function on $X$ is uniformly continuous iff any discrete sequence of subsets of $X$ is uniformly discrete. In [3] the author proved that, with normality, the uniform hypertopology induced via the Hausdorff uniformity from the Nachbin-Hewitt uniformity of $X$, which is the weakest (coarsest, smallest) one satisfying the Atsuji property, coincides with a Vietoris-type hypertopology.

For any infinite cardinal number $\mathfrak{N}$ we will show that any continuous function from $X$ to any metric space of weight $\leq \mathfrak{N}$ is uniformly continuous iff any locally finite covering of cozero-sets of $X$ with cardinality $\leq \mathfrak{N}$ is uniform. The $\mathfrak{N}$-uniformity [9] of $X$ generated by all open normal coverings of $X$ with cardinality $\leq \mathfrak{N}$ is the weakest uniformity compatible with $X$ for which any continuous function from $X$ to any metric space of weight $\leq \mathfrak{N}$ is uniformly continuous. Further, we will consider the topology on $2^X$, denoted by $\mathfrak{N}'$, generated from the base $\{G^i \cap U \mid G^i \in \mathcal{G}, U \in \mathcal{U} \}$, where $G^i$ ranges over the open subsets of $X$ and $U$ ranges over the collection of all locally finite families of open subsets of $X$ of cardinality $\leq \mathfrak{N}$ (see for notations Section 3). The Hausdorff uniform topology induced on $2^X$ from the $\mathfrak{N}$-uniformity of $X$ is weaker than $\mathfrak{N}'$. With $\mathfrak{N}$-collectionwise normality, any continuous function from $X$ to any metric space of density $\leq \mathfrak{N}$ and uniform dimension $\leq 1$ is uniformly continuous iff any discrete family of subsets of $X$ with cardinality $\leq \mathfrak{N}$ is uniformly discrete. When $(X, \tau)$ is a completely regular space this characterization will play a leading role in proving that the uniform hypertopologies induced on $2^X$ via the Hausdorff uniformity from the $\mathfrak{N}'$-uniformity of $X$, generated by the family of all continuous functions from $X$ to any metric space of density $\leq \mathfrak{N}$ and uniform dimension $\leq 1$, and from the $\mathfrak{N}$-uniformity of $X$ agree. Finally, both agree with $\mathfrak{N}'$ iff $X$ is normal.

The paper contains four sections. In Section 1 we give necessary preliminaries and essential definitions. In Section 2 we show that the $\mathfrak{N}$-uniformity is determined by the class of metric spaces of weight $\leq \mathfrak{N}$, characterizing the uniformities for which any continuous function to any metric space of weight $\leq \mathfrak{N}$ is uniformly continuous and proving that the $\mathfrak{N}$-uniformity is the weakest one. Further, with $\mathfrak{N}$-collectionwise normality, we prove the equivalence between the following property of uc-ness: Any continuous function to any metric space of density $\leq \mathfrak{N}$ and uniform dimension $\leq 1$ is uniformly continuous and the following uniform separation property: Any discrete $\mathfrak{N}$-family of subsets is uniformly discrete. As a corollary an intrinsic characterization by coverings of the $\mathfrak{N}'$-uniformity is given. In Section 3 we introduce $\mathfrak{N}'$ a Vietoris-type topology on the hyperspace by the means of open locally finite families of cardinality $\leq \mathfrak{N}$. In Section 4 we show that the uniform hypertopologies
induced via the Hausdorff uniformity from the $\mathcal{U}$-uniformity and $\mathcal{U}'$-uniformity agree. Then we compare $\mathcal{U}'$ with Hausdorff uniform topologies and prove that $\mathcal{U}'$ and the Hausdorff uniform topology deriving from the $\mathcal{U}$-uniformity agree iff $X$ is normal.

1. Preliminaries

In considering uniformities we mostly follow Tukey's definition by coverings. For connections among diagonal nhbds, uniform coverings and uniform pseudometrics we refer to Engelking [4] and Isbell [5].

If $X$ is a topological space and $f$ is real-valued continuous function on $X$, we denote by $\text{supp} f = \{x \in X : f(x) \neq 0\}$. We remark that if $\{\varphi_i : i \in I\}$ is a locally finite partition of unity on $X$, then $\{\text{supp} \varphi_i : i \in I\}$ is a locally finite covering of cozero-sets of $X$ and vice versa any locally finite covering of cozero-sets of $X$ $\{\text{supp} f_i : i \in I\}$ generates a locally finite partition of unity $\{\varphi_i : i \in I\}$ on $X$, where $\varphi_i = f_i/\sum f_i$, for any $i \in I$.

In the following we essentially use the basic results:

**Proposition 1.1** (Morita). *Any locally finite open covering of cozero-sets is normal.*

**Proposition 1.2.** (A.H. Stone). *If $X$ is normal, then any open locally finite covering is normal.*

**Proposition 1.3** (Tukey). *$X$ is normal iff any binary open covering is normal.*

**Proposition 1.4** (A.H. Stone). *Any open normal covering of $X$ with cardinality $\mathcal{U}$ can be star-refined by a locally finite open covering with cardinality $\mathcal{U}$.*

**Proposition 1.5** (Isbell [5, IV]). *Any normal open $\mathcal{U}$-covering of finite order $n$ is the first of a normal sequence of open $\mathcal{U}$-coverings each of order $\leq n$.*

**Proposition 1.6** (Isbell [5, IV]). *A uniform covering has a uniform refinement of order $n$ iff it has a uniform refinement which is a union of $n + 1$ uniformly discrete subcollections.*

We remind that for a metric space $X$ the density of $X$, the least cardinal of a dense subset of $X$, is $\leq \mathcal{U}$ iff the weight of $X$, the least cardinal of a base of $X$, is $\leq \mathcal{U}$. Further, we remark that the class of metric spaces of density $\leq \mathcal{U}$ and uniform dimension $\leq 1$ admits a uniformly universal metric space of density $\leq \mathcal{U}$ and uniform dimension 1 (Kulpa).

Let $\{A_\lambda : \lambda \in \Lambda\}$ and $\{B_\lambda : \lambda \in \Lambda\}$ be both discrete collections of subsets of $X$. We say that $\{A_\lambda\}$ is discretely normally separated from $\{B_\lambda\}$ iff for each $\lambda \in \Lambda$ there exists a continuous function $f_\lambda : X \to [0, 1]$ such that $f_\lambda(A_\lambda) = 1$ and $f_\lambda(X - B_\lambda) = 0$. If $X$ is a uniform space we say that $(A_\lambda)$ is uniformly separated from $\{B_\lambda\}$ iff there exists a diagonal nhbd $V$ such that $V[A_\lambda] \subset B_\lambda$ for each $\lambda$.

We list the known relations between uc-ness and uniform separation properties.
Theorem 1.7 (Nagata). Any real-valued continuous and bounded function on $X$ is uniformly continuous if for any pair of disjoint zero-sets $A$, $B$ of $X$ there exists a diagonal nbhd $V$ such that $V[A] \cap B = \emptyset$.

Theorem 1.8 (Atsuji). Any real-valued continuous function on $X$ is uniformly continuous if any sequence $\{A_n\}$ of subsets of $X$ discretely normally separated from $\{B_n\}$ is uniformly separated from $\{B_n\}$.

2. Uc-ness and uniform separation properties

Let $X$ be a completely regular space and $\aleph_0$ an infinite cardinal number. We recall that the family of all open normal coverings of $X$ with cardinality $\leq \aleph_0$ generates a uniformity compatible with $X$, which is called the $\aleph_0$-uniformity of $X$ [9]. The $\aleph_0$-uniformity is the Tukey-Shirota uniformity.

Proposition 2.1. The $\aleph_0$-uniformity is generated by all open locally finite coverings of $X$ with cardinality $\leq \aleph_0$ if $X$ is normal.

Proof. Any open normal covering of $X$ with cardinality $\aleph_0$ can be star-refined by an open locally finite covering of $X$ with cardinality $\aleph_0$ (Proposition 1.4). But, if $X$ is normal, any open locally finite covering of $X$ is normal. Conversely, since any binary open covering is uniform and then normal, $X$ is normal. \qed

As the $\aleph_0$-uniformity is generated by the class of separable metric spaces, so any $\aleph_0$-uniformity is determined by the class of metric spaces of weight $\leq \aleph_0$.

Theorem 2.2. For any uniform covering $U$ of the $\aleph_0$-uniformity of $X$ there is a uniformly continuous function from $X$ to a metric space of weight $\leq \aleph_0$ for which $U$ is refined from the preimage of a uniform covering.

Proof. Choose a normal sequence $\{U_n: n \in \mathbb{N}\}$ of open coverings of the $\aleph_0$-uniformity whose first element refines $U$ and a pseudometric $\rho$ on $X$ such that for each $n \in \mathbb{N}$ the covering of open spheres $\{S_\rho(x, 1/2^n): x \in X\}$ is refined from $U_n$ and refines $U_{n-1}$. The space $X/\rho$ has weight $\leq \aleph_0$. For suppose $U_n = \{A^*_\lambda: \lambda \in \Lambda_n\}, \text{card} (\Lambda_n) \leq \aleph_0$. For each $n \in \mathbb{N}$ and each $\lambda \in \Lambda_n$ pick $x^*_n$ in $A^*_\lambda$. The set $\{x^*_n\}$ which has cardinality $\leq \aleph_0$ is dense in $(X, \rho)$. Observe that diam$(A^*_\lambda) < 1/2^{n-2}$ for each $n$ and each $\lambda$ and for each integer $n$ and each point $x \in X$, $S_\rho(x, 1/2^{n+3}) \subseteq A^*_{\lambda+2}$ for some $\lambda \in \Lambda_{n+2}$. Thus $x^*_{n+2} \in S_\rho(x, 1/2^n)$. Finally, since the immersion of $X$ in $X/\rho$ is uniformly continuous then $U$ is refined from the uniform covering $\{S_\rho(x, 1): x \in X\}$. \qed

Theorem 2.3. The gage of the $\aleph_0$-uniformity is generated from all continuous pseudometrics $\rho$ of $X$ for which $X/\rho$ has weight $\leq \aleph_0$.

Proof. Since any pseudometric $\rho$ in the gage of the $\aleph_0$-uniformity is relative to a normal sequence of open coverings of cardinality $\leq \aleph_0$ the result follows from
Theorem 2.2. Conversely, suppose $\rho$ is a continuous pseudometric of $X$ for which $X/\rho$ has a dense subset $\{x_\lambda : \lambda \in \Lambda\}$, $\text{card}(\Lambda) \leq \aleph_0$. For each $\lambda \in \Lambda$ pick $y_\lambda$ in $x_\lambda$. Then any covering of open spheres $\{S_\rho(x, r) : x \in X\}$ is refined from $\{S_\rho(y_\lambda, r) : \lambda \in \Lambda\}$ which is a normal covering of cardinality $\leq \aleph_0$. \qed

Let $(X, \mathcal{U})$ be a uniform space.

Theorem 2.4. Any continuous function from $X$ to any metric space of weight $\leq \aleph_0$ is uniformly continuous iff any locally finite covering of cozero-sets of $X$ with cardinality $\leq \aleph_0$ is uniform.

Proof. "If" follows from Proposition 1.1 and the proof of Theorem 2.2. Conversely, let $f$ be a continuous function from $X$ to a metric space $(Y, \rho)$ of density $\leq \aleph_0$. For each $\varepsilon > 0$, consider the covering of open spheres $\{S_\rho(z, \varepsilon/2) : z \in Y\}$ which admits a subordinate locally finite partition of unity $\{f_\lambda : \lambda \in \Lambda\}$, $\text{card}(\Lambda) \leq \aleph_0$. But then the locally finite covering of cozero-sets of $X$ $\{\text{supp}(f_\lambda \circ f) : \lambda \in \Lambda\}$ is uniform. Now, if $x, y$ both belong to $\text{supp}(f_\lambda \circ f)$ for some $\lambda$, then $f(x), f(y)$ both belong to $S_\rho(z, \varepsilon/2)$ for some $z \in Y$ and $\rho(f(x), f(y)) < \varepsilon$. \qed

Theorem 2.5. The $\aleph_0$-uniformity of $X$ is the weakest uniformity compatible with $X$ for which any continuous function from $X$ to any metric space of weight $\leq \aleph_0$ is uniformly continuous.

Proof. Let $\mathcal{U}$ be a uniformity for which any continuous function from $X$ to any metric space of weight $\leq \aleph_0$ is uniformly continuous. For each covering $V$ in the $\aleph_0$-uniformity, there exists a continuous pseudometric $\rho$ such that $X/\rho$ has weight $\leq \aleph_0$ and $\{S_\rho(x, 1) : x \in X\}$ refines $V$. But the immersion of $X$ in $X/\rho$, which is continuous, is uniformly continuous w.r.t. $\mathcal{U}$. So $\{S_\rho(x, 1) : x \in X\}$ is uniform. Then $V$ is uniform. \qed

Theorem 2.6. The following properties are equivalent:
(a) Any continuous function from $X$ to any metric space of density $\leq \aleph_0$ and uniform dimension $\leq 1$ is uniformly continuous.
(b) Any locally finite covering of cozero-sets with cardinality $\leq \aleph_0$ and order $\leq 2$ is uniform.
(c) If $\{A_\lambda : \lambda \in \Lambda\}$ is discretely normally separated from $\{B_\lambda : \lambda \in \Lambda\}$ and $|\Lambda| \leq \aleph_0$, then $\{A_\lambda\}$ is uniformly separated from $\{B_\lambda\}$.

Proof. (a) $\Rightarrow$ (b) Any locally finite covering $V$ of cozero-sets of order $\leq 2$ and cardinality $\leq \aleph_0$ is the first of a normal sequence $\{V_n\}$ of open coverings each of cardinality $\leq \aleph_0$ and order $\leq 2$. Let $\rho$ be a continuous pseudometric with the following
property: \( V_{n+1} \) refines \( \{ S(x, 1/2^n) : x \in X \} \), which refines \( V_n \). The metric space \( X/\rho \) has density \( \leq \aleph_0 \) (Theorem 2.2) and uniform dimension \( \leq 1 \) (Proposition 1.5). But then the immersion \( i: X \to X/\rho \) is uniformly continuous. So \( V \) is refined by \( i^{-1}\{ S(x, 1) : x \in X \} \) which is uniform.

(b) \( \Rightarrow \) (c) Suppose \( \{ A_\lambda : \lambda \in \Lambda \} \) is discretely normally separated from \( \{ B_\lambda : \lambda \in \Lambda \} \) and \( |\Lambda| \leq \aleph_0 \). Then we can find for any \( \lambda \in \Lambda \) a continuous function \( f\lambda \) from \( X \) to \( [0, 1] \) such that \( A_\lambda \subset f\lambda^{-1}(1) \) and \( X - f\lambda^{-1}(0) \subset B_\lambda \). If we put \( f = \sum f\lambda \) we have that \( \bigcup f\lambda^{-1}(1) = f^{-1}(1) \). So we can consider the \( \aleph_0 \)-covering of cozero-sets \( \{ X - f\lambda^{-1}(1), X - f\lambda^{-1}(0) : \lambda \in \Lambda \} \) which is uniform. Let \( V \) be the diagonal nhbd related to it. It is easy to show that \( V[X - f\lambda^{-1}(0)] \cap X - f\lambda^{-1}(0) = \emptyset, \forall \lambda \neq \mu \) and \( V[A_\lambda] \cap X - f\lambda^{-1}(0) = B_\lambda, \forall \lambda \in \Lambda \).

(c) \( \Rightarrow \) (a) Suppose \( (Y, d) \) is a metric space of density \( \aleph_0 \) and uniform dimension \( \leq 1 \). Let \( \{ x_\lambda : \lambda \in \Lambda \}, |\Lambda| \leq \aleph_0 \), be a dense subset of \( Y \). For each integer \( k \) consider the uniform covering \( \{ S(x_\lambda, 1/2k) : \lambda \in \Lambda \} \). It can be refined by a uniform covering which is the union of two uniformly discrete collections both of cardinality \( \leq \aleph_0 \), \( \{ A_\lambda \}, \{ B_\lambda \} \). Choose \( V_\lambda \) in such a way that \( V_\lambda[A_\lambda] \cap A_\lambda = \emptyset, \forall \lambda \neq \kappa \) and \( V_\lambda \) in such a way that \( V_\lambda[B_\lambda] \cap B_\lambda = \emptyset, \forall \rho \neq \sigma \). Put \( V = V_\lambda \cap V_\nu \). Suppose \( \{ S(x_\lambda, \varepsilon) : \lambda \in \Lambda \} \) refines \( V \). Let \( m = \min(\varepsilon/3, 1/2k) \) and \( W = \{ S(x_\lambda, m) : \lambda \in \Lambda \} \). Then \( W[A_\lambda] \cap W[B_\lambda] = \emptyset, \forall \lambda \neq \mu, W[B_\lambda] \cap W[B_\mu] = \emptyset, \forall \rho \neq \sigma \), and \( \{ W[A_\lambda], W[B_\lambda] : \lambda, \mu \} \) is a refinement of \( \{ S(x_\lambda, 1/k) : \lambda \in \Lambda \} \). Because of the Dowker Lemma [4, 5.1.17], since \( \{ A_\lambda \} \) \( \{ B_\lambda \} \) is separated from \( \{ W[A_\lambda] \} \{ W[B_\lambda] \} \) which is a family of disjoint open sets it is separated from a discrete family \( \{ C_\lambda \} \) \( \{ D_\mu \} \) of open sets with \( A_\lambda \subset C_\lambda \) \( B_\mu \subset D_\mu \). Let \( f \) be a continuous function from \( X \) to \( Y \). Since \( \{ f^{-1}(A_\lambda) \}, \{ f^{-1}(B_\lambda) \} \) are discretely normally separated from \( \{ f^{-1}(C_\lambda) \}, \{ f^{-1}(D_\mu) \} \) respectively, then they are uniformly separated by the means of two diagonal nhbds of \( X, U_1, U_2 \). Put \( U = U_1 \cap U_2 \). Thus, if \( (x, y) \in U \) and \( f(x) \in A_\lambda \) or \( f(x) \in B_\mu \), then \( f(y) \in C_\lambda \) or \( f(y) \in D_\mu \). So \( f(x), f(y) \) both belong to \( C_\lambda \) for some \( \lambda \) or both belong to \( D_\mu \) for some \( \mu \). In any case \( d(f(x), f(y)) < 2/k \).

Now, consider \( \aleph_1 \) the weak uniformity generated by all continuous functions from \( X \) to any metric space of density \( \aleph_0 \) and uniform dimension \( \leq 1 \). It can easily be shown that:

Theorem 2.7. The uniformity \( \aleph_1 \) admits as a base the family of diagonal nhbds of the type:

\[
V = \left( X - \bigcup \lambda A_\lambda \times X - \bigcup \lambda A_\lambda \right) \cup \left( \bigcup \lambda B_\lambda \times B_\lambda \right)
\]

where \( \{ A_\lambda : \lambda \in \Lambda \} \) is discretely normally separated from \( \{ B_\lambda : \lambda \in \Lambda \} \) and \( |\Lambda| \leq \aleph_0 \).

Theorem 2.8. If \( X \) is \( \aleph_0 \)-collectionwise normal, then any continuous function from \( X \) to any metric space of density \( \leq \aleph_0 \) and uniform dimension \( \leq 1 \) is uniformly continuous if any discrete family of subsets of \( X \) with cardinality \( \leq \aleph_0 \) is uniformly discrete.
3. A Vietoris-type topology on $2^X$:

Let $(X, \tau)$ be a $T_1$-space and $2^X$ the hyperspace of $X$, the set of all closed nonempty subsets of $X$. For each subset $G$ of $X$ denote by $G^+ = \{ E \in 2^X : E \subseteq G \}$. For each family $U = \{ U_i : i \in I \}$ of subsets of $X$ denote by

$$U^- = \{ U_i : i \in I \}^- = \{ E \in 2^X : E \cap U_i \neq \emptyset, \forall i \in I \}$$

and

$$\langle U_i : i \in I \rangle = \{ E \in 2^X : E \subseteq \bigcup U_i \text{ and } E \cap U_i \neq \emptyset, \forall i \in I \}. $$

The family $\{ G^+ \cap U^- \}$, where $G$ ranges over all open subsets of $X$ and $U$ ranges over all finite families of open subsets of $X$ is a base for the finite or Vietoris topology $2^*$ on $2^X$ [6].

The family $\{ G^+ \cap U^- \}$, where $G$ ranges over all open subsets of $X$ and $U$ ranges over all (countable) locally finite families of open subsets of $X$ is a base for the (countable) locally finite topology $e^*$ [8] ($e^*$ [3]) on $2^X$.

Beer and others proved in [2] that for a metrizable space $X$ the locally finite topology on $2^X$ coincides with the sup of all Hausdorff metric topologies induced from all equivalent metrics of $X$. Also results contained in [8] and [3] emphasize the interest of Vietoris-type topologies deriving from open locally finite families. The choice of locally finite families, which are closure-preserving, has been revealed right because of the following properties:

**Lemma 3.1.** Let $\{ V_i : i \in I \}$ and $\{ W_j : j \in J \}$ be both locally finite families of subsets of $X$. Then $\langle V_i : i \in I \rangle \subseteq \langle W_j : j \in J \rangle$ iff (a) $\bigcup V_i \subseteq \bigcup W_j$ and (b) for each $j \in J$ there exists $i \in I$ such that $V_i \subseteq W_j$.

**Lemma 3.2.** Let $X$ be a $T_1$-space and $\{ U_i : i \in I \}$ a locally finite family of open subsets of $X$. If for each $i \in I$ we pick $x_i$ in $U_i$, we can find a discrete family $\{ V_i : i \in I \}$ of open subsets of $X$ such that $x_i \in V_i \subseteq U_i$ for each $i \in I$.

We introduce now a Vietoris-type topology by the means of open locally finite families of cardinality $\leq \aleph_0$.

**Proposition 3.3.** The family $\{ G^+ \cap U^- \}$, where $G$ ranges over all open subsets of $X$ and $U$ ranges over all locally finite families of open subsets of $X$ with cardinality $\leq \aleph_0$ is a base for a topology on $2^X$, which we will denote by $\aleph^*$.

**Theorem 3.4.** If $X$ is $T_1$, then $\aleph^*$ has as subbase the collection $\{ G^+ \cap U^- \}$, where $G$ ranges over all open subsets of $X$ and $U$ ranges over all discrete families of open subsets of $X$ with cardinality $\leq \aleph_0$. 

From previous lemmas it follows that:

**Proposition 3.5.** If \( \{V_i : i \in I\} \) is a locally finite family of open subsets of \( X \), then the \( \mathcal{N}^* \)-closure of \( \langle V_i : i \in I \rangle \) and \( \langle V_i^- : i \in I \rangle \), where \( - \) denotes closure, coincide.

**Proposition 3.6.** If \( (\mathcal{X}, \mathcal{N}) \) is regular, then \( (X, \tau) \) is normal.

**Proof.** Let \( A, B \in \mathcal{X} \) and \( A \cap B = \emptyset \). By regularity there exists an open locally finite family \( \{V_i : i \in I\} \) of cardinality \( \leq \aleph_0 \) such that

\[
A \subseteq \bigcup \{V_i : i \in I\} \subseteq \bigcup \{V_i^- : i \in I\} \subseteq X - B.
\]

By Lemma 3.1 and Proposition 3.5, \( A \subseteq \bigcup V_i \subseteq \bigcup V_i^- \subseteq X - B \). \( \square \)

### 4. Comparison with uniform topologies

Now let \( (X, \mathcal{U}) \) be a uniform space. For each \( U \in \mathcal{U} \) put

\[
U = \{(A, B) \in \mathcal{X} \times \mathcal{X} : A \subseteq U[B] \text{ and } B \subseteq U[A]\}.
\]

Then \( \{U : U \in \mathcal{U}\} \) is a base for a uniformity on \( \mathcal{X} \) which is called the **Hausdorff uniformity** induced from \( \mathcal{U} \). Its underlying topology \( |\mathcal{U}| \) is called the **Hausdorff uniform topology** induced from \( \mathcal{U} \). We remark that from distinct but equivalent uniformities on \( X \) may arise distinct uniform topologies on \( \mathcal{X} \).

Let \( X \) be a completely regular space.

**Theorem 4.1.** The Hausdorff uniform topology \( |\mathcal{U}| \) induced from \( \mathcal{U} \) is weaker than \( \mathcal{N}^* \) iff \( \mathcal{U} \) is generated from a base of \( \aleph_0 \)-coverings.

**Proof.** Let \( A \in \mathcal{X} \) and \( U \in \mathcal{U} \). Choose a symmetric open basic nhbd \( V = \cup \{U_A \times U_x : \lambda \in \Lambda\} \), card(\( \Lambda \)) \( \leq \aleph_0 \), such that \( V \subseteq U \). Let \( E \) be a subset of \( A \) maximal w.r.t. the property \( x, y \in E \) implies \( (x, y) \notin V^2 \). For each \( x \in E \) pick an index \( \lambda(x) \) such that \( x \in U_{\lambda(x)} \). Then the family \( \{U_{\lambda(x)} : x \in X\} \) is a discrete open family of cardinality \( \leq \aleph_0 \) such that \( A \subseteq \langle U_{\lambda(x)} : x \in X\rangle^- \subseteq U[A] \). Conversely, let \( \{U[x] : x \in X\} \) be a uniform covering and \( W \) a symmetric open diagonal nhbd such that \( W^2 \subseteq U \). Then \( W[X] \) must contain an \( \aleph_0 \)-nhbd of \( X \), \( \langle U_\lambda : \lambda \in \Lambda\rangle \), card(\( \Lambda \)) \( \leq \aleph_0 \). But \( W[X] = \langle W[x] : x \in X\rangle \). For each \( \lambda \in \Lambda \) pick \( x_\lambda \) in \( U_\lambda \). From Lemma 3.1 any \( W[x] \) contains some \( U_\lambda \) and then \( x_\lambda \). So it is contained in \( W^2[x_\lambda] \). Thus \( \{W^2[x_\lambda] : \lambda \in \Lambda\} \) is an open normal refinement of \( \{U[x] : x \in X\} \) with cardinality \( \leq \aleph_0 \). \( \square \)

**Theorem 4.2.** The uniform topologies induced on \( \mathcal{X} \) via the Hausdorff uniformity from the \( \aleph_0 \)-uniformity and from the \( \mathcal{N}^* \)-uniformity of \( X \) agree, i.e., \( |\mathcal{N}| = |\mathcal{N}^*| \).

**Proof.** We have to show that \( |\mathcal{N}| = |\mathcal{N}^*| \). Let \( A \in \mathcal{X} \), \( U \) belong to the \( \aleph_0 \)-uniformity and \( \{U_\lambda\} \) be a discrete open family of cardinality \( \leq \aleph_0 \) such that \( A \subseteq \langle U_\lambda \rangle^- \subseteq U[A] \).
Choose for each $\lambda$ a point $x_\lambda \in A \cap U_\lambda$. Then there exists a symmetric diagonal nhbd $W$ belonging to the $\mathcal{N}$-uniformity such that $W[x_\lambda] \subset U_\lambda$, $\forall \lambda \in \Lambda$. If $B \in W[A]$, then $A \subset W[B]$. So for each $\lambda \in \Lambda$ we can find $b_\lambda \in B$ and $b_\lambda \in W[x_\lambda] \subset U_\lambda$. It follows that $W[A]$ which is an $|2^{|X|}|$-nhbd of $A$ is contained in $U[A]$. □

**Theorem 4.3.** Let $X$ be normal and $\mathcal{U}$ finer than the $\mathcal{N}$-uniformity. Then $\mathcal{N}$ is weaker than the Hausdorff uniform topology $|2^{|X|}|$.

**Proof.** Let $A \in 2^X$, $(V_\lambda : \lambda \in \Lambda)$ be an $\mathcal{N}$-nhbd of $A$. Since $X$ is normal, then there is a continuous function $f : X \to [0, 1]$ such that $f(A) = 1$, $f(X - \bigcup V_\lambda) = 0$. Put $W = \{(x, y) \in X \times X : |f(x) - f(y)| < 1\}$. Then $W$ belongs to $\mathcal{U}$ and $W[A] \subset \bigcup \{V_\lambda : \lambda \in \Lambda\}$. Now, for each $\lambda \in \Lambda$ pick $x_\lambda \in A \cap V_\lambda$ and choose a discrete open family $\{W_\lambda : \lambda \in \Lambda\}$, where $W_\lambda \subset V_\lambda$. For each $\lambda \in \Lambda$ consider a continuous function $f_\lambda : X \to [0, 1]$ such that $f_\lambda(x_\lambda) = 1$ and $f_\lambda(X - W_\lambda) = 0$. Observe that $\bigcup f_\lambda^{-1}(1) = f^{-1}(1)$, where $f = \sum f_\lambda$, and $X - f_\lambda^{-1}(0)$ is contained in $W_\lambda$. The collection $\{X - \bigcup f_\lambda^{-1}(1), X - f_\lambda^{-1}(0) : \lambda \in \Lambda\}$ is a locally finite covering of cozero-sets of order 2. Thus it belongs to $\mathcal{U}$. Let $V$ be its associated diagonal nhbd and $U = W \cap V$. $U[A]$ is a $|2^{|X|}|$-nhbd of $A$ contained in $\langle V_\lambda : \lambda \in \Lambda \rangle$. Suppose $B \in U[A]$, then $B \subset U[A] \subset \bigcup V_\lambda$. Since $A \subset U[B]$, then for each $\lambda \in \Lambda$ there exists $b_\lambda \in B$ such that $(x_\lambda, b_\lambda) \in U$. But $f_\lambda(x_\lambda) = 1$ and $x_\lambda$ doesn’t belong to any $X - f_\mu^{-1}(0)$ when $x_\mu \neq x_\lambda$. So $b_\lambda \in X - f_\mu^{-1}(0) \subset U_\lambda$. □

**Theorem 4.4.** The Hausdorff uniform topology deriving on $2^X$ from the $\mathcal{N}$-uniformity of $X$ and $\mathcal{N}$ agree iff $X$ is normal.

**Proof.** It follows from Theorems 4.1-4.3 □

**Corollary 4.5.** $(2^X, \mathcal{N})$ is completely regular iff $(X, \tau)$ is normal.

*References*