JOURNAL OF NUMBER THEORY 9, 499–501 (1977)

Solution of a Problem of Skolem

ANDREW BREMNER

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109, and Emmanuel College, Cambridge, England

Communicated by D. J. Lewis

Received December 15, 1976

T. Skolem shows that there are at most six integer solutions to the Diophantine equation $x^5 + 2y^5 + 4z^5 - 10xy^3z + 10x^2yz^2 = 1$. The author shows here that there are precisely three integer solutions.

Skolem [1–3] shows that the equation Norm $(x + y\theta + z\theta^2) = x^5 + 2y^5 + 4z^5 - 10\,xy^3z + 10x^2yz^2 = 1$, where $\theta^5 = 2$, has at most six solutions in integers x, y, z of which he gives three, $(x, y, z) = (1, 0, 0), (-1, 1, 0), (1, -2, 1)$. We show here that there are no further solutions.

We have to find all integers m, n such that

$$\pm(x + y\theta + z\theta^2) = \epsilon_1{}^m\epsilon_2{}^n,$$

where ϵ_1, ϵ_2 are fundamental units of $\mathbf{Q}(\theta)$. By the calculations of Skolem [1], we may take $\epsilon_1 = -1 + \theta$, $\epsilon_2 = 1 + \theta + \theta^3$. We work p-adically, but it is expedient to be judicious in the choice of p; in fact we take $p = 251$, one of the first rational primes to split completely into first-degree prime factors in $\mathbf{Q}(\theta)$: Such primes include 5, 151, 241, 251,.... A small computer calculation shows that

$$\epsilon_1^{250} = 1 + 251\xi \qquad \text{with} \quad \xi \equiv 81\theta - 16\theta^2 + 76\theta^3 - 78\theta^4 \bmod 251,$$

$$\epsilon_2^{50} = 1 + 251\eta \qquad \text{with} \quad \eta \equiv 107\theta + 17\theta^2 - 14\theta^3 + 68\theta^4 \bmod 251,$$

and also that the only terms $\epsilon_1{}^r\epsilon_2{}^s$, $0 \leqslant r \leqslant 249$, $0 \leqslant s \leqslant 49$, having the coefficients of θ^3 and θ^4 both divisible by 251, are given by $(r, s) = (0, 0), (1, 0), (2, 0)$. Writing $m = 250M + r$, $n = 50N + s$, we immediately have

$$\pm(x + y\theta + z\theta^2) = \epsilon_1{}^r(1 + 251\xi)^M (1 + 251\eta)^N$$

with $r = 0, 1,$ or 2.

ISSN 0022-314X

$$\text{Write } (1+251\xi)^M(1+251\eta)^N = 1 + 251(M\xi + N\eta) + 251^2(\ ) + \cdots$$
$$= K_0 + K_1\theta + K_2\theta^2 + K_3\theta^3 + K_4\theta^4$$

with

$$\begin{aligned}
K_0 &= 1 + 251(0\cdot M + 0\cdot N) + 251^2(\ ) + \cdots,\\
K_1 &= \quad 251(81M + 107N) + 251^2(\ ) + \cdots,\\
K_2 &= \quad 251(-16M + 17N) + 251^2(\ ) + \cdots,\\
K_3 &= \quad 251(76M - 14N) + 251^2(\ ) + \cdots,\\
K_4 &= \quad 251(-78M + 68N) + 251^2(\ ) + \cdots.
\end{aligned}$$

Equating coefficients of θ^3 and θ^4 to zero gives

(i) $K_3 = K_4 = 0$ when $r = 0$,

(ii) $K_2 - K_3 = K_3 - K_4 = 0$ when $r = 1$,

(iii) $K_1 - 2K_2 + K_3 = K_2 - 2K_3 + K_4 = 0$ when $r = 2$.

Point (i) implies

$$\begin{aligned}
0 &= \quad (76M - 14N) + 251(\ ) + \cdots,\\
0 &= (-78M + 68N) + 251(\ ) + \cdots,
\end{aligned}$$

and since

$$\begin{vmatrix} 76 & -14 \\ -78 & 68 \end{vmatrix} \equiv 60 \bmod 251,$$

we have[1] [by 3, remark at end of proof of Theorem 11] that there is at most one solution, which is clearly $M = N = 0$.

Point (ii) implies

$$\begin{aligned}
0 &= (-92M + 31N) + 251(\ ) + \cdots,\\
0 &= \quad (154M - 82N) + 251(\ ) + \cdots,
\end{aligned}$$

and

$$\begin{vmatrix} -92 & 31 \\ 154 & -82 \end{vmatrix} \equiv 9 \bmod 251,$$

so as above there is at most one solution, which is $M = N = 0$.

[1] For completeness, we state this remark in the form that we need:

Let $F_j(x, y) = \sum_{i=0}^{\infty} p^i f_{i,j}(x, y)$, $j = 1, 2$, where $f_{i,j}(x, y)$ are polynomials with integer coefficients, and p is prime. Suppose that $f_{0,1} = ax + by$, $f_{0,2} = cx + dy$ with

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \not\equiv 0 \bmod p.$$

Then $F_1(x, y) = F_2(x, y) = 0$ has at most one solution in integers x, y.

Point (iii) implies

$$0 = (189M + 59N) + 251(\ ) + \cdots,$$
$$0 = (-246M + 113N) + 251(\ ) + \cdots,$$

and

$$\begin{vmatrix} 189 & 59 \\ -246 & 113 \end{vmatrix} \equiv -22 \bmod 251,$$

so at most one solution, which is $M = N = 0$.

Accordingly, the only solutions are given by

$$(x + y\theta + z\theta^2) = (\theta - 1)^r,\ r = 0, 1, 2,$$

as required.

REFERENCES

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