Gravitational conformal invariance and coupling constants in Kaluza–Klein theory

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Abstract

We introduce a generalized gravitational conformal invariance in the context of non-compactified 5D Kaluza–Klein theory. It is done by assuming the 4D metric to be dependent on the extra non-compactified dimension. It is then shown that the conformal invariance in 5D is broken by taking an absolute cosmological scale R_0 over which the 4D metric is assumed to be dependent weakly on the 5th dimension. This is equivalent to Deser's model for the breakdown of the conformal invariance in 4D by taking a constant cosmological mass term $\mu^2 \sim R_0^{-2}$ in the theory. We set the scalar field to its background cosmological value leading to Einstein equation with the gravitational constant G_N and a small cosmological constant. A dual Einstein equation is also introduced in which the matter is coupled to the higher-dimensional geometry by the coupling G_N^{-1}. Relevant interpretations of the results are also discussed.

1. Introduction

The theory of conformal invariance has been playing a particularly important role in the investigation of gravitational models since Weyl, who introduced the notion of conformal rescaling of the metric tensor. Afterwards, it was promoted to the conformal transformations in scalar–tensor theories, in which another transformation on the scalar field was required to represent the conformal invariance in modern gravitational models. There is an open possibility that the gravitational coupling of matter may have its origin in an invariance breaking effect of this conformal invariance. In fact, since the ordinary coupling of matter to gravity is a dimensional coupling (mediated by the gravitational constant), the local conformal transformations which could change the strength of this dimensional coupling, by affecting the local standards of length and time, are expected to play a key role. In a system which includes matter, conformal invariance

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requires the vanishing of the trace of the stress tensor in the absence of dimensional parameters. However, in the presence of dimensional parameters, the conformal invariance can be also established for a large class of theories [1] if the dimensional parameters are conformally transformed according to their dimensions. One general feature of conformally invariant theories is, therefore, the presence of varying dimensional coupling constants. In particular, one can say that the introduction of a constant dimensional parameter into a conformally-invariant theory breaks the conformal invariance in the sense that a preferred conformal frame is singled out, namely that in which the dimensional parameters have the assumed (constant) configuration. The determination of the corresponding preferred conformal frame depends on the nature of the problem at hand. In a conformally-invariant gravitational model, the symmetry breaking may be considered as a cosmological effect. This means that one breaks the conformal symmetry by defining a preferred conformal frame in terms of the large-scale characteristics of cosmic matter distributed in a universe with finite scale factor R_0. In this way, the breakdown of conformal symmetry becomes a framework in which one can look for the origin of the gravitational coupling of matter, both classical [2] and quantum [3], at large cosmological scales.

The purpose of this Letter is to show that one may look for the origins of both conformal invariance and its breakdown, leading to gravitational couplings, in a 5-dimensional Kaluza–Klein type gravity theory [4]. In this popular non-compactified approach to Kaluza–Klein gravity, known as "Space–Time–Matter" theory (STM), the gravitational field is unified with its source through a new type of 5D manifold in which space and time are augmented by an extra non-compactified dimension which induces 4D matter. Unlike the usual Kaluza–Klein theory in which a cyclic symmetry associated with the extra dimension is assumed, the new approach removes the cyclic condition and derivatives of the metric with respect to the extra coordinate are retained. This induces non-trivial matter on the hypersurface of $l = $ constant. This theory basically involves writing the Einstein field equations with matter as a subset of the Kaluza–Klein field equations without matter [4], a procedure which is guaranteed by an old theorem of differential geometry due to Campbell [8].

This view is also inherent in the membrane theory which is mathematically equivalent, since the canonical metric in STM theory [4] is basically the warp metric of Randall–Sundrum [5]. In this membrane theory the large non-compactified dimension are not in conflict with observation if the Standard Model fields are confined to a 3-brane in the extra dimensions [6]. In fact, the only reason to compactify the extra dimension is to reproduce 4D Newtonian gravity at long distances, and if gravity is somehow trapped into the 3-brane then 4D gravity can be reproduced even if the extra dimensions are infinitely large [6]. Generically, physical phenomena in the non-compact scenarios have to show higher-dimensional character. So, one can test them by detecting the departures from the four-dimensional physics, such as the appearance of the extra force [10]. Even, if these theories with large extra dimensions cannot correctly describe our present universe their potential for describing the early universe is obvious since there is strong evidence that the early universe underwent a phase where it was five-dimensional [11].

We show that in the context of STM theory, i.e. $\widehat{R}_{AB} = 0$ in 5D, one may find a generalized conformally-invariant gravitational model. The well-known conformally-invariant model of Deser [2] in 4D is shown to be a special case when we drop the dependence of the 4D metric on the extra dimension. Moreover, we show that the breakdown of conformal invariance which was introduced in [2] by an ad hoc non-conformal invariant term inserted into the action naturally emerges here by (i) assuming a weak (cosmological) dependence of the 4D metric on the 5th dimension[1] and (ii) approximating the scalar field with its cosmological background value using the well-known cosmological coincidence usually referred to Mach or Wheeler.

This geometric approach to the subject of conformal invariance and its breakdown in gravitational models accounts properly for coupling of the gravitational field with its source in 5D gravity. It also gives an explanation for the origin of a small cosmological constant emerging from non-compactified extra di-

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[1] This assumption is reasonable since 4D general relativity is known to be in a very good agreement with present observations.

mension. This subject is the most recent interest in theories with large extra dimensions [6].

The Letter is organized as follows: in Section 2, we briefly review the conformal invariant gravitational model and its breakdown in 4 dimensions due to Deser [2]. In Section 3, we introduce a generalized conformal invariant gravitational model in 5 dimensions. In Section 4, we study the breakdown of conformal invariance in 5 dimensions and discuss on some relevant interpretations. The Letter ends with a conclusion.

2. Breakdown of conformal invariance in 4D

In this section we briefly revisit the standard work in 4D conformal invariance due to Deser [2]. Consider the action functional

$$S[\phi] = \frac{1}{2} \int d^4x \sqrt{-g} \left(g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{6} R \phi^2 \right), \quad (1)$$

which describes a system consisting of a real scalar field ϕ non-minimally coupled to gravity through the scalar curvature R. Variations with respect to ϕ and $g_{\alpha\beta}$ lead to the equations

$$\left(\Box - \frac{1}{6} R \right) \phi = 0, \quad (2)$$

$$G_{\alpha\beta} = 6\phi^{-2} \tau_{\alpha\beta}(\phi), \quad (3)$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ is the Einstein tensor and

$$\tau_{\alpha\beta}(\phi) = - \left[\nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \phi \nabla^\gamma \phi \right] - \frac{1}{6} (g_{\alpha\beta} \Box - \nabla_\alpha \nabla_\beta) \phi^2, \quad (4)$$

with ∇_α denoting the covariant derivative. Taking the trace of (3) gives

$$\left(\Box - \frac{1}{6} R \right) \phi = 0, \quad (5)$$

which is consistent with Eq. (2). This is a consequence of the conformal symmetry of action (1) under the conformal transformations

$$\phi \to \tilde{\phi} = \Omega^{-1}(x)\phi, \qquad g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \Omega^2(x) g_{\alpha\beta}, \quad (6)$$

where the conformal factor $\Omega(x)$ is an arbitrary, positive and smooth function of space–time. Adding

a matter source S_m independent of ϕ to the action (1) in the form

$$S = S[\phi] + S_m, \quad (7)$$

yields the field equations

$$\left(\Box - \frac{1}{6} R \right) \phi = 0, \quad (8)$$

$$G_{\alpha\beta} = 6\phi^{-2} \left[\tau_{\alpha\beta}(\phi) + T_{\alpha\beta} \right], \quad (9)$$

where $T_{\alpha\beta}$ is the matter energy–momentum tensor. The following algebraic requirement

$$T = 0, \quad (10)$$

then emerges as a consequence of comparing the trace of (9) with (8) which implies that only traceless matter can couple consistently to such gravity models.

We may break the conformal symmetry by adding a dimensional mass term $-\frac{1}{2} \int d^4x \sqrt{-g} \, \mu^2 \phi^2$, with μ being a constant mass parameter, to the action (7). This leads to the field equations

$$\left(\Box - \frac{1}{6} R + \mu^2 \right) \phi = 0, \quad (11)$$

$$G_{\alpha\beta} + 3\mu^2 g_{\alpha\beta} = 6\phi^{-2} \left[\tau_{\alpha\beta}(\phi) + T_{\alpha\beta} \right] \quad (12)$$

and we obtain as a result of comparing the trace of (12) with (11)

$$\mu^2 \phi^2 = T. \quad (13)$$

Now, the basic input is to consider the invariance breaking as a cosmological effect. This would mean that one may take μ^{-1} as the length scale characterizing the typical size of the universe R_0 and T as the average density of the large scale distribution of matter $\overline{T} \sim M R_0^{-3}$, where M is the mass of the universe. This leads, as a consequence of (13) to the estimation of the constant background value of ϕ

$$\bar{\phi}^{-2} \sim R_0^{-2} \left(M/R_0^3 \right)^{-1} \sim R_0/M \sim G_N, \quad (14)$$

where the well-known empirical cosmological relation $G_N M / R_0 \sim 1$ (due to Mach or Wheeler) has been used. In order to well-justify the results we will approximate the correspondence $\bar{\phi}^{-2} \sim G_N$ with $\bar{\phi}^{-2} \approx \frac{8\pi}{6} G_N$. This estimation for the constant background value of the scalar field is usually considered in Brans–Dicke type scalar–tensor gravity theories. Inserting this background value of ϕ into the field Eq. (11) leads

to the following set of Einstein equations

$$G_{\alpha\beta} + 3\mu^2 g_{\alpha\beta} = 6\bar{\phi}^{-2}T_{\alpha\beta} \approx 8\pi G_N T_{\alpha\beta}, \qquad (15)$$

with a correct coupling constant $8\pi G_N$, and a term $3\mu^2$ which is interpreted as the cosmological constant Λ of the order of R_0^{-2}. The field Eq. (11) for $\bar{\phi}$ contains no new information. This is because it is not an independent equation, namely it is the trace of Einstein equations (15). One may easily check that using $\Box\bar{\phi} = 0$ and $\overline{T} = \mu^2\bar{\phi}^2$, Eq. (11) and the trace of Eq. (15) result in the same equation as $-\frac{1}{6}R + \mu^2 = 0$.

3. 5D gravity and generalized conformal invariance

Consider the 5D metric given by

$$dS^2 = \hat{g}_{AB}\,dx^A\,dx^B = G\phi^2 g_{\alpha\beta}\,dx^\alpha\,dx^\beta + dl^2 \quad (16)$$

where the 5D line interval is written as the sum of a 4D part relevant to scalar–tensor theory and an extra part due to the 5th dimension. The capital Latin indices $A, B, \ldots$ run over 0, 1, 2, 3, 4, Greek indices $\alpha, \beta, \ldots$ run over 0, 1, 2, 3, and five-dimensional quantities are denoted by hats. A constant G is also introduced to leave $G\phi^2$ dimensionless. We proceed keeping $g_{\alpha\beta} = g_{\alpha\beta}(x^\alpha, l)$ and $\phi = \phi(x^\alpha)$ as in modern Kaluza–Klein theory [4]. The metric is general, since we have only used 4 of the available 5 coordinate degree of freedom to set the electromagnetic potentials, $g_{4\alpha}$ to zero. Once we use the 5th coordinate degree of freedom, we may choose a background value for the field ϕ. Then the metric will be generally in the form of Randall–Sundrum warp metric [5], or the canonical metric in STM theory [4].

The corresponding Christoffel symbols are obtained

$$\widehat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \phi^{-1}\big(\delta^\alpha_\gamma \nabla_\beta\phi + \delta^\alpha_\beta \nabla_\gamma\phi - g_{\beta\gamma}\nabla^\alpha\phi\big),$$

$$\widehat{\Gamma}^\alpha_{\beta\alpha} = \Gamma^\alpha_{\beta\alpha} + 4\phi^{-1}\nabla_\beta\phi,$$

$$\widehat{\Gamma}^4_{\beta\gamma} = -\frac{1}{2}\partial_4\hat{g}_{\beta\gamma},$$

$$\widehat{\Gamma}^\alpha_{4\alpha} = \frac{1}{2}\hat{g}^{\alpha\beta}\partial_4\hat{g}_{\alpha\beta},$$

$$\widehat{\Gamma}^\alpha_{\beta 4} = \frac{1}{2}\hat{g}^{\alpha\delta}\partial_4\hat{g}_{\delta\beta},$$

$$\widehat{\Gamma}^4_{\alpha 4} = \widehat{\Gamma}^\alpha_{44} = \widehat{\Gamma}^4_{44} = 0, \qquad (17)$$

where $\hat{g}_{\alpha\beta} = G\phi^2 g_{\alpha\beta}$. The 5D Ricci tensor can be written in terms of the 4D one plus other terms

$$\widehat{R}_{\alpha\beta} = R_{\alpha\beta} - 2\phi^{-1}\nabla_\alpha\nabla_\beta\phi + 4\phi^{-2}\nabla_\alpha\phi\nabla_\beta\phi$$
$$\quad - \phi^{-2}\big[\phi\Box\phi + \nabla^\alpha\phi\nabla_\alpha\phi\big]g_{\alpha\beta}$$
$$\quad + \frac{1}{2}G\phi^2\Big[g^{\gamma\delta}\partial_4 g_{\delta\alpha}\partial_4 g_{\beta\gamma} - \frac{1}{2}g^{\lambda\delta}\partial_4 g_{\alpha\beta}\partial_4 g_{\lambda\delta}$$
$$\quad - \partial_4^2 g_{\alpha\beta}\Big]. \qquad (18)$$

The field equations $\widehat{R}_{AB} = 0$ then give

$$R_{\alpha\beta} = 2\phi^{-1}\nabla_\alpha\nabla_\beta\phi - 4\phi^{-2}\nabla_\alpha\phi\nabla_\beta\phi$$
$$\quad + \phi^{-2}\big[\phi\Box\phi + \nabla^\alpha\phi\nabla_\alpha\phi\big]g_{\alpha\beta}$$
$$\quad - \frac{1}{2}G\phi^2\Big[g^{\gamma\delta}\dot{g}_{\delta\alpha}\dot{g}_{\beta\gamma} - \frac{1}{2}g^{\lambda\delta}\dot{g}_{\alpha\beta}\dot{g}_{\lambda\delta} - \ddot{g}_{\alpha\beta}\Big], \qquad (19)$$

$$\widehat{R}_{4\alpha} = \nabla_\alpha\big(k^\alpha_\beta - \delta^\alpha_\beta k\big) = 0$$
$$\text{with } k^\alpha_\beta = \frac{1}{2}\hat{g}^{\alpha\delta}\dot{\hat{g}}_{\delta\beta} = \frac{1}{2}g^{\alpha\delta}\dot{g}_{\delta\beta}, \qquad (20)$$

$$\widehat{R}_{44} = 2\big(\dot{k} - 4k^\alpha_\beta k^\beta_\alpha\big) = 0, \qquad (21)$$

where an overdot denotes differentiation with respect to 5th coordinate l (see [7]). Eq. (19) may lead to a set of 10 Einstein equations. Eq. (20) which have the form of conservation law may also lead to a set of 4 Gauss–Codazzi equations for the extrinsic curvature k^α_β of a 4D hypersurface Σ_l foliating in 5th dimension. Finally, Eq. (21) is one equation for the scalar combinations of the extrinsic curvature. The Ricci scalar for the space–time part is obtained by contracting Eq. (19) with the metric $g_{\alpha\beta}$

$$R = 6\phi^{-1}\Box\phi - \frac{1}{2}G\phi^2$$
$$\quad \times \Big[g^{\alpha\beta}g^{\gamma\delta}\dot{g}_{\delta\alpha}\dot{g}_{\beta\gamma} - \frac{1}{2}g^{\alpha\beta}g^{\lambda\delta}\dot{g}_{\alpha\beta}\dot{g}_{\lambda\delta}$$
$$\quad - g^{\alpha\beta}\ddot{g}_{\alpha\beta}\Big]. \qquad (22)$$

Combining Eqs. (19) and (22) we obtain the Einstein-like equations with Einstein tensor $G_{\alpha\beta}$ in the left-hand side and some terms of scalar field together with 4D metric and their covariant derivatives in the right-hand side as follows

$$G_{\alpha\beta} = 6\phi^{-2}\tau_{\alpha\beta}(\phi) + \frac{1}{2}G\phi^2\Big[\mathcal{T}_{\alpha\beta} - \frac{1}{2}\mathcal{T}g_{\alpha\beta}\Big], \quad (23)$$

where

$$\tau_{\alpha\beta}(\phi) = -\frac{2}{3}\nabla_\alpha\phi\nabla_\beta\phi + \frac{1}{6}g_{\alpha\beta}\nabla^\gamma\phi\nabla_\gamma\phi$$
$$-\frac{1}{3}\phi\Box\phi g_{\alpha\beta} + \frac{1}{3}\phi\nabla_\alpha\nabla_\beta\phi \qquad (24)$$

and

$$\mathcal{T}_{\alpha\beta} = g^{\gamma\delta}\dot{g}_{\delta\alpha}\dot{g}_{\beta\gamma} - \frac{1}{2}g^{\lambda\delta}\dot{g}_{\alpha\beta}\dot{g}_{\lambda\delta} - \ddot{g}_{\alpha\beta}. \qquad (25)$$

It is easy to show that the tensor $\tau_{\alpha\beta}$ in Eq. (24) is exactly the same one in Eq. (4). The field equation for the scalar field may be obtained by contracting Eq. (23) with $g_{\alpha\beta}$ or $\hat{g}_{\alpha\beta}$ as

$$\left(\Box - \frac{1}{6}R + \frac{1}{12}G\phi^2\mathcal{T}\right)\phi = 0. \qquad (26)$$

We notice that Eq. (26) has a dynamical mass term $\frac{1}{12}G\phi^2\mathcal{T}$ with the dimension of (mass)^2. In the presence of dimensional parameters, the conformal invariance can be established for a large class of theories [1] if the dimensional parameters are conformally transformed according to their dimensions. In this regard, Eq. (26), although modified by the mass term compared to (5), but is still invariant under the generalized conformal transformations

$$\phi \to \tilde{\phi} = \Omega^{-1}(x,l)\phi,$$
$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta} = \Omega^2(x,l)g_{\alpha\beta}. \qquad (27)$$

This is simply because the 5D metric (16) is invariant under the above conformal transformations. Obviously, the following combination

$$\hat{G}_{\alpha\beta} \equiv \hat{R}_{\alpha\beta} - \frac{1}{2}\hat{g}^{\gamma\lambda}\hat{R}_{\gamma\lambda}\hat{g}_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{2}g^{\gamma\lambda}\hat{R}_{\gamma\lambda}g_{\alpha\beta}$$

is invariant under (27) due to the invariance of the metric $\hat{g}_{\alpha\beta}$. Therefore, Eq. (23) which arises as a result of $\hat{G}_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{2}g^{\gamma\lambda}\hat{R}_{\gamma\lambda}g_{\alpha\beta} = 0$ is invariant under (27). And Eq. (26) as a consequence of $\hat{g}^{\alpha\beta}\hat{G}_{\alpha\beta} = 0$ or $g^{\alpha\beta}\hat{G}_{\alpha\beta} = 0$ is invariant under (27) as well, regardless of which metric is used to contraction since the right-hand side is zero. Note that although the initial l-independent scalar field ϕ transforms to an l-dependent one $\tilde{\phi}$, but the l-dependent function $\Omega^{-1}(x,l)$ will not appear in the transformed scalar field equation because the metric also transforms in such a way that the function $\Omega^{-1}(x,l)$ is factored out throughout the transformed equation rendering the initial scalar field equation. Therefore, by

pure 5D approach we are able to introduce a generalized conformal invariant gravitational model defined by Eqs. (23), (26) and (27) subject to the subsidiary Eqs. (20) and (21).

4. Breakdown of conformal invariance in 5D

Now, we are in a position to compare Eqs. (26), (23) with the corresponding Eqs. (11), (12). By this comparison it turns out that we are able to revisit the breakdown of conformal invariance in 4D by a 5D approach since we have derived the field Eqs. (26), (23) which can be identified with (11), (12) in the broken phase of the conformal invariance in 4D.

To this end, we take a dimensional analysis. The dimension of $\mathcal{T}_{\alpha\beta}$ or $\mathcal{T}$ will no doubt be (length)$^{-2}$. Now, we assume the cosmological effect $\dot{g}_{\alpha\beta} \sim \frac{1}{R_0}$ which fixes a very slow variation of $g_{\alpha\beta}$ over the absolute cosmological scale R_0. This assumption leads to the breakdown of the conformal invariance since it means that we have fixed our standard of length by the scale of the universe and that (comparing Eq. (26) with (11) and using $G\phi^2 \sim 1$) $\mathcal{T}$ may be identified with $12\mu^2$ which is a constant mass term breaking the conformal invariance. Now, we put the above identification into the Einstein-like equation (23). We then have

$$G_{\alpha\beta} + 3\mu^2 g_{\alpha\beta} = 6\phi^{-2}\left[\tau_{\alpha\beta}(\phi) + \frac{1}{12}\phi^2\mathcal{T}_{\alpha\beta}\right], \qquad (28)$$

which, comparing with Eq. (13), leads to the identification

$$T_{\alpha\beta} = \frac{1}{12}\phi^2\mathcal{T}_{\alpha\beta}, \qquad (29)$$

which is the desired result in the context of induced matter theory since the matter energy–momentum tensor $T_{\alpha\beta}$ is dynamically induced by the scalar field ϕ and higher dimension, namely $\mathcal{T}_{\alpha\beta}$. Taking the trace of (29) we find

$$T = \frac{1}{12}\phi^2\mathcal{T}, \qquad (30)$$

and by taking $\mathcal{T} = 12\mu^2$ we obtain the Eq. (13). Now, according to (30) we may discuss on the background value $\bar{\phi}$ corresponding to the absolute cosmological scale R_0. We have already fixed $\mathcal{T}$ (or μ^2) by cosmological considerations, namely $\mathcal{T} \sim R_0^{-2}$. This

was achieved by the 5th coordinate degree of freedom through $\dot{g}_{\alpha\beta} \sim R_0^{-1}$. The 5th coordinate degree of freedom accounts for the scalar field in the general metric (16). Thus, (see Eq. (14) and the following discussion) we may take a background value $\bar{\phi}$, using this coordinate degree of freedom, as

$$\bar{\phi}^{-2} \approx \frac{8\pi}{6} G_N, \tag{31}$$

which identifies G with $\frac{8\pi}{6} G_N$ such that $G\bar{\phi}^2 \approx 1$. This condition reduces the general metric (16) to the one which is mathematically equivalent to the warp metric of Randall–Sundrum [5], or the canonical metric of STM theory [4]. If we now insert this constant background value $\bar{\phi}^{-2}$ into Eq. (28) and use (29) we find

$$G_{\alpha\beta} + 3\mu^2 g_{\alpha\beta} \approx 8\pi G_N T_{\alpha\beta}, \tag{32}$$

in which

$$T_{\alpha\beta} = \frac{1}{16\pi} G_N^{-1} \mathcal{T}_{\alpha\beta}. \tag{33}$$

Eq. (32) is the well-known Einstein equation in the broken phase of the conformal invariance with a cosmological constant $\Lambda = 3\mu^2$ and a coupling of matter to gravity, G_N. The scalar field Eq. (26) is the trace of Einstein equations, so its information is already included in them (see the discussion in Section 2).

Now, the relevance of 5D approach manifests. It relates the current upper bound value of the cosmological constant $\Lambda \sim R_0^{-2}$ to a geometric phenomenon in which *the cosmological constant is generated by the very slow variation of 4D metric with respect to 5th dimension.*[2] Moreover, it unifies the origins of the matter and the cosmological constant in that they appear as "two manifestations of higher-dimensional geometry".

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[2] In a recent work of Arkani-Hamed et al. [6], a small effective cosmological constant is emerged from a large extra dimension in a non-compactified approach to 5D Kaluza–Klein gravity. Also, in a compactified model of Kaluza–Klein cosmology [12], smallness of the cosmological constant is related to smallness of the compactified dimension. Therefore, it seems that the subject of cosmological constant in higher-dimensional (at least in 5D) models is inevitably involved with extra dimension.

The traditional Einstein equation (32) may alternatively be written in its pure geometric form

$$G_{\alpha\beta} + 3\mu^2 g_{\alpha\beta} \approx \frac{1}{2} \mathcal{T}_{\alpha\beta}, \tag{34}$$

in which the coupling constant G_N is removed from theory. To say, although the Einstein tensor $G_{\alpha\beta}$ couples to the matter $T_{\alpha\beta}$ by G_N but the matter itself couples by G_N^{-1} to the geometry $\mathcal{T}_{\alpha\beta}$ (33) and so the coupling G_N is removed. In this level, the appearance of G_N in the traditional Einstein equation seems to be a mathematical tool only for dimensional consistency. However, in the physical level Eqs. (32) and (33) exhibit an interesting phenomenon, with varying G_N, in that if G_N decreases with time leading to a weakly coupling of gravity $G_{\alpha\beta}$ to the matter $T_{\alpha\beta}$ (32), the matter itself will then be coupled strongly to the hidden geometry $\mathcal{T}_{\alpha\beta}$ (33). Regarding the present small value of G_N we find an strong coupling of matter $T_{\alpha\beta}$ to the higher-dimensional geometry $\mathcal{T}_{\alpha\beta}$. This strong coupling may account for non-observability of the 5th dimension. In other words, the effects of the 5th dimension may be hidden behind this strong coupling and what we observe as the matter may be *the manifestation of a weak effect of 5th dimension which is strengthened by a strong coupling G_N^{-1}.* This means that going back in time in G_N varying scenarios we will encounter with an era $G_N \sim 1$ in which $T_{\alpha\beta}$ may decouple from $\mathcal{T}_{\alpha\beta}$ leading to a naked geometry of 5th dimension without the concept of matter, as indicated in Eq. (34). In conclusion it may be said that two Eqs. (32) and (33) define *dual weak-strong* regimes, in 5D approach to coupling constants, and that Eq. (33) defines a *dual-Einstein* equation coupling matter to higher dimension.

It is worth noting that the conformal invariance in 4D may be easily recovered in this 5D approach by restricting the 4D metric $g_{\alpha\beta}$ to be independent of 5th dimension (simply by assuming Kaluza–Klein compactification condition for higher dimension). The relevant field equations in this choice are

$$\begin{aligned} R_{\alpha\beta} = {} & 2\phi^{-1}\nabla_\alpha\nabla_\beta\phi - 4\phi^{-2}\nabla_\alpha\phi\nabla_\beta\phi \\ & + \phi^{-2}\big[\phi\Box\phi + \nabla^\alpha\phi\nabla_\alpha\phi\big]g_{\alpha\beta}, \end{aligned} \tag{35}$$

where by taking the trace of (35) and combining it with (35) we obtain the conformal invariant Eqs. (2) and (3). The origin of this conformal invariance in 4D

turns out to be the invariance of the 4D part of 5D metric

$$\hat{g}_{AB}(x^A) = \begin{pmatrix} G\phi^2(x^\alpha)g_{\alpha\beta}(x^\alpha) & 0 \\ 0 & 1 \end{pmatrix}, \qquad (36)$$

under the conformal transformations (6).

Conclusion

A key feature of any fundamental theory consistent with a given symmetry is that its breakdown would lead to effects which can have various manifestations of physical importance. Therefore, in the case of conformal symmetry in gravitational models, one would expect that the corresponding cosmological invariance breaking would have important effects generating the gravitational coupling and the cosmological constant. In this Letter we have introduced a generalized conformally-invariant gravitational model of 5D gravity theory $\widehat{R}_{AB} = 0$, with 4D part that is dependent on the extra dimension. The conformal invariance in 4D then becomes a special case when we take the 4D metric to be independent of the extra dimension. Moreover, we have shown that the cosmological breakdown of conformal symmetry in a conformally-invariant gravitational model in 4D may be naturally derived in this context if we assume a weak (cosmological) dependence of the 4D metric on the higher dimension and use the cosmological coincidence due to Mach or Wheeler [9] to approximate the scalar field by its cosmological background value. This approach to the issue of couplings and parameters in gravity leads to a geometric interpretation for the small cosmological constant Λ. Moreover, a dual coupling G_N^{-1} is introduced by which the matter couples strongly to the geometric effects of higher dimension through a *dual* Einstein equation, and non-observability of higher dimension is then justified.

We also mention to the generality of the 5D conformal invariance. In Deser's model the conformal symmetry is broken once a *constant mass term* is introduced. However, in 5D approach a *dynamical mass term* is appeared without breaking the conformal symmetry. This generalized symmetry is broken when we take a preferred conformal frame by introducing an *absolute length scale* R_0 through $\dot{g}_{\alpha\beta} \sim R_0^{-1}$. In other words, what we call the conformal invariance

in Deser's model is not really a conformal invariance; it is just *scale invariance* which is a special case of conformal invariance. This is because the dimensional constant mass term could not transform conformally.[3]

Some remarks should be reminded about the choice of the metric (16), its stability and (possible) conflict with observation. We have taken (16) because we would expect to obtain a 5D metric of Randall–Sundrum or canonical type which seems to be able to describe the universe after the breakdown of conformal invariance. Referring to a previous Letter [13] in which a conformally related metric to (16) was taken and stabilized, quantum mechanically, close to $l = 0$ constant hypersurface, we may consider a same stabilization procedure with the metric (16). This is because the key features in [13] to stabilize the non-compact dimension, close to $l = 0$, namely "a weak dependence of $g_{\alpha\beta}(x^\alpha, l)$ on l" and "l-independence of the scalar field" are admitted in the present paper. In this regard, we may live in a 5D universe but very close to the 4D reference hypersurface $l = 0$, so that the evolution of the scalar field ϕ (varying G_N) or the slowly varying metric $\dot{g}_{\alpha\beta} \sim R_0^{-1}$, does not affect practically this stabilized hypersurface.

As we have mentioned in the introduction, referring to many papers in membrane theory, the large non-compactified dimension are not in conflict with observations. The same applies for STM theory if we confine our universe very close to the $l = 0$ reference hypersurface in which case the size of 5th dimension has no observable effect. In particular, we have argued in this Letter that the strong coupling G_N^{-1} may account for the non-observability of the 5th large dimension, so that the effects of this dimension may be hidden behind this strong coupling.

There is a natural question in the context of induced matter theory about its possible connection to quantum theory. This is because we can induce the matter geometrically from the 5th dimension whereas we know the matter has a underlying quantum structure. Therefore, it deserves to pay attention to this issue. First, it is well-known that the existence of a dimen-

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[3] The conformal invariance is more general than scale invariance which is used in Deser's model. If scale invariance is characterized by vanishing of the trace of the energy–momentum tensor, conformal invariance implies scale invariance in the absence of dimensional parameters in the theory.

sional gravitational constant G_N is the main source of non-renormalizability of quantum gravity. On the other hand, the quantum theory approach to the traditional Einstein equation suffers from the problem that the left-hand side is geometry and the right-hand side is the matter. Eq. (34), however, as a pure geometric Einstein equation is free of G_N. Moreover, both sides of this equation has the geometric structure. Perhaps, it is helpful to study the 4D quantum gravity in this pure geometric 5D approach. Second, in a study of 4D conformal invariance in QFT in [3], the following equation like our scalar field Eq. (26) is obtained

$$\left(\Box - \frac{1}{6} + \phi^{-2} \mathcal{S}_\alpha^\alpha \{\omega\}\right)\phi = 0$$

in which $\mathcal{S}_\alpha^\alpha \{\omega\}$ is the trace of the tensor $\mathcal{S}_{\alpha\beta}\{\omega\}$ describing the distribution of matter due to local quantum effects. It is therefore very appealing to think about how the higher-dimensional effects in 5D may play the role of quantum effects in 4D.

Acknowledgements

References

[1] J.D. Bekenstein, A. Meisel, Phys. Rev. D 26 (1980) 1313.
[2] S. Deser, Ann. Phys. 59 (1970) 248.
[3] H. Salehi, Int. J. Theor. Phys. 37 (1998) 1253.
[4] P.S. Wesson, B. Mashhoon, H. Liu, W.N. Sajko, Phys. Lett. B 456 (1999) 34;
H. Liu, B. Mashhoon, Ann. Phys. 4 (1995) 565;
B. Mashhoon, H. Liu, P.S. Wesson, Phys. Lett. B 331 (1994) 305;
W.N. Sajko, Phys. Rev. D 60 (1999) 1040381;
P.S. Wesson, Space, Time, Matter: Modern Kaluza–Klein Theory, World Scientific, Singapore, 1999;
J.M. Overduin, P.S. Wesson, Phys. Rep. 283 (1997) 303;
J.E. Lidsey, C. Romero, R. Tavakol, S. Rippl, Class. Quantum Grav. 14 (1997) 865.
[5] L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690;
L. Randall, R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370.
[6] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper, R. Sundrum, Phys. Lett. B 480 (2000) 193;
N. Arkani-Hamed, S. Dimopoulos, G. Dvali, N. Kaloper, Phys. Rev. Lett. 84 (2000) 586;
M. Gogberashvili, Mod. Phys. Lett. A 14 (1999) 2025;
M. Gogberashvili, hep-ph/9908347.
[7] P.S. Wesson, J. Ponce de Leon, J. Math. Phys. 33 (1992) 3883;
W.N. Sajko, P.S. Wesson, J. Math. Phys. 39 (1998) 2193.
[8] J.E. Campbell, A Course of Differential Geometry, Clarendon Press, Oxford, 1926.
[9] W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation, Freeman, San Francisco, 1973.
[10] D. Youm, Phys. Rev. D 62 (2000) 084002;
P.S. Wesson, B. Mashhoon, H. Liu, W.N. Sajko, Phys. Lett. B 456 (1999) 34;
P.S. Wesson, B. Mashhoon, H. Liu, Mod. Phys. Lett. A 12 (1997) 2309;
B. Mashhoon, Phys. Lett. B 331 (1994) 305;
B. Mashhoon, Gen. Rel. Grav. 30 (1998) 555;
D. Kalligas, P.S. Wesson, C.W.F. Everitt, Astrophys. J. 439 (1994) 548;
Y.M. Cho, D.H. Park, Gen. Rel. Grav. 23 (1991) 741.
[11] P. Horava, E. Witten, Nucl. Phys. B 475 (1996) 94;
E. Witten, Nucl. Phys. B 471 (1996) 135;
J.E. Lidsey, D. Wands, E.J. Copeland, Phys. Rep. 337 (2000) 343.
[12] F. Darabi, H.R. Sepangi, Class. Quantum Grav. 16 (1999) 1565.
[13] F. Darabi, W.N. Sajko, P.S. Wesson, Class. Quantum Grav. 17 (2000) 4357.