Some new algorithms for the spectral dichotomy methods

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Abstract

Given a regular matrix pencil \( \lambda B - A \) having no eigenvalues on the circle \( C_r \) of center 0 and radius \( r \), we describe a new algorithm to compute iteratively the deflating subspaces of \( \lambda B - A \) corresponding to the eigenvalues inside and outside \( C_r \) along with “a spectral condition number” that indicates the numerical quality of the computed deflating subspaces. We then generalize the proposed algorithm to the case where \( C_r \) is replaced by a straight line or an ellipse.

Keywords: Invariant subspace; Matrix pencil; Spectral dichotomy; Spectral condition number

1. Introduction

Given a square matrix \( A \) and a positively oriented contour \( \gamma \) in the complex plane, the spectral dichotomy methods applied to \( A \) and \( \gamma \) consist in determining whether \( A \) possesses eigenvalues on or in a neighborhood of \( \gamma \). In the case where no such eigenvalues exist, these methods compute iteratively the projector \( P \), and hence the invariant subspaces of \( A \) corresponding to the eigenvalues inside and outside \( \gamma \). The numerical computation of the projector is accompanied with the spectral norm \( \omega = \| H \| \) of a Hermitian positive definite matrix \( H \). This quantity, called the dichotomy (or spectral) condition number, indicates the numerical quality of the computed projector. More precisely, the smaller \( \omega \), the better the numerical quality of the projector \( P \).
The spectral dichotomy methods have been introduced by Godunov in [5]. Bulgakov [3] and Godunov and Bulgakov [6] developed convergence estimates and stopping criteria based on the quantity $\omega$ in the case where $\gamma$ is the unit circle or the imaginary axis. Malyshev [12–14] adapted the spectral dichotomy methods to linear matrix pencils. More precisely, he developed a new computational tool to compute the deflating subspaces of a regular matrix pencil of the form $\lambda B - A$ corresponding to the eigenvalues inside and outside the unit disk. We mention that some improvements of these methods are proposed by Bai et al. [1] (see also [2]) and that other approaches based on the Schur factorization are proposed by Kågström and Van Dooren [10], and Kågström and Poromaa [11].

Generalizations of the spectral dichotomy methods to the cases where $\gamma$ is an ellipse, a parabola or a polygon have been proposed by Godunov and Sadkane [7,8], and Malyshev and Sadkane [15].

In this paper, we present new algorithms for the spectral dichotomy methods. In Section 2, we give a detailed description of the spectral dichotomy of a matrix with respect to a circle. This section contains all the ingredients that will be used in the remaining sections of the paper. In Section 3, the dichotomy of a matrix with respect to the imaginary axis is transformed, via exponential transformations, to a circular dichotomy problem where the results of Section 2 are used. Section 4 discusses the spectral dichotomy of regular matrix pencils with respect to a circle. We show that the results of Section 2 can be used again. Finally, Section 5 deals with the problem of spectral dichotomy of matrix pencils with respect to an ellipse. By using a special matrix pencil, we show that this problem is transformed to that of Section 4 and the results of Section 2 are used again. In each case, a computable expression of $H$ and hence of $\omega = \|H\|$ is given.

The transformations used in Sections 3 and 5 were already used by the authors, for example, in [7,15]. However, the algorithm presented in Section 2 (Algorithm 1) and used throughout this paper is new.

Each section terminates with an algorithm easily implementable that determines the projector $P$ associated with the eigenvalues enclosed by (excluded from) the region of interest and the dichotomy condition number $\omega = \|H\|$ that describes the numerical quality of the computed projector.

Throughout this paper, we will use the following notations: the symbol $\| \|$ denotes the spectral norm (the 2-norm) for vectors and matrices, and $A^*$ stands for the conjugate transpose of the matrix $A$. The notation $A = A^* > 0$ means that $A$ is Hermitian and positive definite. The identity matrix of order $n$ will be denoted by $I_n$ or just $I$ if the order is clear from the context.

2. Spectral dichotomy of a matrix with respect to a circle

Let $A \in \mathbb{C}^{N \times N}$ be a matrix having no eigenvalues on the positively oriented circle $\gamma = C_r$ of center 0 and radius $r$. Then the matrix

- $A$ is Hermitian and positive definite.
- The identity matrix of order $n$ will be denoted by $I_n$ or just $I$ if the order is clear from the context.
\[
P = \frac{1}{2\pi} \int_{C_{r}} (zI - A)^{-1} \, dz = \frac{1}{2\pi} \int_{0}^{2\pi} \left( I - \frac{e^{-i\theta} A}{r} \right)^{-1} \, d\theta \quad (2.1)
\]
is a projector onto the invariant subspace of \( A \) corresponding to the eigenvalues of \( A \) enclosed by \( \gamma \).

The behavior of the norm of the resolvent \( \|(zI - A)^{-1}\| \) over the circle \( \gamma \) can easily be described by the norm \( \omega = \|H\| \) of the matrix

\[
H \equiv H(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \left( I - \frac{e^{-i\theta} A}{r} \right)^{-1} H^{(0)} \left( I - \frac{e^{-i\theta} A}{r} \right)^{-1} \, d\theta, \quad (2.2)
\]
where \( H^{(0)} = H^{*(0)} > 0 \) is a matrix used for scaling purposes. Contrary to \( (zI - A)^{-1} \), the matrix \( H \) is Hermitian positive definite and the quantity \( \omega = \|H\| \) can be computed in a reliable way. This quantity plays a significant role in the numerical quality of the projector \( P \). See [5,9].

It can be shown that the projector \( P \) and the matrix \( H \) satisfy the following relations [9]:

\[
\begin{align*}
\begin{cases}
r^2 H - A^* HA &= P^* H^{(0)} P - (I - P)^* H^{(0)} (I - P), \\
PA &= AP, \\
P^2 &= P, \\
PH &= (PH)^*.
\end{cases}
\end{align*}
\]

(2.3)

2.1. Computation of \( P \) and \( H \)

The \( 2\pi \)-periodic function

\[
\theta \rightarrow \left( I - \frac{e^{-i\theta} A}{r} \right)^{-1}
\]
can be decomposed in the Fourier series

\[
\left( I - \frac{e^{-i\theta} A}{r} \right)^{-1} = \sum_{k=-\infty}^{+\infty} Z_k e^{ik\theta} \quad (2.4)
\]
with

\[
Z_k = \frac{1}{2\pi} \int_{0}^{2\pi} \left( I - \frac{e^{-i\theta} A}{r} \right)^{-1} e^{-ik\theta} \, d\theta. \quad (2.5)
\]

From (2.5), we have

\[
\sup_{k \in \mathbb{Z}} \|Z_k\| < +\infty \quad (2.6)
\]
and

\[
Z_0 = P. \quad (2.7)
\]
From (2.2) and (2.4), we obtain the “Parseval” formula

\[ H = \sum_{k=-\infty}^{+\infty} Z_k^* H^{(0)} Z_k. \]  

(2.8)

We see that if the Fourier coefficients \( Z_k \) are known, then the projector \( P \) and the matrix \( H \) will be known. Actually, we will show that not all of the coefficients \( Z_k \) are needed to compute \( H \).

From (2.4), it is easy to see that the matrices \( Z_k \) satisfy the following infinite linear system:

\[
\begin{align*}
Z_k - \frac{1}{r} Z_{k+1} &= 0 \quad \text{if } k \neq 0, \\
Z_0 - \frac{1}{r} Z_1 &= I.
\end{align*}
\]  

(2.9)

Moreover, we notice that if we set

\[ Z_k^{(s)} = \sum_{l=-\infty}^{+\infty} Z_{k+ls}, \]  

(2.10)

then the sequence \( k \to Z_k^{(s)} \) is \( s \)-periodic and for sufficiently large \( s \), \( Z_k^{(s)} \) and \( Z_k \) are very close. In other words

\[ Z_{k+s}^{(s)} = Z_k^{(s)} \]  

(2.11)

and

\[ \lim_{|s| \to +\infty} Z_k^{(s)} = Z_k. \]  

(2.12)

Relation (2.11) is obvious. Let us prove (2.12). Since the eigenvalues of \( A/r \) are not on the unit circle, this matrix can be decomposed as

\[ \frac{A}{r} = T \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} T^{-1}, \]  

(2.13)

where the eigenvalues of the matrices \( A_1 \) and \( A_2 \) are, respectively, outside and inside the unit disk. The sequence of matrices \( Z_k \) is then obtained from (2.5) as

\[ Z_k = T \left( \frac{1}{2\pi} \int_0^{2\pi} (I - e^{-i\theta} A_1)^{-1} e^{-ik\theta} d\theta \right) T^{-1} \]

\[ = \begin{cases} 
T \left( -A_1^{-k} \right) T^{-1} & \text{if } k \geq 1, \\
T \left( 0 \right) T^{-1} & \text{if } k \leq 0.
\end{cases} \]  

(2.14)

Thus,

\[ Z_k^{(s)} = \sum_{l=-\infty}^{+\infty} Z_{k+ls} \]  

(2.15)
\[
Z_k + \sum_{l=1}^{+\infty} (Z_{k-ls} + Z_{k+ls}). \tag{2.16}
\]

For large positive \( s \) we have \( k - ls < 0 \) and \( k + ls > 0 \) and
\[
\sum_{l=1}^{+\infty} (Z_{k-ls} + Z_{k+ls}) = T \begin{pmatrix}
-\sum_{l=1}^{+\infty} A_1^{-k-ls} & 0 \\
\sum_{l=1}^{+\infty} A_2^{-k+ls} & 0
\end{pmatrix} T^{-1}
\[
= T \begin{pmatrix}
A_1^{-k-s} (A_1^{-s} - I)^{-1} & 0 \\
A_2^{-k+s} (I - A_2^{-s})^{-1} & 0
\end{pmatrix} T^{-1} \tag{2.17}
\]

which goes to 0 when \( s \) goes to \(+\infty\).

Relations (2.11) and (2.12) allow us to seek the solution of (2.9) in the form \( Z_k^{(s)} \) for large \( s \) during one period. In other words, instead of solving (2.9), it is sufficient to solve for example the cyclic linear system
\[
\begin{cases}
Z_k^{(2j+1)} - \frac{A}{r} Z_{k+1}^{(2j+1)} = 0 & \text{for } k = 1, 2, \ldots, 2^{j+1} - 1, \\
Z_{2j+1}^{(2j+1)} - \frac{A}{r} Z_1^{(2j+1)} = I,
\end{cases} \tag{2.18}
\]

whose unknowns are \( Z_0^{(s)}, Z_1^{(s)} \), \( Z_2^{(s)} \), \ldots, \( Z_s^{(s)} \) with \( s = 2^{j+1} \).

It is easy to see that this system has a unique solution if and only if the eigenvalues of \( A \) are not on the circle \( \gamma = C_r \).

The following theorem shows the connection between the matrices \( P, H \) and the sequence \( Z_k^{(2j+1)} \).

**Theorem 2.1.** We have
\[
\lim_{j \to \infty} Z_0^{(2j+1)} = \lim_{j \to \infty} Z_{2j+1}^{(2j+1)} = P, \tag{2.19}
\]
\[
\lim_{j \to \infty} \frac{A}{r} Z_1^{(2j+1)} = P - I, \tag{2.20}
\]
\[
\lim_{j \to \infty} H_j = H. \tag{2.21}
\]

where
\[
H_j = \sum_{k=1}^{2j+1} (Z_k^{(2j+1)})^* H^{(0)} Z_k^{(2j+1)}. \tag{2.22}
\]

**Proof.** From (2.12) and the last equation of (2.9), we directly obtain
\[
\lim_{j \to +\infty} Z_0^{(2j+1)} = Z_0 = P
\]
and
\[
\lim_{j \to +\infty} \frac{A}{r} Z_1^{(2j+1)} = \frac{A}{r} Z_1 = Z_0 - I = P - I.
\]

To prove (2.22), we use the expressions of \( Z_k \) for \( k \geq 1 \) and \( Z_k^{(2j+1)} \) given in (2.14), (2.15) and (2.17)
\[
Z_k^{(2j+1)} = Z_k + T \begin{pmatrix}
A_1^{k-2j+1} (A_1^{-2j+1} - I)^{-1} & 0 \\
0 & A_2^{-k+2j+1} (I - A_2^{2j+1})^{-1}
\end{pmatrix} T^{-1}
\]
\[
= T \begin{pmatrix}
-A_1^{-k} & 0 \\
0 & 0
\end{pmatrix}
+ \begin{pmatrix}
-A_1^{-k-2j+1} (I - A_1^{-2j+1})^{-1} & 0 \\
0 & A_2^{-k+2j+1} (I - A_2^{2j+1})^{-1}
\end{pmatrix} T^{-1}
\]
\[
= T \begin{pmatrix}
-A_1^{-k} (I - A_1^{-2j+1})^{-1} & 0 \\
0 & A_2^{-k+2j+1} (I - A_2^{2j+1})^{-1}
\end{pmatrix} T^{-1}.
\]

Hence,
\[
H_j = \sum_{k=1}^{2j+1} Z_k^{(2j+1)} * H^{(0)} Z_k^{(2j+1)} = T^{-1} \begin{pmatrix}
E_j & F_j & L_j
\end{pmatrix} T^{-1}
\]

with
\[
E_j = \left( I - A_1^{-2j+1} \right)^{-1} \left( \sum_{k=1}^{2j+1} (A_1^{-k})^* H^{(0)}_{11} A_1^{-k} \right) \left( I - A_1^{-2j+1} \right)^{-1}
\]
\[
L_j = \left( I - A_2^{2j+1} \right)^{-1} \left( \sum_{k=0}^{2j+1} (A_2^{2j+1-k})^* H^{(0)}_{22} A_2^{2j+1-k} \right) \left( I - A_2^{2j+1} \right)^{-1}
\]
\[
F_j = - \left( I - A_1^{-2j+1} \right)^{-1} \left( \sum_{k=1}^{2j+1} (A_1^{-k})^* H^{(0)}_{12} A_2^{2j+1-k} \right) \left( I - A_2^{2j+1} \right)^{-1}
\]

and
\[
\begin{pmatrix}
H^{(0)}_{11} & H^{(0)}_{12} \\
H^{(0)}_{12}^* & H^{(0)}_{22}
\end{pmatrix} = T^* H^{(0)} T.
\]
It is clear that

\[ \lim_{j \to +\infty} E_j = \sum_{k=1}^{+\infty} (A_1^{-k})^* H_{11}^{(0)} A_1^{-k} \]

\[ \lim_{j \to +\infty} L_j = \sum_{k=0}^{+\infty} (A_2^{+k})^* H_{22}^{(0)} A_2^{-k} \]

\[ = \sum_{k=-\infty}^{0} (A_2^{-k})^* H_{22}^{(0)} A_2^{-k} \]

\[ \lim_{j \to +\infty} F_j = -\lim_{j \to +\infty} 2j+1 \sum_{k=1}^{2j+1} (A_1^{-k})^* H_{12}^{(0)} A_2^{2j+1-k} \]

We now show that \( \lim_{j \to +\infty} F_j = 0. \)

Since the eigenvalues of \( A_1 \) and \( A_2 \) are, respectively, outside and inside the unit disk, there exists a matrix norm \( ||| \cdot ||| \) such that \( ||| A_1^{-1} ||| < 1 \) and \( ||| A_2 ||| < 1 \).

If \( ||| A_1^{-1} ||| = ||| A_2 ||| \), then

\[ \left\| \sum_{k=1}^{2j+1} (A_1^{-k})^* H_{12}^{(0)} A_2^{2j+1-k} \right\| \leq \| H_{12}^{(0)} \| \sum_{k=1}^{2j+1} ||| A_1^{-1} ||| ||| A_2 ||| \right\|^{2j+1-k} \]

and

\[ \lim_{j \to +\infty} \sum_{k=1}^{2j+1} (A_1^{-k})^* H_{12}^{(0)} A_2^{2j+1-k} = 0. \]

If \( ||| A_1^{-1} ||| \neq ||| A_2 ||| \), then

\[ \left\| \sum_{k=1}^{2j+1} (A_1^{-k})^* H_{12}^{(0)} A_2^{2j+1-k} \right\| \leq \| H_{12}^{(0)} \| \sum_{k=1}^{2j+1} ||| A_1^{-1} ||| ||| A_2 ||| \right\|^{2j+1-k} \]

\[ = \| H_{12}^{(0)} \| \frac{||| A_2 |||^{2j+1} - ||| A_1^{-1} ||| ||| A_2 |||^{2j+1}}{||| A_1^{-1} ||| - 1} \]

and

\[ \lim_{j \to +\infty} \sum_{k=1}^{2j+1} A_1^{-k} H_{12}^{(0)} A_2^{2j+1-k} = 0. \]

Thus, we have proved that

\[ \lim_{j \to +\infty} H_j = T^{-*} \begin{pmatrix} \sum_{k=1}^{+\infty} (A_1^{-k})^* H_{11}^{(0)} A_1^{-k} & 0 \\ 0 & \sum_{k=-\infty}^{0} A_2^{-k} H_{22}^{(0)} A_2^{-k} \end{pmatrix} T^{-1}. \]

(2.23)
Using the expression of $Z_k$ in (2.14), we easily find that (2.23) is equal to

$$\sum_{k=-\infty}^{+\infty} Z_k^* H^{(0)} Z_k \equiv H. \quad \Box$$

From Theorem 2.1, we see that only the matrix $Z_{1j}^{(2j+1)}$ (or $Z_{2j}^{(2j+1)}$), for large $j$, suffices to compute $P$. We will see that the matrix $H$ can be determined only from $Z_1^{(2j+1)}$, $Z_{2j}^{(2j+1)}$, $Z_{2j+1}^{(2j+1)}$ and $Z_{2j+2}^{(2j+1)}$. This is the subject of the following subsection.

### 2.2. An iterative method for computing $Z_{1j}^{(2j+1)}$, $Z_{2j}^{(2j+1)}$, $Z_{2j+1}^{(2j+1)}$ and $Z_{2j+2}^{(2j+1)}$

We denote by $B_0 = I$ and $A_0 = -A/r$. System (2.18) can then be written as

$$\begin{cases}
B_0 Z_{1k}^{(2j+1)} + A_0 Z_{2k}^{(2j+1)} = 0 & \text{for } k = 1, 2, \ldots, 2^{j+1} - 1, \\
B_0 Z_{2j+1}^{(2j+1)} + A_0 Z_{1j}^{(2j+1)} = I.
\end{cases} \quad (2.24)$$

We first explain the idea used for solving (2.24) in the simple cases where $j = 0, 1$.

For $j = 0$, this system becomes

$$\begin{pmatrix}
B_0 & A_0 & 0 & 0 \\
A_0 & B_0 & 0 & 0 \\
0 & B_0 & 0 & A_0 \\
0 & 0 & B_0 & A_0
\end{pmatrix}
\begin{pmatrix}
Z_1^{(2)} \\
Z_2^{(2)} \\
Z_3^{(2)} \\
Z_4^{(2)}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}. \quad (2.25)$$

from which we get $Z_1^{(2)}$ and $Z_2^{(2)}$.

For $j = 1$, system (2.24) becomes

$$\begin{align*}
B_0 Z_{11}^{(4)} + A_0 Z_{21}^{(4)} &= 0, \\
B_0 Z_{31}^{(4)} + A_0 Z_{41}^{(4)} &= 0, \\
B_0 Z_{21}^{(4)} + A_0 Z_{12}^{(4)} &= 0, \\
B_0 Z_{41}^{(4)} + A_0 Z_{24}^{(4)} &= I.
\end{align*} \quad (2.26)$$

The three first equations of (2.26) can be written in matrix form as

$$\begin{pmatrix}
B_0 & A_0 & 0 & 0 \\
A_0 & B_0 & 0 & 0 \\
0 & B_0 & 0 & A_0 \\
0 & 0 & B_0 & A_0
\end{pmatrix}
\begin{pmatrix}
Z_1^{(4)} \\
Z_2^{(4)} \\
Z_3^{(4)} \\
Z_4^{(4)}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}. \quad (2.27)$$

By using a $QR$ factorization of the form

$$\begin{pmatrix}
B_0 & A_0 & 0 & 0 \\
A_0 & B_0 & 0 & 0 \\
0 & B_0 & 0 & A_0 \\
0 & 0 & B_0 & A_0
\end{pmatrix} = Q^{(0)}
\begin{pmatrix}
R_{11}^{(0)} & R_{12}^{(0)} & R_{13}^{(0)} & R_{14}^{(0)} \\
0 & R_{22}^{(0)} & R_{23}^{(0)} & R_{24}^{(0)} \\
0 & 0 & B_1 & A_1
\end{pmatrix}, \quad (2.28)$$
where we have denoted by $R_{ij}^{(0)}$, $B_1$ and $A_1$ the elements created during the QR factorization, we obtain the new system equivalent to (2.27)

$$
\begin{pmatrix}
R_{11}^{(0)} & R_{12}^{(0)} \\
0 & R_{22}^{(0)}
\end{pmatrix}
\begin{pmatrix}
Z_2^{(4)} \\
Z_3^{(4)}
\end{pmatrix}
+
\begin{pmatrix}
R_{13}^{(0)} & R_{14}^{(0)} \\
R_{23}^{(0)} & R_{24}^{(0)}
\end{pmatrix}
\begin{pmatrix}
Z_1^{(4)} \\
Z_4^{(4)}
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0
\end{pmatrix},
$$
(2.29)

$$
B_1 Z_1^{(4)} + A_1 Z_4^{(4)} = 0.
$$

Now, if we combine the last equation of (2.29) with the last equation of (2.26), we get $Z_1^{(4)}$ and $Z_4^{(4)}$. Then we obtain $Z_2^{(4)}$ and $Z_3^{(4)}$ from (2.29).

This idea can be generalized as follows: the linear system (2.24) is written as

$$
B_0 Z_k^{(2j+1)} + A_0 Z_{k+1}^{(2j+1)} = 0
$$
for $k = 1, 2, \ldots, 2^j - 1$, (2.30)

$$
B_0 Z_k^{(2j+1)} + A_0 Z_{k+1}^{(2j+1)} = 0
$$
for $k = 2^j + 1, \ldots, 2^{j+1} - 1$, (2.31)

where we have denoted by $B_j$ and $A_j$ as in (2.34)

$$
B_j Z_1^{(2j+1)} + A_j Z_2^{(2j+1)} = 0.
$$
(2.34)

Because of the similarities between Eqs. (2.30) and (2.32), the unknowns $Z_2^{(2j+1)}, Z_3^{(2j+1)}, \ldots, Z_{2^j}^{(2j+1)}$ from Eqs. (2.30) associated with $k = 1, 2, \ldots, 2^j - 1$. We obtain a new equation with the unknowns $Z_1^{(2j+1)}$ and $Z_2^{(2j+1)}$, which we denote by

$$
B_j Z_1^{(2j+1)} + A_j Z_2^{(2j+1)} = 0.
$$
(2.35)

We thus have the following cyclic linear system:

$$
\begin{align*}
B_j Z_1^{(2j+1)} + A_j Z_2^{(2j+1)} &= 0, \\
B_0 Z_2^{(2j+1)} + A_0 Z_{2j+1}^{(2j+1)} &= 0, \\
B_j Z_{2j}^{(2j+1)} + A_j Z_{2j+1}^{(2j+1)} &= 0, \\
B_0 Z_{2^j}^{(2j+1)} + A_0 Z_1^{(2j+1)} &= I.
\end{align*}
$$
(2.36)

or equivalently
\[
\begin{pmatrix}
B_0 & A_0 & 0 & 0 \\
A_j & 0 & B_j & 0 \\
0 & B_j & 0 & A_j \\
0 & 0 & A_0 & B_0
\end{pmatrix}
\begin{pmatrix}
Z_{2j+1}^{(2j+1)} \\
Z_{2j+1}^{(2j+1)} \\
Z_1^{(2j+1)} \\
Z_{2j+1}^{(2j+1)}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
I
\end{pmatrix}.
\] (2.37)

As in the case \( j = 1 \), by using a \( QR \) factorization of the matrix
\[
\begin{pmatrix}
B_0 & A_0 & 0 & 0 \\
A_j & 0 & B_j & 0 \\
0 & B_j & 0 & A_j \\
0 & 0 & A_0 & B_0
\end{pmatrix}
= Q(j)
\begin{pmatrix}
R_{11}^{(j)} & R_{12}^{(j)} & R_{13}^{(j)} & R_{14}^{(j)} \\
0 & R_{22}^{(j)} & R_{23}^{(j)} & R_{24}^{(j)} \\
0 & 0 & B_{j+1} & A_{j+1}
\end{pmatrix}
\] (2.38)

we obtain the new equation
\[
B_{j+1}Z_{1}^{(2j+1)} + A_{j+1}Z_{2j+1}^{(2j+1)} = 0,
\] (2.39)

which is then combined with the last equation of (2.36) to obtain \( Z_{1}^{(2j+1)} \) and \( Z_{2j+1}^{(2j+1)} \).

The matrices \( Z_{2j}^{(2j)} \) and \( Z_{2j+1}^{(2j+1)} \) are then obtained from (2.37) and (2.38).

The following theorem shows how the matrices \( H_j \) (and hence the matrix \( H \)) can be computed iteratively.

**Theorem 2.2.** Let

\[
A_j = B_0 Z_{2j}^{(2j+1)} + A_0 Z_{1}^{(2j+1)},
\] (2.40)

\[
\nabla_j = B_0 Z_{2j+1}^{(2j+1)} + A_0 Z_{1}^{(2j+1)}.
\] (2.41)

Then

\[
H_j = A_j^* H_{j-1} A_j + \nabla_j^* H_{j-1} \nabla_j.
\] (2.42)

**Proof.** Eqs. (2.30) and relation (2.40) are grouped in the linear system

\[
\begin{align*}
B_0 Z_{2j}^{(2j+1)} + A_0 Z_{k+1}^{(2j+1)} &= 0 & \text{for } k = 1, 2, \ldots, 2^j - 1, \\
B_0 Z_{2j}^{(2j+1)} + A_0 Z_{1}^{(2j+1)} &= A_j,
\end{align*}
\] (2.43)

Similarly to (2.24), the linear system (2.43) has a unique solution (since the eigenvalues of \( A \) are not on the circle \( \gamma \)).

We now write the system (2.24) at step \( j - 1 \):

\[
\begin{align*}
B_0 Z_{k}^{(2j)} + A_0 Z_{k+1}^{(2j)} &= 0 & \text{for } k = 1, 2, \ldots, 2^j - 1, \\
B_0 Z_{2j}^{(2j)} + A_0 Z_{1}^{(2j)} &= I
\end{align*}
\] (2.44)

and multiply each equation in (2.44) on the right by \( A_j \),
\[
\begin{align*}
B_0 Z_k^{(2^i)} A_j + A_0 Z_{k+1}^{(2^i)} A_j &= 0 \quad \text{for } k = 1, 2, \ldots, 2^j - 1, \\
B_0 Z_{2^j}^{(2^i)} A_j + A_0 Z_1^{(2^i)} A_j &= A_j.
\end{align*}
\tag{2.45}
\]

System (2.45) is nothing but (2.43) with the unknowns \(Z_k^{(2^i)} A_j, k = 1, \ldots, 2^j\). The uniqueness of the solution ensures that
\[
Z_k^{(2^{i+1})} = Z_k^{(2^i)} A_j \quad \forall k = 1, \ldots, 2^j.
\]

We show in a similar way that
\[
Z_k^{(2^{j+1})} = Z_k^{(2^j)} \nabla_j \quad \forall k = 2^j + 1, \ldots, 2^{j+1}.
\]

Therefore,
\[
H_j = \sum_{k=1}^{2^{j+1}} \left( Z_k^{(2^{j+1})} \right)^* H^{(0)} Z_k^{(2^{j+1})}
\]
\[
= \sum_{k=1}^{2^j} A_j^* \left( Z_k^{(2^j)} \right)^* H^{(0)} Z_k^{(2^j)} A_j + \sum_{k=2^j+1}^{2^{j+1}} \nabla_j^* \left( Z_k^{(2^j)} \right)^* H^{(0)} Z_k^{(2^j)} \nabla_j
\]
\[
= A_j^* H_{j-1} A_j + \nabla_j^* H_{j-1} \nabla_j. \quad \Box
\]

All the above discussion is summarized in the following algorithm:

**Algorithm 1** (Spectral dichotomy of \(A\) with respect to the circle \(\gamma = C_r\)).

1. **Initialize**: \(B_0 = I, A_0 = -A/r\):
   - Solve for \(Z_1^{(2^i)}, Z_2^{(2^i)}\) the system
     \[
     \begin{pmatrix} B_0 & A_0 \\ A_j & 0 \end{pmatrix} \begin{pmatrix} Z_1^{(2^i)} \\ Z_2^{(2^i)} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}
     \]
   - Compute \(H_0 = Z_1^{(2^i)^*} H^{(0)} Z_1^{(2^i)} + Z_2^{(2^i)^*} H^{(0)} Z_2^{(2^i)}\)

2. **Iterate**: computation of \(Z_1^{(2^{j+1})}, Z_2^{(2^{j+1})}, Z_2^{(2^{j+1})}, Z_2^{(2^{j+1})}\) and \(H_j\):
   - for \(j = 1, 2, \ldots\)
   - Use the QR factorization
     \[
     \begin{pmatrix} B_0 & A_0 \\ A_j & 0 \end{pmatrix} \begin{pmatrix} Z_1^{(2^j)} \\ Z_2^{(2^j)} \end{pmatrix} = \begin{pmatrix} R_{11}^{(j)} & R_{12}^{(j)} & R_{13}^{(j)} & R_{14}^{(j)} \\ 0 & R_{22}^{(j)} & R_{23}^{(j)} & R_{24}^{(j)} \\ 0 & 0 & B_j & A_j \end{pmatrix}
     \]
   - \(Q^{(j-1)}\) is the unitary matrix.
• Solve for $Z_{1}^{(2j+1)}$, $Z_{2j+1}^{(2j+1)}$ the system

$$
\begin{pmatrix} B_{j} & A_{j} \\ A_{0} & B_{0} \end{pmatrix} \begin{pmatrix} Z_{1}^{(2j+1)} \\ Z_{2j+1}^{(2j+1)} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}
$$

• Solve for $Z_{1}^{(2j+1)}$, $Z_{2j+1}^{(2j+1)}$ the system

$$
\begin{pmatrix} R^{(j-1)}_{11} & R^{(j-1)}_{12} \\ 0 & R^{(j-1)}_{22} \end{pmatrix} \begin{pmatrix} Z_{1}^{(2j+1)} \\ Z_{2j+1}^{(2j+1)} \end{pmatrix} = - \begin{pmatrix} R^{(j-1)}_{13} & R^{(j-1)}_{14} \\ R^{(j-1)}_{23} & R^{(j-1)}_{24} \end{pmatrix} \begin{pmatrix} Z_{1}^{(2j+1)} \\ Z_{2j+1}^{(2j+1)} \end{pmatrix}
$$

• Compute

$$
\begin{align*}
A_{j} &= B_{0}Z_{2j}^{(2j+1)} + A_{0}Z_{1}^{(2j+1)} \\
\nabla_{j} &= B_{0}Z_{2j+1}^{(2j+1)} + A_{0}Z_{1}^{(2j+1)} \\
H_{j} &= A_{j}^{*}H_{j-1}A_{j} + \nabla_{j}^{*}H_{j-1}\nabla_{j}
\end{align*}
$$

end

Fig. 1 shows the behavior of $\|H(r)\|$ when the parameter $r$ varies. Here the matrix $A$ is defined by

$$
A = \begin{pmatrix}
-2 & 25 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 10 & 3 & 3 & 3 & 0 \\
0 & 0 & 2 & 15 & 3 & 3 & 0 \\
0 & 0 & 0 & 0 & 3 & 10 & 0 \\
0 & 0 & 0 & 0 & -2 & 25 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0
\end{pmatrix}
$$

Fig. 1. Behavior of $\|H(r)\|$ computed by Algorithm 1 with $H(0) = I$ (solid curve) and by the method described in [14] (dotted curve).
This matrix has ill-conditioned eigenvalues. The double eigenvalues $-2$ and $-3$ are
defective (see e.g., [9]). The solid curve is obtained with Algorithm 1 and $H^{(0)} = I$
whereas the dotted curve is obtained with Malyshev's algorithm as described in [14].
In the neighborhoods of $|r| = |\lambda|$, where $\lambda$ is an eigenvalue of $A$, we see that $\|H(r)\|$ goes
to infinity. The more ill-conditioned the eigenvalue, the larger this neighbor-
hood. Both the algorithms give the following information: In the interval $r \in [0, 2[$
the trace of the projector $P$ is equal to: trace($P$) = 1, $\|P\| = \|I - P\| = 5.93 \times 10^4$
and $\|P^2 - P\| = 5.20 \times 10^{-11}$ with Algorithm 1 and $\|P^2 - P\| = 6.13 \times 10^{-9}$
with the algorithm in [14]. In the interval $r \in [2, 3[$, we have trace($P$) = 4, $\|P\| =$
$\|I - P\| = 3.307 \times 10^5$ and $\|P^2 - P\| = 1.16 \times 10^{-10}$ with Algorithm 1 and
$\|P^2 - P\| = 7.69 \times 10^{-8}$ with the algorithm in [14]. When $r > 3$, we have
trace($P$) = 7 and $P = I$.

3. Spectral dichotomy of a matrix with respect to the imaginary axis

In this section, we discuss the spectral dichotomy of a matrix $A$ with respect to
the imaginary axis. By using transformations of the form $\alpha I + e^{i\beta} A$, with $\alpha, \beta \in \mathbb{R}$,
the imaginary axis can be replaced by any straight line in the complex plane.
Throughout this section, we assume that $A$ does not have eigenvalues on the imag-
inary axis. Suppose that we are interested in the invariant subspace of $A$ correspond-
ing to the eigenvalues with negative real parts. Let $\gamma$ be a positively oriented contour
enclosing the eigenvalues of $A$ with negative real parts and excluding the other ones.
Then
$$P = \frac{1}{2\pi} \int_\gamma (zI - A)^{-1} \, dz = \frac{1}{2} I + \lim_{\alpha \to +\infty} \frac{1}{2\pi} \int_{-\alpha}^{+\alpha} (itI - A)^{-1} \, dt,$$
(3.1)
is the desired projector.

The dichotomy condition number is described by the norm $\omega = \|H\|$ of the
Hermitian positive definite matrix
$$H = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (itI - A)^{-*} C (itI - A)^{-1} \, dt,$$
(3.2)
where the matrix $C = C^* > 0$ is used for normalization purposes (it plays the same
role as the matrix $H(0)$ in (2.2)).

3.1. Computation of $P$ and $H$

By using exponential transformation of the form $B = e^{\tau A}$ with $\tau > 0$ a scaling
parameter, the spectral dichotomy with respect to the imaginary axis boils down to
the spectral dichotomy with respect to the unit circle. Indeed, the eigenvalues $\lambda$ of $A$
and $\mu$ of $B$ are such that $\mu = e^{\tau \lambda}$. Thus $\Re(\lambda) < 0$ if and only if $|\mu| < 1$. Moreover,
the exponential transformation does not change the invariant subspaces. In other words, the projector onto the invariant subspace corresponding to the eigenvalues in the unit disk is equal to $P$.

It can be shown [9] that the matrix pair $(P, H)$ satisfies the relation (compare with (2.3))

$$H - B^*HB = P^*H^{(0)}P - (I - P)^*H^{(0)}(I - P),$$

with

$$H^{(0)} = \int_0^\tau e^{sA^*}Ce^{sA} \, ds.$$  

The algorithm that computes $P$ and $H$ is then composed of the following steps:

1. Computation of $B = e^{\tau A}$.
2. Computation of $H^{(0)} = \int_0^\tau e^{sA^*}Ce^{sA} \, ds$.
3. Application of Algorithm 1 to the matrix $B$ and $r = 1$.

The first two steps can be accomplished in a number of ways (see e.g., [9] for more details). For example, these steps may be done as follows:

The scaling parameter $\tau$ in $e^{\tau A}$ can be chosen equal to $\tau = 1/2\|A\|$ so that $\|\tau A\| = 1/2$. Then the exponential $e^{\tau A}$ can be computed efficiently, for example, with Taylor series.

For computing the integral $H^{(0)} \equiv H^{(0)}(\tau) = \int_0^\tau e^{sA^*}Ce^{sA} \, ds$, we notice first that the derivatives of $H^{(0)}(\tau)$ as functions of $\tau$ are given by

$$\frac{d^jH^{(0)}(\tau)}{d\tau^j} = e^{\tau A^*}C_{j-1}e^{\tau A}, \quad j = 1, 2, \ldots,$$

where the matrices $C_j$ are obtained by the following recurrence:

$$C_0 = C \quad \text{and} \quad C_j = A^*C_{j-1} + C_{j-1}A, \quad j = 1, 2, \ldots$$

Then

$$H^{(0)}(\tau) = \sum_{j=0}^\infty \frac{\tau^j}{j!} \frac{d^jH^{(0)}(0)}{d\tau^j} = \sum_{j=0}^\infty \frac{\tau^{j+1}}{(j+1)!}C_j.$$

We finally obtain the following algorithm:

**Algorithm 2** (Spectral dichotomy of $A$ with respect to the imaginary axis).

1. **Computation of $B = e^{\tau A}$ with Taylor series:**
   - $\tau = \frac{1}{2\|A\|}$, $B_0 = I$, $\Omega_0 = I$.
   - for $j = 1, 2, \ldots$
     - $\Omega_j = (\tau/j)A\Omega_{j-1}$
     - $B_j = B_{j-1} + \Omega_j$
   - end
2. **Computation of $H^{(0)} = \int_0^\tau e^{sA^*}Ce^{sA} \, ds$:**
   - $E_0 = \tau C$, $H_0^{(0)} = E_0$. 


for $j = 1, 2, \ldots$

\begin{align*}
E_j &= \frac{1}{j+1} (A^* E_{j-1} + E_j A)
\end{align*}

\begin{align*}
H_j^{(0)} &= H_{j-1}^{(0)} + E_j
\end{align*}

end

3. Apply Algorithm 1 to $B$ with $\gamma = C_1$.

4. Spectral dichotomy of a regular matrix pencil with respect to a circle

In this section, we generalize the results of Section 2 to regular matrix pencils of the form $\lambda B - A$, where $A, B \in \mathbb{C}^{N \times N}$ with $\det(\lambda B - A) \neq 0 \forall \lambda \in C_r$.

We first notice that by changing $A$ to $(rB - A)^{-1} A$ and $B$ to $r(rB - A)^{-1} B$, we can assume that

\begin{align*}
B - A &= I, \quad (4.1) \\
\det(\lambda B - A) &= 0 \quad \forall \lambda \in C_1. \quad (4.2)
\end{align*}

In other words, the regular pencil $\lambda B - A$ does not have eigenvalues on the unit circle $\gamma = C_1$ and satisfy the normalization condition (4.1). Throughout this section, we assume that $\lambda B - A$ satisfies conditions (4.1) and (4.2). The normalization condition (4.1) allows us to simplify the notations (see the expression of the projector in (4.5)). It is of course only optional.

The Kronecker decomposition [4] applied to $A$ and $B$ gives

\begin{align*}
A &= T \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix} Q \\
B &= T \begin{pmatrix} I & 0 \\ 0 & B_2 \end{pmatrix} Q, \quad (4.3)
\end{align*}

where the eigenvalues of $A_1$ and $B_2$ lie inside the unit disk.

The projector onto the deflating subspace of $\lambda B - A$ corresponding to the eigenvalues inside the unit disk is given by

\begin{align*}
P &= \frac{1}{2\pi i} \int_{C_1} (\lambda B - A)^{-1} B \, d\lambda. \quad (4.4)
\end{align*}

Using the expressions of $A$ and $B$ in (4.3), we directly obtain (see also [14])

\begin{align*}
P &= Q^{-1} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} Q = T \begin{pmatrix} I & 0 \\ 0 & B_2 \end{pmatrix} T^{-1}. \quad (4.5)
\end{align*}

The last expression of $P$ follows from the normalization condition (4.1).

We now introduce the Hermitian positive definite matrix

\begin{align*}
H &= \frac{1}{2\pi} \int_0^{2\pi} \left( B - e^{-i\theta} A \right)^{-*} H^{(0)} \left( B - e^{-i\theta} A \right)^{-1} d\theta, \quad (4.6)
\end{align*}

where $H^{(0)} = H^{(0)*} > 0$ is, as previously, used for scaling purposes.
4.1. Computation of P and H

The computation of $P$ and $H$ will be done in a similar way to the case of the circular dichotomy of a matrix and will briefly be described. The $2\pi$-periodic function $\theta \mapsto (B - e^{-i\theta} A)^{-1}$ is decomposed in the Fourier series

$$
(B - e^{-i\theta} A)^{-1} = \sum_{k=-\infty}^{+\infty} Z_k e^{ik\theta} \quad (4.7)
$$

with

$$
Z_k = \frac{1}{2\pi} \int_0^{2\pi} (B - e^{-i\theta} A)^{-1} e^{-ik\theta} d\theta. \quad (4.8)
$$

From (4.8), we have

$$
sup_k \|Z_k\| < +\infty \quad (4.9)
$$

and

$$
P = Z_0 B. \quad (4.10)
$$

Now from (4.6) and (4.7) we have

$$
H = \frac{1}{2\pi} \int_0^{2\pi} (B - e^{-i\theta} A)^{-1} H^{(0)} (B - e^{-i\theta} A)^{-1} d\theta
$$

$$
= \sum_{k=-\infty}^{+\infty} Z_k^* H^{(0)} Z_k. \quad (4.11)
$$

Thus the matrices $P$ and $H$ can be computed from the Fourier coefficients $Z_k$. From (4.7), we obtain the linear system

$$
\begin{align*}
B Z_k - A Z_{k+1} &= 0, \quad \text{if } k \neq 0, \\
B Z_0 - A Z_1 &= I,
\end{align*} \quad (4.12)
$$

which is of the same kind as system (2.9). All the arguments used in Section 2 remain valid. In particular, the sequence $Z_k$ satisfies properties (2.10)–(2.12); the projector $P$ is such that $P = \lim_{j \to \infty} Z_{0}^{(2^{j+1})}$ $B = \lim_{j \to \infty} Z_{2}^{(2^{j+1})}$ $B$ and the matrix $H$ satisfies the recurrences (2.22), (2.40)–(2.42) with $B_0 = B$ and $A_0 = -A$. The matrices $B_0$, $A_0$ and the sequence $Z_{k}^{(2^{j+1})}$, $k = 1, 2, \ldots, 2^{j+1}$, satisfy system (2.24) whose solution is unique because of property (4.2). So Algorithm 1 can be used as it is.

**Remark 4.1.** We would like to point out that Malyshev developed in [14] a method to compute the integral (4.6) in the case, where $H^{(0)} = A A^* + B B^*$. Therefore the integrals computed by Algorithm 1 and by the method in [14] are different. A detailed comparison between these two methods will be done in a future work.
5. Spectral dichotomy of a regular matrix pencil with respect to an ellipse

Let \( \lambda B - A \) be a regular matrix pencil of order \( N \) having no eigenvalues on the ellipse \( \gamma \) of equation

\[
\left( \frac{\Re(\lambda)}{a} \right)^2 + \left( \frac{\Im(\lambda)}{b} \right)^2 = 1
\]

(5.1)

with \( a \geq b > 0 \).

In this section, we are interested in computing the projector \( P \) onto the deflating subspace of \( \lambda B - A \) associated to the eigenvalues inside the ellipse \( \gamma \):

\[
P = \frac{1}{2\pi i} \int_{\gamma} (\lambda B - A)^{-1} B \, d\lambda.
\]

(5.2)

With the change of variable

\[
\lambda = \frac{a + b}{2} e^{i\phi} + \frac{a - b}{2} e^{-i\phi},
\]

the projector \( P \) can be written

\[
P = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{a + b}{2} e^{i\phi} - \frac{a - b}{2} e^{-i\phi} \right) \times \left[ \left( \frac{a + b}{2} e^{i\phi} + \frac{a - b}{2} e^{-i\phi} \right) B - A \right]^{-1} B \, d\phi.
\]

(5.3)

It was shown in [7] that by using the special pencil \( \lambda \mathcal{B} - \mathcal{A} \) of order \( 2N \) with

\[
\mathcal{B} = \begin{pmatrix} \frac{a+b}{2} B & -A \\ 0 & \frac{a+b}{2} B \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} -\frac{a-b}{2} B & 0 \\ -\frac{a-b}{2} B & -\frac{a-b}{2} B \end{pmatrix},
\]

(5.4)

the spectral dichotomy of \( \lambda B - A \) with respect to \( \gamma \) reduces to that of \( \lambda \mathcal{B} - \mathcal{A} \) with respect to the unit circle \( C_1 \).

More precisely, if we denote by

\[
\mathcal{H} = \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{B} - e^{-i\theta} \mathcal{A})^{-1} \mathcal{B} \, d\theta \equiv \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{pmatrix}, \quad \text{with} \ \mathcal{H}_{ij} \in \mathbb{C}^{N \times N}
\]

the projector onto the deflating subspace of \( \lambda \mathcal{B} - \mathcal{A} \) associated to the eigenvalues inside the unit circle \( C_1 \), by \( \mathcal{H} \) the matrix

\[
H = \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{B} - e^{-i\theta} \mathcal{A})^{-1} \mathcal{B} \, d\theta \equiv \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},
\]

with \( H_{ij} \in \mathbb{C}^{N \times N} \).
associated with the projector $\mathcal{P}$ and by $H$ the matrix

$$H = \frac{1}{2\pi} \int_0^{2\pi} \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-*}$$

$$\times \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-1} d\varphi$$

(5.5)

associated with the projector $P$, then the following theorem fully characterizes the projector $P$ and the dichotomy condition number $\omega = \|H\|$ from $\mathcal{P}$ and $\|H\|$, respectively.

**Theorem 5.1.** We have

$$P = \mathcal{P}_{11} + \mathcal{P}_{22} - I,$$

(5.6)

$$H = \mathcal{H}_{11} = \mathcal{H}_{22},$$

(5.7)

$$\|H\| \leq \|\mathcal{H}\| \leq 2\|H\|,$$

(5.8)

$$\|\mathcal{H}_{12}\| \leq \|H\|.$$  

(5.9)

**Proof.** The matrix $(\mathcal{B} - e^{-i\theta} \mathcal{A})^{-1}$ can be decomposed as

$$(\mathcal{B} - e^{-i\theta} \mathcal{A})^{-1} = \begin{pmatrix} \left( \frac{a+b}{2} + \frac{a-b}{2} e^{-i\theta} \right) B & -A \\ -e^{-i\theta} A & \left( \frac{a+b}{2} + \frac{a-b}{2} e^{-i\theta} \right) B \end{pmatrix}^{-1}$$

$$= e^{i\frac{\theta}{2}} \Theta \mathcal{X} \Theta^*$$

with

$$\Theta = \begin{pmatrix} e^{i\frac{\theta}{2}} I & 0 \\ 0 & I \end{pmatrix}, \quad \Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$$

and

$$\mathcal{X} = \begin{pmatrix} \left( \frac{a+b}{2} e^{i\frac{\theta}{2}} + \frac{a-b}{2} e^{-i\frac{\theta}{2}} \right) B - A \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & \left( \frac{a+b}{2} e^{i\frac{\theta}{2}} + \frac{a-b}{2} e^{-i\frac{\theta}{2}} \right) B + A \end{pmatrix}^{-1}.$$

Thus

$$(\mathcal{B} - e^{-i\theta} \mathcal{A})^{-*} (\mathcal{B} - e^{-i\theta} \mathcal{A})^{-1} = \Theta \Omega \mathcal{X}^* \Theta^* \Theta^*.$$

From these decompositions, we directly obtain

$$\mathcal{P}_{11} = \frac{a+b}{8\pi} \int_0^{2\pi} \left[ (\lambda(\theta) B - A)^{-1} + (\lambda(\theta) B + A)^{-1} \right] e^{i\frac{\theta}{2}} B \, d\theta$$

and

$$\mathcal{P}_{22} = \frac{1}{4\pi} \int_0^{2\pi} - \left[ (\lambda(\theta) B - A)^{-1} - (\lambda(\theta) B + A)^{-1} \right] A \, d\theta$$

$$+ \frac{a+b}{8\pi} \int_0^{2\pi} \left[ (\lambda(\theta) B - A)^{-1} + (\lambda(\theta) B + A)^{-1} \right] e^{i\frac{\theta}{2}} B \, d\theta.$$
where
\[
\lambda(\theta) = \frac{a + b}{2}e^{i\theta} + \frac{a - b}{2}e^{-i\theta}.
\]

Therefore
\[
\mathcal{P}_{11} + \mathcal{P}_{22} = \frac{1}{4\pi} \int_0^{2\pi} (\lambda(\theta)B - A)^{-1} \left[ (a + b)e^{i\theta} B - A \right] d\theta
+ \frac{1}{4\pi} \int_0^{2\pi} (\lambda(\theta)B + A)^{-1} \left[ (a + b)e^{i\theta} B + A \right] d\theta
= \frac{1}{2\pi} \int_0^{\pi} \left[ \left( \frac{a + b}{2}e^{i\varphi} + \frac{a - b}{2}e^{-i\varphi} \right) B - A \right]^{-1}
\times \left( (a + b)e^{i\varphi} B - A \right) d\varphi
+ \frac{1}{2\pi} \int_{\pi}^{2\pi} \left[ \left( \frac{a + b}{2}e^{i\varphi} + \frac{a - b}{2}e^{-i\varphi} \right) B - A \right]^{-1}
\times \left( (a + b)e^{i\varphi} B - A \right) d\varphi
= \frac{1}{2\pi} \int_0^{2\pi} \left[ \left( \frac{a + b}{2}e^{i\varphi} + \frac{a - b}{2}e^{-i\varphi} \right) B - A \right]^{-1}
\times \left( (a + b)e^{i\varphi} B - A \right) d\varphi,
\]
and
\[
\mathcal{P}_{11} + \mathcal{P}_{22} - I = \frac{1}{2\pi} \int_0^{2\pi} \left[ \left( \frac{a + b}{2}e^{i\varphi} + \frac{a - b}{2}e^{-i\varphi} \right) B - A \right]^{-1}
\times \left( (a + b)e^{i\varphi} B - A \right) d\varphi
- \frac{1}{2\pi} \int_0^{2\pi} \left[ \left( \frac{a + b}{2}e^{i\varphi} + \frac{a - b}{2}e^{-i\varphi} \right) B - A \right]^{-1}
\times \left( \frac{a + b}{2}e^{i\varphi} B - A \right) d\varphi
= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{a + b}{2}e^{i\varphi} - \frac{a - b}{2}e^{-i\varphi} \right)
\times \left[ \left( \frac{a + b}{2}e^{i\varphi} + \frac{a - b}{2}e^{-i\varphi} \right) B - A \right]^{-1} B d\varphi = P.
\]

On the other hand, we have
\[
\mathcal{H}_{11} = \mathcal{H}_{22} = \frac{1}{4\pi} \int_0^{2\pi} \left[ \left( \frac{a + b}{2}e^{i\varphi} + \frac{a - b}{2}e^{-i\varphi} \right) B - A \right]^{-1}
\times \left( \frac{a + b}{2}e^{i\varphi} + \frac{a - b}{2}e^{-i\varphi} \right) B - A \right]^{-1} d\theta
\]
\[
\begin{align*}
&= \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-s} d\varphi \\
&\quad \times \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-1} d\varphi \\
&\quad + \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-s} d\varphi \\
&\quad \times \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-1} d\varphi \\
&= H
\end{align*}
\]

and

\[
\mathcal{H}_{12} = \frac{1}{4\pi} \int_{0}^{2\pi} e^{i\varphi} \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-s} \\
\times \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-1} d\varphi \\
\quad + \frac{1}{4\pi} \int_{0}^{2\pi} e^{i\varphi} \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-s} \\
\times \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-1} d\varphi \\
= \frac{1}{2\pi} \int_{0}^{2\pi} e^{i\varphi} \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-s} \\
\times \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-1} d\varphi \\
\quad + \frac{1}{2\pi} \int_{0}^{2\pi} e^{i\varphi} \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-s} \\
\times \left[ \left( \frac{a+b}{2} e^{i\varphi} + \frac{a-b}{2} e^{-i\varphi} \right) B - A \right]^{-1} d\varphi
\]
From (5.5), the last expression of $\mathcal{H}_{12}$ and the Cauchy–Schwartz inequality, we obtain

$$\forall x, y \in \mathbb{C}^N \quad |(\mathcal{H}_{12} x, y)| \leq \sqrt{(H x, x)} \sqrt{(H y, y)}.$$

So

$$\|\mathcal{H}_{12}\| = \max_{x, y \neq 0} \frac{|(\mathcal{H}_{12} x, y)|}{\|x\| \|y\|} \leq \|H\|.$$ 

Since $\mathcal{H}$ and $H$ are Hermitian positive definite, we have

$$\|\mathcal{H}\| = \max_{x, y \neq 0} \frac{(\mathcal{H} (x \ y), (x \ y))}{\|x\|^2 + \|y\|^2} = \max_{x \neq 0} \frac{(H x, x) \|x\|^2}{\|x\|^2 + \|y\|^2} \leq \|H\|.$$ 

and

$$\left( \mathcal{H} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) = (H x, x) + (H y, y) + 2 \Re((H_{12} y, x)) \leq (H x, x) + (H y, y) + 2 \sqrt{(H x, x)} \sqrt{(H y, y)} = \left( \sqrt{(H x, x)} + \sqrt{(H y, y)} \right)^2.$$

Thus

$$\|\mathcal{H}\| \leq \max_{x, y \neq 0} \frac{\left( \sqrt{(H x, x)} + \sqrt{(H y, y)} \right)^2}{\|x\|^2 + \|y\|^2} \leq 2 \|H\|. \quad \square$$

The corresponding algorithm is summarized as follows:

**Algorithm 3 (Spectral dichotomy of $\lambda B - A$ with respect to the ellipse (5.1)).**

1. Set:

$$\mathcal{B} = \begin{pmatrix} \frac{a+b}{2} & -A \\ 0 & \frac{a-b}{2} B \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} -\frac{a+b}{2} B & 0 \\ A & -\frac{a-b}{2} B \end{pmatrix}.$$ 

2. Apply the spectral dichotomy to $\lambda \mathcal{B} - \mathcal{A}$ with respect to the unit circle $C_1$. 

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## Notes

- The inequality $\|\mathcal{H}_{12}\| \leq \|H\|$ follows from the Cauchy–Schwartz inequality and the fact that $\mathcal{H}_{12}$ is Hermitian.
- The algorithm provides a method to find the spectral dichotomy of the matrix $\lambda B - A$ with respect to the ellipse defined by (5.1).
- The expressions involving $\mathcal{H}$ and $H$ are derived using the properties of Hermitian matrices and the Cauchy–Schwartz inequality.
Remark 5.1. For the sake of simplicity, we have chosen to work with the ellipse (5.1) satisfying the conditions $a \geq b$. For the general case, it suffices to replace the matrix $A$ by

$$A = \begin{pmatrix}-\frac{a-b}{2} & 0 \\ \text{sign}(a-b)A & -\frac{a-b}{2} \end{pmatrix}.$$

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References